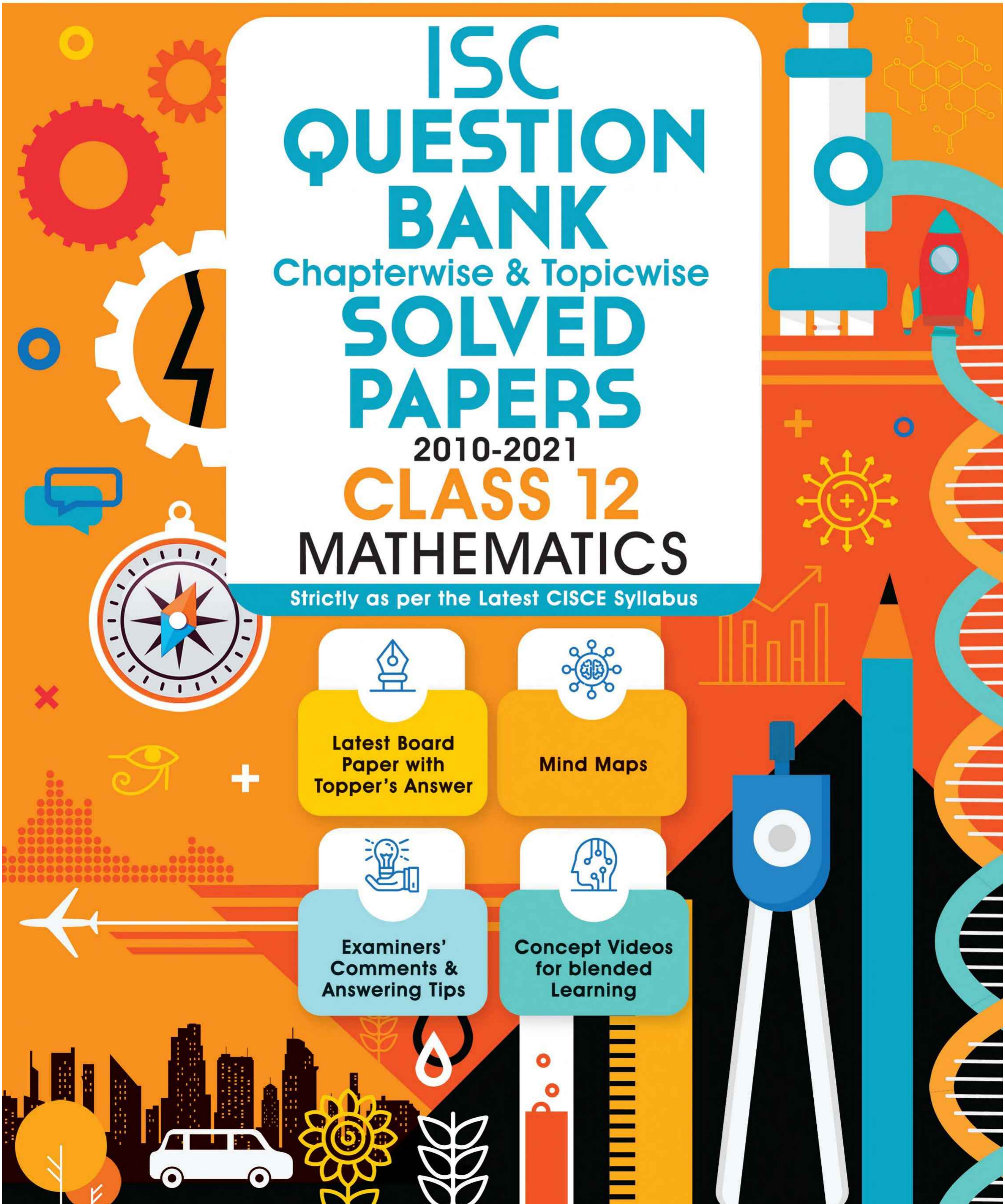
Best Seller

FOR
2023 EXAM

OSWAAL BOOKS®

LEARNING MADE SIMPLE



ISC QUESTION BANK

Chapterwise & Topicwise

SOLVED PAPERS

2010-2021

CLASS 12

MATHEMATICS

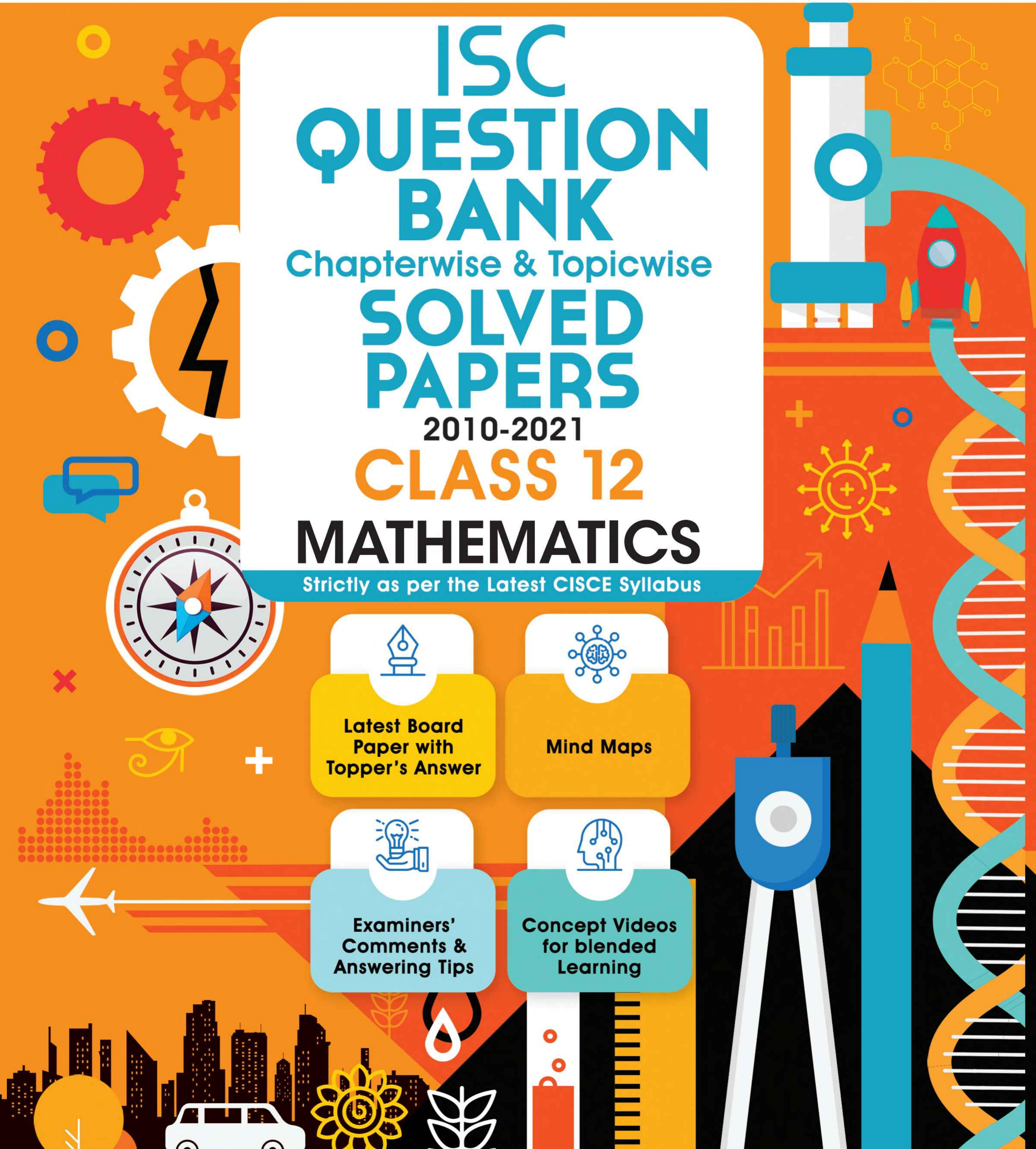
Strictly as per the Latest CISCE Syllabus

- Latest Board Paper with Topper's Answer
- Mind Maps
- Examiners' Comments & Answering Tips
- Concept Videos for blended Learning

Best Seller

FOR
2023 EXAM

OSWAAL BOOKS®
LEARNING MADE SIMPLE



ISC QUESTION BANK

Chapterwise & Topicwise

SOLVED PAPERS

2010-2021

CLASS 12

MATHEMATICS

Strictly as per the Latest CISCE Syllabus

- Latest Board Paper with Topper's Answer
- Mind Maps
- Examiners' Comments & Answering Tips
- Concept Videos for blended Learning

7th EDITION

YEAR 2022-23

ISBN "978-93-5595-124-3"

SYLLABUS COVERED

ISC–INDIAN SCHOOL CERTIFICATE EXAMINATION

COPYRIGHT RESERVED BY THE PUBLISHERS

All rights reserved. No part of this book may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, without written permission from the publishers. The author and publisher will gladly receive information enabling them to rectify any error or omission in subsequent editions.

PUBLISHED BY

OSWAAL BOOKS & LEARNING PVT. LTD.

1/11, Sahitya Kunj, M.G. Road,
Agra - 282002, (UP) India

1/1, Cambourne Business Centre
Cambridge, Cambridgeshire
CB 236DP, United kingdom

0562-2857671

contact@oswaalbooks.com

www.OswaalBooks.com

DISCLAIMER

Oswaal Books has exercised due care and caution in collecting all the data before publishing this book.

In spite of this, if any omission, inaccuracy or printing error occurs with regard to the data contained in this book, Oswaal Books will not be held responsible or liable.

Oswaal Books will be grateful if you could point out any such error or offer your suggestions which will be of great help for other readers.

Google Play Edition

TABLE OF CONTENTS

- Latest ISC Syllabus issued by CISCE for Academic Year 2022-2023 6 - 12
- Topics found difficult by candidates 13 - 13
- Concepts in which Candidates got Confused 14 - 14
- Suggestions for Students 15 - 15
- ISC Solved Specimen Paper 2021-22 Semester-2 Examination 17 - 23
- ISC Solved Board Paper 2021-22 Semester-1 Examination 24 - 32

UNIT I : Relations and Functions

1. Relations and Functions **1 - 24**
Topic 1. Relations
Topic 2. Functions
Topic 3. Composite Functions
Topic 4. Binary Operations
2. Inverse Trigonometric Functions **25 - 49**
- **Self Assessment Test-Unit-I** **50 - 50**

UNIT II : Algebra

3. Matrices **51 - 82**
Topic 1. Matrices and Operations
Topic 2. Invertible Matrices and Martin's Rule
Topic 3. Elementary Operation or Transformation of a Matrix
4. Determinants **83 - 109**
Topic 1. Determinants and its Properties
Topic 2. Applications of Determinants
- **Self Assessment Test-Unit-II** **110 - 110**

UNIT III : Calculus

5. Continuity, Differentiability & Differentiation **111 - 146**
Topic 1. Continuity and Differentiability
Topic 2. Differentiation
Topic 3. L'Hospital's Rule, Rolle's Theorem and Lagrange's Mean Value Theorem
6. Applications of Derivatives **147 - 177**
Topic 1. Rate of Change of Bodies
Topic 2. Tangents and Normals
Topic 3. Approximate Values, Differentials and Errors
Topic 4. Increasing / Decreasing Functions
Topic 5. Maxima and Minima
7. Integrals **178 - 215**
Topic 1. Indefinite Integral
Topic 2. Definite Integral
8. Differential Equations **216 - 240**
Topic 1. Basic Concepts and Variable Separable Methods
Topic 2. Linear Differential Equations
Topic 3. Homogeneous Differential Equations
Topic 4. Application of Differential Equations
- **Self Assessment Test-Unit-III** **241- 241**

UNIT IV : Probability

9. Probability **242 - 276**
Topic 1. Conditional Probability and Multiplication Theorem on Probability
Topic 2. Random Variable and its Probability Distributions, Bernoulli Trials and Binomial Distribution
Topic 3. Baye's Theorem
- **Self Assessment Test-Unit-IV** **277 - 277**

UNIT V : Vectors

10. Vectors **278 - 301**
Topic 1. Basic Algebra of Vectors and Dot Product
Topic 2. Cross Product
Topic 3. Scalar Triple Product
- **Self Assessment Test-Unit-V** **302 - 302**

UNIT VI : Three Dimensional Geometry

11. Three Dimensional Geometry **303 - 326**
Topic 1. Direction Ratios, Direction Cosines and Lines
Topic 2. Plane and Its Equations in Various Forms
- **Self Assessment Test-Unit-VI** **327- 327**

UNIT VII : Applications of Integrals

12. Applications of Integrals **328 - 339**
- **Self Assessment Test-Unit-VII** **340 - 340**

UNIT VIII : Applications of Calculus

13. Applications of Calculus **341 - 356**
- **Self Assessment Test-Unit-VIII** **357 - 357**

UNIT IX : Linear Regression

14. Linear Regression **358 - 373**
- **Self Assessment Test-Unit-IX** **374 - 374**

UNIT X : Linear Programming

15. Linear Programming **375 - 390**
- **Self Assessment Test-Unit-X** **391 - 391**

❑❑

OSWAAL BOOKS LEARNING MADE SIMPLE

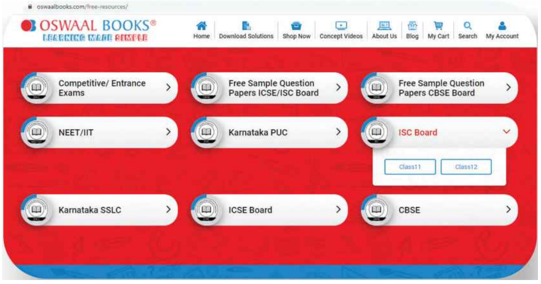
We can't let you miss out on this!

Keep Yourself Updated!
For the latest CISCE updates throughout the Academic Year 2022-23, Scan the QR Code below

Get Unlimited Access to Free Resources

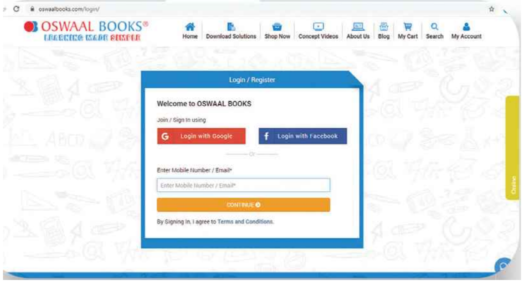
at: oswaalbooks.com/free-resources/

How? It's Easy as


oswaalbooks.com/free-resources/
OSWAAL BOOKS®
LEARNING MADE SIMPLE
Home
Download Solutions
Shop Now
Concept Videos
About Us
Blog
My Cart
Search
My Account
Competitive/ Entrance Exams
Free Sample Question Papers ICSE/ISC Board
Free Sample Question Papers CBSE Board
NEET/IIT
Karnataka PUC
ISC Board
Class11
Class12
Karnataka SSLC
ICSE Board
CBSE

Step 1: Visit oswaalbooks.com/free-resources/ and choose your board and class.

Step 2: Not Registered? Register here quickly for the ocean of knowledge inside. (If registered already, simply login and move ahead)

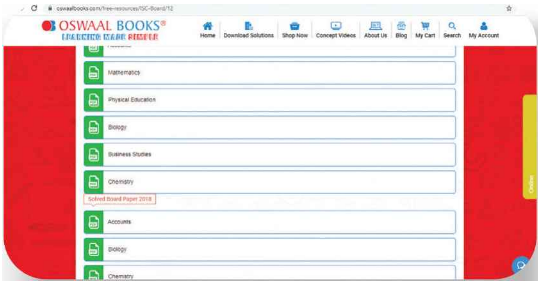

oswaalbooks.com/login/
OSWAAL BOOKS®
LEARNING MADE SIMPLE
Home
Download Solutions
Shop Now
Concept Videos
About Us
Blog
My Cart
Search
My Account
Login / Register
Login with Google
Login with Facebook
Or
CONTINUE
Online

Welcome to OSWAAL BOOKS

Join / Sign In using

Enter Mobile Number / Email*

By Signing In, I agree to Terms and Conditions.


oswaalbooks.com/free-resources/ISC-Board/12
OSWAAL BOOKS®
LEARNING MADE SIMPLE
Home
Download Solutions
Shop Now
Concept Videos
About Us
Blog
My Cart
Search
My Account
Mathematics
Physical Education
Biology
Business Studies
Chemistry
Solved Board Paper 2018
Accounts
Biology
Chemistry
Online

Step 3: You are here! Enjoy this unlimited treat of FREE resources.

LET THE ADVENTURE BEGIN!

The new way of learning; Blended Learning

The pandemic introduced us all to a phenomenon which now seems to be the way forward for learners & teachers alike, it is **blended learning.** In just a span of a year, we have witnessed a rapid advancement in e-learning. Many researchers say that, in no time e-learning will become mainstream. Oswaal Books identified this as an opportunity and thus we decided to prepare students for this turbulent yet a very useful change, hence this book is a hybrid edition. Through this hybrid edition, we aim to help the students learn at their own pace. Hence, we wish to make learning adaptive in order to simplify it for every student.

India is currently one of the youngest economies in the world. Hence, it's imperative for our education system to churn out more learners than ever before. It is this 'learner's mindset' which will set us apart from the rest of the world. Through Education, we must sow the seeds of Curiosity today to reap the benefits of Intellect tomorrow.

What Oswaal Question Banks have for you?

It is exciting to finally see new innovations in learning all over the country. The structural changes in Education that are being brought in by the new National Educational Policy (NEP) resonate with our approach of focusing on deeper understanding instead of rote learning. We have designed this book to make learning simpler for every student. Our **cognitive and innovative** exercises like the Mind Maps make us the leaders of simplified learning. Our study material proves to be so effective because it is always updated (this book is based on the CISCE Reduced Syllabus for 2022 exams) based on **Bloom's Taxonomy** and also helps retain information for a longer period of time. The last year saw a sudden rise in the need for online classes and self-study. So as **blended learning** replaces traditional learning methods, Oswaal Books provide the right material to adapt and learn faster. This book takes an **intuitive approach** for skill development to unleash a student's maximum potential. All the latest Typologies of Questions; based on the board's Specimen Paper 2021-22, have been exclusively developed by the **Oswaal Editorial Board** to help students master every form of question.

Our Heartfelt Gratitude!

We have taken due care in developing this book. There have been a lot of people who have helped us in our journey. We would like to offer heartfelt gratitude to them- our authors, editors, reviewers, and especially students like you who regularly send us suggestions which help us in continuous improvement of this book.

Here are the names of the students who have helped us make this book better:

1. **Sahil** from Patna
2. **Mauli** from Mumbai
3. **Vishaka** from Delhi
4. **Karan** from Ghaziabad
5. **Shipra** from Fatehpur
6. **Nupoor** from Alwar
7. **Muskaan** from Karnal
8. **Prastuti** from Kanpur
9. **Pratham** from Chennai
10. **Nirali** from Delhi

Wish you all Happy Learning and a Successful 2022-23!!

SYLLABUS

MATHEMATICS (860)

CLASS 12

Latest Syllabus issued by CISCE For the Academic Year 2022-2023

There will be ***two*** *papers in the subject:*
Paper I : *Theory (3 hours) ……80 marks*
Paper II: *Project Work ……20 marks*

PAPER I (THEORY) – 80 Marks

The syllabus is divided into ***three*** *sections A, B and C.*

Section A is compulsory for all candidates. Candidates will have a choice of attempting questions from ***EITHER*** *Section B* ***OR*** *Section C.*

There will be one paper of ***three*** *hours duration of 80 marks.*

Section A (65 Marks): *Candidates will be required to attempt* ***all*** *questions. Internal choice will be provided in two questions of two marks, two questions of four marks and two questions of six marks each.*

Section B/ Section C (15 Marks): *Candidates will be required to attempt* ***all*** *questions* ***EITHER*** *from Section B or Section C. Internal choice will be provided in one question of two marks and one question of four marks.*

DISTRIBUTION OF MARKS FOR THE THEORY PAPER

S.No.	UNIT	TOTAL WEIGHTAGE
SECTION A : 65 Marks		
1.	Relations and Functions	10 Marks
2.	Algebra	10 Marks
3.	Calculus	32 Marks
4.	Probability	13 Marks
SECTION B : 15 Marks		
5.	Vectors	5 Marks
6.	Three - Dimensional Geometry	6 Marks
7.	Applications of Integrals	4 Marks
OR **SECTION C : 15 Marks**		
8.	Application of Calculus	5 Marks
9.	Linear Regression	6 Marks
10.	Linear Programming	4 Marks
Total		**80 Marks**

SECTION A

1. Relations and Functions

(i) Types of relations: reflexive, symmetric,transitive and equivalence relations. One to one and onto functions, composite functions, inverse of a function. Binary operations.

- *Relations as :*
    - *Relation on a set A*
    - *Identity relation, empty relation, universal relation.*
    - *Types of Relations : reflexive, symmetric, transitive and equivalence relation.*
- *Binary Operation: all axioms and properties*
- *Functions :*
    - *As special relations, concept ofwriting "y is a function of x" as y =f(x).*
    - *Types: one to one, many to one, into, onto.*
    - *Real Valued function.*
    - *Domain and range of a function.*
    - *Conditions of invertibility.*
    - *Composite functions (algebraic functions only).*

(ii) Inverse Trigonometric Functions
Definition, domain, range, principal value

branch. Graphs of inverse trigonometric functions. Elementary properties of inverse trigonometricfunctions.

- *Principal values.*
- *$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. and their graphs.*
- $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$.
- $\sin^{-1}x = \operatorname{cosec}^{-1}\dfrac{1}{x}$; $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ *and*

  similar relations for $\cot^{-1}x$, $\tan^{-1}x$, etc.

  $$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$$

  $$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \pm \sqrt{1-y^2}\sqrt{1-x^2}\right)$$

  similarly $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}, xy < 1$

  $$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}, xy > -1$$
- *Formulae for $2\sin^{-1}x$, $2\cos^{-1}x$, $2\tan^{-1}x$, $3\tan^{-1}x$ etc. and application of these formulae.*

2. Algebra

Matrices and Determinants

(i) Matrices

Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices. Operation on matrices: Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. Non- commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order upto 3). Concept of elementary row and column operations. Invertible matrices and proof of the uniqueness of inverse, if it exists (here all matrices will have real entries).

(ii) Determinants

Determinant of a square matrix (up to 3 x 3 matrices), properties of determinants, minors, co-factors and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

- *Types of matrices ($m \times n$; $m, n \leq 3$), order; Identity matrix, Diagonal matrix.*
- *Symmetric, Skew symmetric.*
- *Operation – addition, subtraction, multiplication of a matrix with scalar, multiplication of two matrices (the compatibility).*

  E.g. $\begin{bmatrix}1 & 1\\ 0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$ *=AB (say) but BA is not possible.*
- *Singular and non-singular matrices.*
- *Existence of two non-zero matrices whose product is a zero matrix.*
- *Inverse (2×2, 3×3)* $A^{-1} = \dfrac{AdjA}{|A|}$

● *Martin's Rule (i.e. using matrices)*

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

$$A = \begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix} B = \begin{bmatrix}d_1\\ d_2\\ d_3\end{bmatrix} X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$AX = B \Rightarrow X = A^{-1}B$

Problems based on above.

NOTE 1. : The conditions for consistency of equations in two and three variables, using matrices, are to be covered.

NOTE 2. : Inverse of a matrix by elementary operations to be covered.

● *Determinants*

- *Order.*
- *Minors.*
- *Cofactors.*
- *Expansion.*
- *Applications of determinants in finding the area of triangle and collinearity.*
- *Properties of determinants. Problems based on properties of determinants.*

3. Calculus

(i) Continuity, Differentiability and Differentiation. Continuity and differentiability, derivative of composite functions, chain rule, derivatives of inverse trigonometric functions, derivative of implicit functions. Concept of exponential and logarithmic functions.

Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives. Rolle's and Lagrange's Mean Value Theorems (without proof) and their geometricinterpretation.

● *Continuity*

- *Continuity of a function at a point $x = a$.*
- *Continuity of a function in an interval.*
- *Algebra of continues function.*
- *Removable discontinuity.*

SYLLABUS

- *Differentiation*
  - *Concept of continuity and differentiability of* $|x|$, $[x]$, etc.
  - *Derivatives of trigonometric functions.*
  - *Derivatives of exponential functions.*
  - *Derivatives of logarithmic functions.*
  - *Derivatives of inverse trigonometric functions - differentiation by means of substitution.*
  - *Derivatives of implicit functions and chain rule.*
  - *e for composite functions.*
  - *Derivatives of Parametric functions.*
  - *Differentiation of a function with respect to another function e.g. differentiation of* $\sin x^3$ *with respect to* x^3.
  - *Logarithmic Differentiation - Finding dy/dx when* $y = x^{x^x}$.
  - *Successive differentiation up to* 2^{nd} *order.*

NOTE 1. : Derivatives of composite functions using chain rule.

NOTE 2. : Derivatives of determinants to be covered.

- *L' Hospital's theorem.*

  $\frac{0}{0}$, form, $\frac{\infty}{\infty}$ form, 0^0 form, ∞^∞ form etc.

- *Rolle's Mean Value Theorem - its geometrical interpretation.*
- *Lagrange's Mean Value Theorem - its geometrical interpretation.*

(ii) Applications of Derivatives

Applications of derivatives: rate of change of bodies, increasing/decreasing functions, tangents and normals, use of derivatives in approximation, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-lifesituations).

- *Equation of Tangent and Normal*
- *Approximation.*
- *Rate measure.*
- *Increasing and decreasing functions.*
- *Maxima and minima.*
  - *Stationary/turning points.*
  - *Absolute maxima/minima*
  - *local maxima/minima*
  - *First derivatives test and second derivatives test*
  - *Point of inflexion.*
  - *Application problems based on maxima and minima.*

(iii) Integrals

Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts, Evaluation of simple integrals of the following types and problems based on them.

Definite integrals as a limit of a sum, Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

- *Indefinite integral*
  - *Integration as the inverse of differentiation.*
  - *Anti-derivatives of polynomials and functions* $(ax+b)^n$, $\sin x$, $\cos x$, $\sec^2 x$, $\operatorname{cosec}^2 x$ *etc .*
  - *Integrals of the type* $\sin^2 x$, $\sin^3 x$, $\sin^4 x$, $\cos^2 x$, $\cos^3 x$, $\cos^4 x$.
  - *Integration of* $1/x$, e^x.
  - *Integration by substitution.*
  - *Integrals of the type* $f'(x)[f(x)]^n$, $\frac{f'(x)}{f(x)}$
  - *Integration of tanx, cotx, secx, cosecx.*
  - *Integration by parts.*
  - *Integration using partial fractions. Expressions of the form* $\frac{f(x)}{g(x)}$ *when degree of f(x) < degree of g(x).*

    Eg. $\frac{x+2}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1}$

    $$\frac{x+2}{(x-2)(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$$

    $$\frac{x+1}{(x^2+3)(x-1)}=\frac{Ax+B}{x^2+3}+\frac{C}{x-1}$$

  - *When degree of f(x) ≥ degree of g(x), e.g.*

    $$\frac{x^2+1}{x^2+3x+2}=1-\left(\frac{3x+1}{x^2+3x+2}\right)$$

- *Integrals of the type:*

  $$\int\frac{dx}{x^2\pm a^2},\int\frac{dx}{\sqrt{x^2\pm a^2}},\int\frac{px+q}{ax^2+bx+c}dx,\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$$

  and $\int\sqrt{a^2\pm x^2}\,dx,\int\sqrt{x^2-a^2}\,dx,$

  $$\int\sqrt{ax^2+bx+c}\,dx,\int(px+q)\sqrt{ax^2+bx+c}\,dx,$$

  integrations reducible to the above forms.

  $$\int\frac{dx}{a\cos x+b\sin x},$$

  $$\int\frac{dx}{a+b\cos x},\int\frac{dx}{a+b\sin x},\int\frac{dx}{a\cos x+b\sin x+c},$$

  $$\int\frac{(a\cos x+b\sin x)dx}{c\cos x+d\sin x},$$

$$\int \frac{dx}{a\cos^2 x + b\sin^2 x + c}$$

$$\int \frac{1 \pm x^2}{1 + x^4}\,dx,$$

$$\int \frac{1 \pm x^2}{1 + x^4}, \int \sqrt{\tan x dx}, \int \sqrt{\cot x dx},$$

- *Definite Integral*
  - *Definite integral as a limit of the sum.*
  - *Fundamental theorem of calculus (without proof)*
  - *Properties of definite integrals.*
  - *Problems based on the following properties of definite integrals are to be covered.*

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$$

where $a < c < b$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$$

$$\int_0^a f(x)dx = \int f(a \quad x)\,dx$$

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f\,(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f \text{ is an even function} \\ 0, & \text{if } f \text{ is an odd function} \end{cases}$$

(iv) Differential Equations

Definition, order and degree, general and particular solutions of a differential equation. Formation of differential equation whose general solution is given. Solution of differential equations by method of separation of variables solutions of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type $\frac{dy}{dx} + py = q$, where p and q are functions of x or constants. $\frac{dx}{dy} + px = q$, where p and q are functions of y or constants.

- *Differential equations, order and degree.*
- *Formation of differential equation by eliminating arbitrary constant(s).*
- *Solution of differential equations.*
- *Variable separable.*
- *Homogeneous equations.*
- *Linear form* $\frac{dy}{dx} + Py = Q$ *where* P *and* Q *are functions of* x *only. Similarly for* dx/dy.
- *Solve problems of application on growth and decay.*
- *Solve problems on velocity, acceleration, distance and time.*
- *Solve population-based problems on application of differential equations.*
- *Solve problems of application on coordinate geometry.*

NOTE 1: *Equations reducible to variable separable type are included.*

NOTE 2: *The second order differential equations are excluded.*

4. Probability

Conditional probability, multiplication theorem on probability, independent events, total probability, Bayes' theorem, Random variable and its probability distribution, mean and variance of random variable. Repeated independent (Bernoulli) trials and Binomial distribution.

- *Independent and dependent events conditional events.*
- *Laws of Probability, addition theorem, multiplication theorem, conditional probability.*
- *Theorem of Total Probability.*
- *Baye's theorem.*
- *Theoretical probability distribution, probability distribution function; mean and variance of random variable, Repeated independent (Bernoulli trials), binomial distribution – its mean and variance.*

SECTION B

5. Vectors

Vectors and scalars, magnitude and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Definition, Geometrical Interpretation, properties and application of scalar (dot) product of vectors, vector (cross) product of vectors, scalar triple product of vectors.

- *As directed line segments.*
- *Magnitude and direction of a vector.*
- *Types: equal vectors, unit vectors, zero vector.*
- *Position vector.*
- *Components of a vector.*
- *Vectors in two and three dimensions.*

- *$\hat{i}$, $\hat{j}$, $\hat{k}$ as unit vectors along the x, y and the z axes; expressing a vector in terms of the unit vectors.*
- *Operations: Sum and Difference of vectors; scalar multiplication of a vector.*
- *Section formula.*
- *Triangle inequalities.*
- *Scalar (dot) product of vectors and its geometrical significance.*
- *Cross product - its properties - area of a triangle, area of parallelogram, collinear vectors.*
- *Scalar triple product - volume of a parallelepiped, co-planarity.*

NOTE : Proofs of geometrical theorems by using Vector algebra are excluded.

6. Three - dimensional Geometry

Direction cosines and direction ratios of a line joining two points. Cartesian equation and vector equation of a line, coplanar and skew lines, shortest distance between two lines. Cartesian and vector equation of a plane. Angle between (i) two lines, (ii) two planes, (iii) a line and a plane. Distance of a point from a plane.

- *Equation of x-axis, y-axis, z axis and lines parallel to them.*
- *Equation of xy - plane, yz – plane, zx – plane.*
- *Direction cosines, direction ratios.*
- *Angle between two lines in terms of direction cosines / direction ratios.*
- *Condition for lines to be perpendicular/parallel.*

- *Lines*
  - *Cartesian and vector equations of a line through one and two points.*
  - *Coplanar and skew lines.*
  - *Conditions for intersection of two lines.*
  - *Distance of a point from a line.*
  - *Shortest distance between two lines.*

NOTE : Symmetric and non-symmetric forms of lines are required to be covered.

- *Planes*
  - *Cartesian and vector equation of a plane.*
  - *Direction ratios of the normal to the plane.*
  - *One point form.*
  - *Normal form.*
  - *Intercept form.*
  - *Distance of a point from a plane.*
  - *Intersection of the line and plane.*
  - *Angle between two planes, a line and a plane.*
  - *Equation of a plane through the intersection of two planes i.e. $P_1 + kP_2 = 0$.*

7. Application of Integrals

Application in finding the area bounded by simple curves and coordinate axes. Area enclosed between two curves.

- *Application of definite integrals - area bounded by curves, lines and coordinate axes is required to be covered.*
- *Simple curves: lines, circles/ parabolas/ ellipses, polynomial functions, modulus function, trigonometric function, exponential functions, logarithmic functions.*

SECTION C

8. Application of Calculus

Application of Calculus in Commerce and Economics in the following :

- *Cost function,*
- *average cost,*
- *marginal cost and its interpretation*
- *demand function,*
- *revenue function,*
- *marginal revenue function and its interpretation,*
- *Profit function and breakeven point.*
- *Rough sketching of the following curves: AR, MR, R, C, AC, MC and their mathematical interpretation using the concept of increasing- decreasing functions.*
  Self-explanatory

NOTE : *Application involving differentiation, increasing and decreasing function to be covered.*

9. Linear Regression

- *Lines of regression of x on y and y on x.*
- *Scatter diagrams*
- *The method of least squares.*
- *Lines of best fit.*
- *Regression coefficient of x on y and y on x.*
- $b_{xy} \times b_{yx} = r^2, 0 \le b_{xy} \times b_{yx} \le 1$
- *Identification of regression equations*
- *Angle between regression line and properties of regression lines.*
- *Estimation of the value of one variable using the value of other variable from appropriate line of regression.*
  Self-explanatory

10. Linear Programming

Introduction, related terminology such as constraints, objective function, optimization, different types of linear programming (L.P.) problems, mathematical formulation of L.P. problems, graphical method of solution for problems in two variables, feasible and infeasible regions (bounded and unbounded), feasible and infeasible solutions, optimal feasible solutions (up to three non-trivialconstraints).

Introduction, definition of related terminology such as constraints, objective function, optimization, advantages of linear programming; limitations of linear programming; application areas of linear programming; different types of linear programming (L.P.) problems, mathematical formulation of L.P problems, graphical method of solution for problems in two variables, feasible and infeasible regions, feasible and infeasible solutions, optimum feasible solution.

SYLLABUS

PAPER II (PROJECT WORK) – 20 Marks

Candidates will be expected to have completed two projects, one from Section A and one from either Section B or Section C.

The project work will be assessed by the subject teacher and a Visiting Examiner appointed locally and approved by the Council.

Mark allocation for **each** Project [10 marks]:

Overall format	1 mark
Content	4 mark
Findings	2 mark
Viva-voce based on the Project	3 mark
Total	**10 mark**

List of suggested assignments for Project Work:

SECTION A

1. Using a graph, demonstrate a function which is one-one but not onto.
2. Using a graph demonstrate a function which is invertible.
3. Construct a composition table using a binary function addition/multiplication modulo upto 5 and verify the existence of the properties of binary operation.
4. Draw the graph of $y = \sin^{-1} x$ (or any other inverse trigonometric function), using the graph of $y = \sin x$ (or any other relevant trigonometric function). Demonstrate the concept of mirror line (about $y = x$) and find its domain and range.
5. Explore the principal value of the function $\sin^{-1} x$ (or any other inverse trigonometric function) using a unit circle.
6. Find the derivatives of a determinant of the order of 3 x 3 and verify the same by other methods.
7. Verify the consistency of the system of three linear equations of two variables and verify the same graphically. Give its geometrical interpretation.
8. For a dependent system (non-homogeneous) of three linear equations of three variables, identify infinite number of solutions.
9. For a given function, give the geometrical interpretation of Mean Value theorems. Explain the significance of closed and open intervals for continuity and differentiability properties of the theorems.
10. Explain the concepts of increasing and decreasing functions, using geometrical significance of dy/dx. Illustrate with proper examples.
11. Explain the geometrical significance of point of inflexion with examples and illustrate it using graphs.
12. Explain and illustrate (with suitable examples) the concept of local maxima and local minima using graph.
13. Explain and illustrate (with suitable examples) the concept of absolute maxima and absolute minima using graph.
14. Illustrate the concept of definite integral $\int_a^b f(x)dx$, expressing as the limit of a sum and verify it by actual integration.
15. Demonstrate application of differential equations to solve a given problem (example, population increase or decrease, bacteria count in a culture, etc.).
16. Explain the conditional probability, the theorem of total probability and the concept of Bayes' theorem with suitable examples.
17. Explain the types of probability distributions and derive mean and variance of binomial probability distribution for a given function.

SECTION B

18. Using vector algebra, find the area of a parallelogram/ triangle. Also, derive the area analytically and verify the same.
19. Using Vector algebra, prove the formulae of properties of triangles (sine/cosine rule, etc.)
20. Using Vector algebra, prove the formulae of compound angles, e.g. sin (A + B) = Sin A Cos B + Sin B Cos A, etc.
21. Describe the geometrical interpretation of scalar triple product and for a given data, find the scalar triple product.
22. Find the image of a line with respect to a given plane.
23. Find the distance of a point from a given plane measured parallel to a given line.
24. Find the distance of a point from a line measured parallel to a given plane.
25. Find the area bounded by a parabola and an oblique line.
26. Find the area bounded by a circle and an oblique line.
27. Find the area bounded by an ellipse and an oblique line.
28. Find the area bounded by a circle and a circle.
29. Find the area bounded by a parabola and a parabola.
30. Find the area bounded by a circle and a parabola.
    (Any other pair of curves which are specified in the syllabus may also be taken.)

SECTION C

31. Draw a rough sketch of Cost (C), Average Cost (AC) and Marginal Cost (MC)
    Or
    Revenue (R), Average Revenue (AR) and Marginal Revenue (MR).
    Give their mathematical interpretation using the concept of increasing - decreasing functions and maxima-minima.
32. For a given data, find regression equations by the method of least squares. Also find angles between regression lines.
33. Draw the scatter diagram for a given data. Use it to draw the lines of best fit and estimate the value of Y when X is given and vice-versa.
34. Using any suitable data, find the minimum cost by applying the concept of Transportation problem.
35. Using any suitable data, find the minimum cost and maximum nutritional value by applying the concept of Diet problem.
36. Using any suitable data, find the Optimum cost in the manufacturing problem by formulating a linear programming problem (LPP).

NOTE : No question paper for Project Work will be set by the Council.

❑❑

SAMPLE TABLE FOR PROJECT WORK

S. No.	Unique Identification Number (Unique ID) of the candidate	PROJECT 1					PROJECT 2					TOTAL MARKS
		A	B	C	D	E	F	G	H	I	J	
		Teacher	Visiting Examiner	Average Marks (A + B ÷ 2)	Viva-Voce by Visiting Exam-iner	Total Marks (C + D)	Teacher	Visiting Examiner	Average Marks (F + G ÷ 2)	Viva-Voce by Visiting Exam-iner	Total Marks (H + I)	(E + J)
		7 Marks*	7 Marks*	7 Marks	3 Marks	10 Marks	7 Marks*	7 Marks*	7 Marks	3 Marks	10 Marks	20 Marks
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												

*Breakup of 7 Marks to be awarded separately by the Teacher and the Visiting Examiner is as follows:		Name of Teacher: Signature: Date
Overall Format	1 Mark	Name of Visiting Examiner
Content	4 Marks	
Findings	2 Marks	Signature: Date

NOTE: VIVA-VOCE (3 Marks) for each Project is to be conducted only by the Visiting Examiner, and should be based on the Project only

Topics Found Difficult by Candidates

Examination Paper 2020

- Continuity and derivability
- Modulus function
- Applications on derivatives
- Integration by substitution and by parts
- Applications integrals
- 3D geometry
- Inverse circular functions
- Mean value theorems, open and closed interval in Mean Value theorem.

Examination Paper 2019

- Increasing and decreasing functions.
- In proving the function "onto".
- Applying formulae of inverse trigonometric functions.
- Finding the area of a triangle using determinant.
- Proving differentiability.
- Application of derivatives, in particular, rate measurer, Maxima and Minima.
- Conditional Probability and Bayes' theorem.
- Family of planes and equation of straight line of 3D Geometry.
- Area under the curve.
- Application of calculus in Commerce and Economics.

Examination Paper 2018

- Application of derivatives including Maxima and Minima.
- Integrals, and curve sketching.
- Vectors, interchange of vector equation to Cartesian equation and vice-versa.
- Probability and Probability distribution.
- Inverse circular functions.

Examination Paper 2017

- Conic Section in general
- Integrals, definite Integrals and curve sketching
- Vectors and Interchange of vector equation to Cartesian form (vice versa) of plane and Straight-line equations of 3D Geometry and their applications
- Conditional Probability and their applications, Binomial Probability distribution
- Complex numbers and Inverse circular functions
- Maxima and Minima

Examination Paper 2016

- Indefinite Integrals (use of substitution or integration by parts)
- Definite Integrals – use of properties.
- Inverse Circular Functions (formulae and relations)
- Differential Equations (solving Homogeneous and Linear Differential Equations)
- Vectors – in general
- Annuity (Deferred annuities)
- Conics in general.
- Probability – use of sum and product laws and identifying all cases.

Examination Paper 2015

- Determinant properties and their use.
- Conics (parabola, ellipse, hyperbola)
- Application of L Hospital's rule.
- Indefinite Integrals, Definite Integrals.
- Inverse trigonometric functions.
- Area of curves.
- Probability (Both sections) and probability distribution.
- Differential equations.
- Complex numbers.
- Vectors.
- 3D plane & straight-line.
- Annuities.
- Linear programming.
- Regression lines.

OSWAAL BOOKS® LEARNING MADE SIMPLE

Concepts in which Candidates got Confused

Examination Paper 2020

- Probability, mutually exclusive and independent events, Product and sum rule of probability, dependent and independent events.
- Identification of the types of differential equations and their framing
- Between the regression coefficients byx and bxy and regression lines y on x and x on y.
- Linear Programming Problems in identifying the constraints.
- Application of definite integrals: Sketching of curves, Identifying the area of the shaded region and finding the upper and lower limits from the graph.

Examination Paper 2019

- Between the composite functions $f \circ g$ and $g \circ f$.
- Between open and closed interval in Mean Value theorem.
- Between mutually exclusive and independent events.
- Identifying the type of differential equation.
- Definite integrals and their properties.
- Between Marginal cost and average cost
- Between the regression coefficients b_{yx} and b_{xy} and regression lines y on x and x on y.
- Between Dot and Cross product of vectors.

Examination Paper 2018

- Open and closed interval of mean value theorems.
- Product and sum rule of probability and dependent and independent events.
- Dot and Cross product of vectors, Projection of a vector.
- Properties of definite integrals.

Examination Paper 2017

- Hyperbola and Ellipse: Their standard form and other relations
- Lagrange's Mean Value Theorem
- Product and sum rule of probability and concepts of dependent and independent events
- Definite Integrals and their properties
- Dot and Cross product of vectors
- Present value of annuity and Amount of annuity at the end of the period.

Examination Paper 2016

- Regression lines: y on x and x on y
- Sum and product laws of probability
- 3 – D: Image of a given point and perpendicular distance
- Conditional probability property in Baye's theorem
- Price Index by aggregate and Price Relative methods
- Differences between and usage of formulae for BD, TD, BG, DV, etc.

Examination Paper 2015

- Conics (parabola, ellipse, hyperbola)
- Open & closed intervals for Mean value theorem.
- Conversion of inverse trigonometric functions.
- Regression coefficient b_{yx} & b_{xy} and r.
- Differential equations (Linear & Homogeneous form)
- Geometrical problem in vectors.
- Annuity due & ordinary annuity.
- Banker's discount & banker's gain.
- Price relative and aggregate method in Index No.
- Shortest distance between skew lines and parallel lines.
- Probability distribution (conceptual problem)

Suggestions for Students

- ⇨ Study and practice the entire syllabus keeping in view the pattern and the weightage of each chapter in the question paper. Avoid selective study
- ⇨ Stress on learning the concepts of topics like calculus and its applications as its weightage is maximum in the syllabus.
- ⇨ Understand the concepts of each topic and practice an adequate number of problems on a regular basis, taking the assistance of the teacher, wherever required.
- ⇨ Make a list of all formulae and revise them frequently.
- ⇨ Take part in all periodic tests conducted by the school sincerely, identify the areas of weakness and try to resolve your problems.
- ⇨ Revise the concepts of Class XI and integrate them with the Class XII syllabus (factorization, formulae of algebraic expressions, etc.)
- ⇨ Make use of the given reading time of 15 minutes, for reading the questions carefully and high-lighting various key points in the statements of the questions.
- ⇨ Avoid calculation errors and increase accuracy to the highest levels while solving problems based on numerical values.
- ⇨ Manage your time effectively while attempting the question paper. Practice mock/sample papers by strictly adhering to the stipulated time.
- ⇨ Avoid selective study. Study and practice the entire syllabus keeping in view the pattern of and the weightage of each chapter in the question paper.
- ⇨ Revise the concepts of Class XI and integrate them with Class XII syllabus.
- ⇨ Understand the concepts of each topic and practice adequate number of problems on a regular basis, taking the assistance of the teacher, wherever required.
- ⇨ Make chapter/topic wise list of all formulae, comprehend and revise the same at regular intervals.
- ⇨ Utilize the reading time of 15 minutes for reading the questions carefully and make a note of the various key points in the statements of the questions.
- ⇨ Increase the level of accuracy to the highest levels while solving problems based on numerical values.
- ⇨ Manage your time effectively while attempting the question paper. Practise mock/sample papers by strictly adhering to the stipulated time.
- ⇨ Clarify the concepts of each chapter/topic with the help of your teacher.
- ⇨ Learn the Formulae related to every topic after acquiring thorough understanding of each symbol used.
- ⇨ Revise all topics and formulae involved and make a chapter wise or topic-wise list of these.
- ⇨ Time management is important while attempting the paper. Practise solving papers within a stipulated time.
- ⇨ Practise mock papers by following the time management with the guidance of the teacher.
- ⇨ Avoid selective study. Study the entire syllabus thoroughly and revise from time to time.
- ⇨ Concepts of Class XI must be revised and integrated with the Class XII syllabus.
- ⇨ Concepts of each chapter/topic must be clear. Formulae of related topics must be learnt.
- ⇨ Time management is important while attempting the paper. Practice solving papers within a stipulated time. Make wise choices from the options available in the question paper and manage time wisely.
- ⇨ Develop logical and reasoning skills to have a clear understanding.
- ⇨ Revise all topics and formulae involved and make a chapter wise or topic-wise list of these.
- ⇨ Make wise choices from the options available in the question paper and management time wisely.
- ⇨ Be methodical and neat in working.
- ⇨ Learn to use the easiest method with correct formula for solving a problem.
- ⇨ Theorem, rules and laws to be well understood.
- ⇨ Attempt the question with a tension-free approach.

Exclusive School Books Suppliers

OUR DISTRIBUTORS

	ANDHRA PRADESH
HYDERABAD	Sri Balaji Book Depot, (040) 27613300, 9866355473
	ASSAM
GUWAHATI	Book Emporium, 9675972993, 6000763186
	Kayaan Enterprises, (0361) 2630443
	BIHAR
PATNA	Bokaro Student Friends, (0612) 2300600, 2618400, Gyan Ganga Ltd., 9304826651
	Nova Publisher & Distributors, (0612) 2666404,
	Shri Durga Pustak Mandir, 9334477386, Vikas Book Depot, 7004301983, 0612-2304753, Sharda Pustak Bhandar, 9334259293
MUZAFFARPUR	Pustak Bhandar, 9097046555
	DELHI
DELHI	Mittal Books, (011) 23288887, 9899037390, R.D. Chawla & Sons, (011) 23282360-61
	CHANDIGARH
PATIALA	Adarsh Enterprises, Patiala, (0175) 2311431, 9814347613
	GUJARAT
AHMEDABAD	Patel Book, 9898184248, 9824386112
VAPI	Goutam Book Sellers, 9081790813
	JHARKHAND
BOKARO	Bokaro Student Friends, (0654) 2233094, 7360021503, Sahu Pustak Bhandar, 9431378296, 7979845480, Bharati Bhawan, 9835523899
RANCHI	Crown Book Distributor & Publishers, (0651) 2213735, Bokaro Students Friends, 9234628152
	KARNATAKA
HUBLI	Renuka Book Distributors, 0836-2244124, 9513355124
	KERALA
ERNAKULAM	Academic Book House, (0484) 2376613
JOMER SRINILAYAM	H & C Store, (0484) 2351233
KOTTAYAM	H & C Store, (0481) 2304351, Book Centre, (0481) 2566992
TRIVANDRUM	Academic Book House, (0471) 2333349, 9447063349,
	H & C Store, (0471) 2572010, 9446411996
CALICUT	Aman Book Stall, (0495) 2721282,
	MADHYA PRADESH
INDORE	Sushil Prakashan,(0731) 2503333, 2535892, 9425322330, Arun Prakashan, (0731) 2459448, 9424890785
	MAHARASHTRA
MUMBAI	Shivam Books & Stationery, (022) 28230897, 9892935799, Vidyarthi Sales Agencies, 9819776110

NAGPUR	Laxmi Pustakalay Stationers, (0712) 2727354, Novelty Book Depot, 9657690200, Vijay Book Depot, 9860122094, (0712) 2520496, 2534217, Karmveer Book, 9923966466
DHULE	Navjeevan Book Stall, 7020525561
PUNE	Natraj The Book Shopee, (020) 24485054, 9890054092,
	ODISHA
CUTTACK	A. K. Mishra Agencies, 9437025991, 9437081319
BHUBANESHWAR	M/s Pragnya, 8847888616, 9437943777
	PUNJAB
LUDHIANA	Amit Book Depot., 9815807871
JALANDHAR	Cheap Book Store, 9872223458, 9878258592, Gaurav the Book World, 9478821183, Subhash Book Depot, 9876453625, City Book Shop, 9417440753
	RAJASTHAN
JAIPUR	Saraswati Book House, (0141) 2610823, 9829811155
BHILWARA	Nakoda Book Depot, (01482) 243653, 9214983594
	TAMIL NADU
CHENNAI	Indian Book House, (044) 24327784, 9094129595,
	Vijaya Stores, 9381037417
SALEM	Salem Book House & Dealer, 9443324584 (0427-2411315)
	TRIPURA
AGARTALA	Books Corner, 9856358594
	UTTAR PRADESH
AGRA	Sparsh Book Agency, 9412257817, Om Pustak Mandir, (0562) 2464014, 9319117771, Panchsheel Books, 9412257961, 9412257962,
ALLAHABAD	Mehrotra Book Agency, (0532) 2266865, 9415636890
AZAMGARH	Sasta Sahitya Sadan, 9450029674
GORAKHPUR	Atul Pustak Bhandar, 9935210622, Central Book, 9935454590
LUCKNOW	Rama Book Depot (Retail), (0522) 4080133, 9956922433, Vyapar Sadan, 7607102462, Azad Book Depot Pvt. Ltd., 7317000250
MEERUT	Ideal Book Depot, (0121) 4059252, 9837066307
VARANASI	Bokaro Student Friends, (0542) 2401250, 8299344607
	Gupta Books, 9918155500, 8707225564
	UTTARAKHAND
DEHRADUN	Brothers Pustak Bhandar, 9897833882, Mittal Book, 9837223825
	WEST BENGAL
KOLKATA	Oriental Publishers & Distributor (033) 40628367,
	Katha 'O' Kahini, (033) 22196313, 22419071,
	Saha Book House, (033) 22193671, 9333416484
	United Book House, 9231692641, (033) 22418105
SILIGURI	Agarwal Book House, (0353) 2535274, 9832038727

3003

Specimen Question Paper
ISC SEMESTER-2 EXAMINATION
(Issued by CISCE on Date 20th Dec. 2021)
MATHEMATICS
Class-12th

SOLVED

Max. Marks : 40 **Time allowed : 1:30 Hours**

General Instructions:

1. *Candidates are allowed an additional 10 minutes for only reading the paper.*
2. *They must Not start writing during this time.*
3. *The Question Paper consists of three sections A, B and C.*
4. *Candidates are required to attempt all questions from Section A and all questions EITHER from Section B OR Section C.*
5. *All working, including rough work, should be done on the same sheet as, and adjacent to the rest of the answer.*
6. *The intended marks for questions or parts of questions are given in brackets [ ].*
7. *Mathematical tables and graph papers are provided.*

OSWAAL BOOKS® LEARNING MADE SIMPLE

Section-A

32 Marks

1. Choose the correct option for the following questions.

(i) If $\int \frac{(\log x)^2}{x}\,dx = \frac{(\log x)^k}{k} + c$, then the value of k is: **[1]**

(A) 3
(B) 2
(C) 1
(D) None of the above options

(ii) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(k-x)dx$, then the value of k is: **[1]**

(A) a
(B) $2a$
(C) Independent of a
(D) None of the above options

(iii) The degree of the differential equation $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2\left(\frac{d^2y}{dx^2}\right)^2$ is: **[1]**

(A) 1 **(B)** 2
(C) 3 **(D)** 4

(iv) Given $\int e^x\left(\frac{x-1}{x^2}\right)dx = e^x f(x) + c$. Then $f(x)$ satisfying the equation is: **[1]**

(A) x
(B) x^2
(C) $\frac{1}{x}$
(D) None of the above options

(v) Two cards are drawn out randomly from a pack of 52 cards one after the other, without replacement. The probability of first card being a king and second card not being a king is: **[1]**

(A) $\frac{48}{663}$ **(B)** $\frac{24}{663}$
(C) $\frac{12}{663}$ **(D)** $\frac{4}{663}$

(vi) If two balls are drawn from a bag containing 3 white, 4 black and 5 red balls. Then, the probability that the drawn balls are of different colours is: **[1]**

(A) $\frac{1}{66}$ **(B)** $\frac{3}{66}$
(C) $\frac{19}{66}$ **(D)** $\frac{47}{66}$

2. Evaluate: $\int \frac{x^3 - x^2 + x - 1}{x-1}dx$. **[2]**

OR

Evaluate: $\int \log_{10} x\, dx$.

3. Solve the differential equation: $\operatorname{cosec}^3 x\, dy - \operatorname{cosec} y\, dx = 0$. **[2]**

OR

Solve the differential equation: $\frac{dy}{dx} = 2^{-y}$.

4. Evaluate: $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$. **[2]**

5. A bag contains 6 red and 5 blue balls and another bag contains 5 red and 8 blue balls. A ball is drawn from the first bag and without noticing its colour is placed in the second bag. If a ball is drawn from the second bag, then find the probability that the drawn ball is red in colour. **[4]**

OR

A bag contains 3 red and 4 white balls and another bag contains 2 red and 3 white balls. If one ball is drawn from the first bag and 2 balls are drawn from the second bag, then find the probability that all three balls are of the same colour.

6. Evaluate: $\int \frac{1+\sin x}{1-\sin x}dx$. **[4]**

7. In a bolt factory, machines X, Y and Z manufacture 20%, 35% and 45% respectively of the total output. Of their output 8%, 6% and 5% respectively are defective bolts. One bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured in the machine Y? **[6]**

8. Evaluate: $\int \frac{dx}{\sin x+\sin 2x}$. **[6]**

OR

Evaluate: $\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$.

Section-B

8 Marks

9. Choose the correct option for the following questions: [2]

(i) The equation of the plane which is parallel to $2x - 3y + z = 0$ and which passes through $(1, -1, 2)$ is:

(A) $2x - 3y + z - 7 = 0$
(B) $2x - 3y + z + 7 = 0$
(C) $2x - 3y + z - 8 = 0$
(D) $2x - 3y + z + 6 = 0$

(ii) The intercepts made on the coordinate axes by the plane $2x + y - 2z = 3$ are:

(A) $\frac{-3}{2}, -3, \frac{-3}{2}$ **(B)** $\frac{3}{2}, 3, \frac{-3}{2}$

(C) $\frac{3}{2}, -3, \frac{-3}{2}$ **(D)** $\frac{3}{2}, 3, \frac{3}{2}$

10. Find the equation of the plane passing through the point (1, 1, 1) and is perpendicular to the line $\frac{x-1}{3} = \frac{y-2}{0} = \frac{z-3}{4}$. Also, find the distance of this plane from the origin. **[2]**

11. Using integration, find the area of the region bounded between the line $x = 4$ and the parabola $y^2 = 16x$. **[4]**

Section-C

8 Marks

12. Choose the correct option for the following questions: [2]

(i) If the regression line of x on y is, $9x + 3y - 46 = 0$ and y on x is, $3x + 12y - 7 = 0$, then the correlation coefficient 'r' is equal to:

(A) $\frac{-1}{12}$ **(B)** $\frac{1}{12}$

(C) $\frac{-1}{2\sqrt{3}}$ **(D)** $\frac{1}{2\sqrt{3}}$

(ii) If $\overline{X} = 40$, $\overline{Y} = 6$, $\sigma_x = 10$, $\sigma_y = 1.5$ and $r = 0.9$ for the two sets of data X and Y, then the regression line of X on Y will be:

(A) $x - 6y - 4 = 0$ **(B)** $x + 6y - 4 = 0$
(C) $x - 6y + 4 = 0$ **(D)** $x + 6y + 4 = 0$

13. For 5 observations of pairs (x, y) of variables X and Y, the following results are obtained:

$\Sigma x = 15, \Sigma y = 25, \Sigma x^2 = 55, \Sigma y^2 = 135, \Sigma xy = 83.$

Calculate the value of b_{xy} and b_{yx}. **[2]**

14. A manufacturer wishes to produce two commodities A and B. The number of units of material, labour and equipment needed to produce one unit of each commodity is shown in the table given below. Also shown is the available number of units of each item, material, labour, and equipment.

Items	**Commodity A**	**Commodity B**	**Available no. of Units**
Material	1	2	8
Labour	3	2	12
Equipment	1	1	10

Find the maximum profit if each unit of commodity A earns a profit of ₹ 2 and each unit of B earns a profit of ₹ 3. **[4]**

■■

ANSWERS

Section-A

1. (i) Option (A) is correct.

Explanation:

Given, $\int \frac{(\log x)^2}{x}dx = \frac{(\log x)^k}{k}+C$

Taking $\int \frac{(\log x)^2}{x}dx$

Let $\log x = t$

Differentiating w.r.t. x,

$$\frac{1}{x}dx = dt$$

$$\therefore \quad \int \frac{(\log x)^2}{x}dx = \int t^2 dt$$

$$= \frac{t^3}{3}+C$$

$$= \frac{(\log x)^3}{3}+C$$

Hence, $\frac{(\log x)^3}{3}+C = \frac{(\log x)^k}{k}+C$

Comparing on both sides, we get

$\Rightarrow \quad k = 3.$

(ii) Option (B) is correct.

Explanation:

Given, $\int_0^{2a} f(x)dx = \int_0^a f(x)dx+\int_0^a f(k-x)dx$ ...(i)

$$\because \quad \int_0^{2a} f(x)dx = \int_0^a f(x)dx+\int_a^{2a} f(x)dx$$

$$\therefore \quad \int_0^{2a} f(x)dx = \int_0^a f(x)dx-\int_a^0 f(2a-t)dt$$

$[\because x = 2a-t \Rightarrow dx = -dt]$

$$= \int_0^a f(x)dx+\int_0^a (2a-t)dt$$

$$= \int_0^a f(x)dx+\int_0^a (2a-x)dx \quad \text{...(ii)}$$

From eq. (i) and (ii), we get $k = 2a$.

(iii) Option (B) is correct.

Explanation: Given differential equation

$$\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2 = x^2\left(\frac{d^2y}{dx^2}\right)^2$$

Here, the highest differential coefficient is $\frac{d^2y}{dx^2}$, then its degree = 2.

(iv) Option (C) is correct.

Explanation: Given,

$$\int e^x\left(\frac{x-1}{x^2}\right)dx = e^x f(x) + C \quad \text{...(i)}$$

$$\text{L.H.S.} = \int e^x\left(\frac{x-1}{x^2}\right)dx$$

$$= \int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$

$$= \int e^x.\frac{1}{x}dx-\int e^x.\frac{1}{x^2}dx$$

$$= \frac{1}{x}.e^x-\int -\frac{1}{x^2}e^x dx-\int e^x.\frac{1}{x^2}dx+C$$

[Integrating the first integral by parts by taking $\frac{1}{x}$ as the first function]

$$= \frac{1}{x}.e^x+\int \frac{1}{x^2}e^x dx-\int e^x\frac{1}{x^2}dx+C$$

$$= \frac{1}{x}.e^x+C$$

From eq. (i)

$$f(x) = \frac{1}{x}$$

(v) Option (A) is correct.

Explanation: Total number of cards = 52

Number of king cards = 4

$\therefore$ Probability of king card $= \frac{4}{52} = \frac{1}{13}$

Remaining cards = 52 – 1 = 51

$\therefore$ Probability of drawing second card $= \frac{52-4}{51}$

$$= \frac{48}{51}$$

Hence, Probability of first and second card

$$= \frac{1}{13}\times\frac{48}{51} = \frac{48}{663}.$$

(vi) Option (D) is correct.

Explanation: Required probability that the drawn balls are of different colour

$$= \frac{{}^3C_1\times{}^4C_1}{{}^{12}C_2}+\frac{{}^3C_1\times{}^5C_1}{{}^{12}C_2}+\frac{{}^4C_1\times{}^5C_1}{{}^{12}C_2}$$

$$= \frac{3\times4}{\frac{12\times11}{2}}+\frac{3\times5}{\frac{12\times11}{2}}+\frac{4\times5}{\frac{12\times11}{2}}$$

$$= \frac{12}{66}+\frac{15}{66}+\frac{20}{66}$$

$$= \frac{47}{66}.$$

2. We have,

$$\int\frac{x^3-x^2+x-1}{x-1}dx = \int\frac{[x^2(x-1)+1(x-1)]}{x-1}dx$$

$$= \int\frac{(x-1)(x^2+1)}{(x-1)}dx$$

$$= \int(x^2+1)dx$$

$$= \int x^2dx+\int dx$$

$$= \frac{x^3}{3}+x+C.$$

OR

$$\int(\log_{10} x)dx = \int(\log_{10} x).1\,dx$$

$$= (\log_{10} x).x-\int\left(\frac{1}{x}\log_{10} e\right)x\,dx$$

[Using integration by parts]

$$= x\log_{10} x-\log_{10} e\int 1dx$$

$$= x\log_{10} x-x\log_{10} e + C.$$

3. Given differential equation,

$\text{cosec}^3 x\,dy - \text{cosec}\,y\,dx = 0$

$$\Rightarrow \quad \text{cosec}^3 x\,dy = \text{cosec}\,y\,dx$$

$$\Rightarrow \quad \frac{dy}{\text{cosec}\,y} = \frac{dx}{\text{cosec}^3 x}$$

$$\Rightarrow \quad \sin y\,dy = \sin^3 x\,dx$$

OSWAAL BOOKS LEARNING MADE SIMPLE

Integrating on both sides, we get

$$\int \sin y\, dy = \int \sin^3 x\, dx$$

$$\Rightarrow \quad \int \sin y\, dy = \int \frac{3\sin x - \sin 3x}{4} dx$$

[Using $\sin 3x = 3\sin x - 4\sin^3 x$]

$$\Rightarrow \quad -\cos y = \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx$$

$$\Rightarrow \quad -\cos y = \frac{3}{4}(-\cos x) - \frac{1}{4}\times\left(-\frac{\cos 3x}{3}\right) + C$$

$$\Rightarrow \quad -\cos y = -\frac{3}{4}\cos x + \frac{1}{4}\cos 3x + C\,.$$

OR

Given, $\dfrac{dy}{dx} = 2^{-y}$

$$\Rightarrow \quad \frac{dy}{2^{-y}} = dx$$

$$\Rightarrow \quad 2^y dy = dx$$

Integrating on both sides, we get

$$\int 2^y dy = \int dx$$

$$\Rightarrow \quad \frac{2^y}{\log_e 2} = x + C$$

or $\dfrac{2^y}{\log_e 2} - x = C.$

4. Let $I = \displaystyle\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}} dx \quad$ ...(i)

$$= \int_2^8 \frac{\sqrt{10-(8+2-x)}}{\sqrt{8+2-x}+\sqrt{10-(8+2-x)}} dx$$

[Using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$]

$$= \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}} dx \quad \text{...(ii)}$$

On adding eq. (i) and (ii), we get

$$2I = \int_2^8 \left(\frac{\sqrt{10-x}+\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\right) dx$$

$$2I = \int_2^8 dx = [x]_2^8$$

$$\Rightarrow \quad I = \frac{1}{2}[8-2]$$

$$= \frac{1}{2}\times 6 = 3.$$

5. Let E_1, E_2 and A denote the following events:

E_1: Red ball is transferred from Ist bag to IInd bag.

E_2: Red ball is transferred from IInd bag to Ist bag.

A: Drawn ball from IInd bag is red in colour. It is given that

$$P(E_1) = \frac{6}{11},\ P(E_2) = \frac{5}{13}$$

E_1 has occurred 6 red balls and 8 blue balls,

So, $P\left(\dfrac{A}{E_1}\right) = \dfrac{6}{14} = \dfrac{3}{7}$

$$P\left(\frac{A}{E_2}\right) = \frac{4}{12}$$

$$\therefore \quad P(A) = P(E_1)\times P\left(\frac{A}{E_1}\right) + P(E_2)\times P\left(\frac{A}{E_2}\right)$$

$$= \frac{6}{11}\times\frac{3}{7} + \frac{5}{13}\times\frac{4}{12}$$

$$= \frac{18}{77} + \frac{5}{39}$$

$$= 0.36$$

OR

Let E_1, E_2, E_3, E_4 and E denote the following events:

E_1: Taking out 1 red ball from first bag

E_2: Taking out 2 red balls from second bag

E_3: Taking out 1 white ball from first bag

E_4: Taking out 2 white balls from second bag

and E: 3 balls are of same colour.

$\therefore$ Required probability,

$$P(E) = P(E_1).P(E_2) + P(E_3).P(E_4)$$

$$= \frac{3}{7}\times\frac{{}^2C_2}{{}^5C_2} + \frac{4}{7}\times\frac{{}^3C_2}{{}^5C_2}$$

$$= \frac{3}{7}\times\frac{1}{10} + \frac{4}{7}\times\frac{3}{10}$$

$$= \frac{3}{70} + \frac{12}{70}$$

$$= \frac{15}{70} = \frac{3}{14}.$$

6. Given, $\displaystyle\int \frac{1+\sin x}{1-\sin x} dx = \int \frac{1+\cos\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)} dx$

[$\because$ $\sin x = \cos(90° - x)$]

$$= \int \frac{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)} dx$$

[$\because$ $1 + \cos\theta = 2\cos^2\dfrac{\theta}{2}$

and $1 - \cos\theta = \sin^2\dfrac{\theta}{2}$]

$$= \int \cot^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$= \int\left[\text{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right)-1\right]dx$$

$$= \int \text{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx-\int dx$$

$$= -\frac{\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)}{-\frac{1}{2}}-x+C$$

$$= 2\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)-x+C.$$

7. Let event E_1: bolt manufactured by machine X

event E_2: bolt manufactured by machine Y

event E_3: bolt manufactured by machine Z

and A: defective bolt is drawn.

$$\therefore \quad P(E_1) = \frac{20}{100} = 0.2$$

$$P(E_2) = \frac{35}{100} = 0.35$$

$$P(E_3) = \frac{45}{100} = 0.45$$

$$P\left(\frac{A}{E_1}\right) = \frac{8}{100} = 0.08$$

$$P\left(\frac{A}{E_2}\right) = \frac{6}{100} = 0.06$$

$$P\left(\frac{A}{E_3}\right) = \frac{5}{100} = 0.05$$

Using Bayes' theorem,

Required Probability,

$$P\left(\frac{E_2}{A}\right)$$

$$= \frac{P(E_2).P\left(\frac{A}{E_2}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2).P\left(\frac{A}{E_2}\right)+P(E_3).P\left(\frac{A}{E_3}\right)}$$

$$= \frac{0.35\times0.06}{0.2\times0.08+0.35\times0.06+0.45\times0.05}$$

$$= \frac{0.021}{0.016+0.021+0.0225}$$

$$= \frac{0.021}{0.0595} = \frac{210}{595}$$

$$= \frac{42}{119} = \frac{6}{17}.$$

8.

$$\int\frac{dx}{\sin x+\sin 2x} = \int\frac{dx}{\sin x+2\sin x\cos x}$$

$$= \int\frac{dx}{\sin x(1+2\cos x)}$$

$$= \int\frac{\sin x\,dx}{\sin^2 x(1+2\cos x)}$$

$$= \int\frac{\sin x\,dx}{(1-\cos^2 x)(1+2\cos x)}$$

Let $\cos x = t$

$\Rightarrow \quad -\sin x\,dx = dt$

$$= -\int\frac{dt}{(1-t^2)(1+2t)}$$

$$= -\int\frac{dt}{(1-t)(1+t)(1+2t)}$$

Using method of partial fractions,

$$\frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-t}+\frac{B}{1+t}+\frac{C}{1+2t} \qquad ...(i)$$

$$\Rightarrow \frac{1}{(1-t)(1+t)(1+2t)}$$

$$= \frac{A(1+t)(1+2t)+B(1-t)(1+2t)+C(1-t^2)}{(1-t)(1+t)(1+2t)}$$

$\Rightarrow 1 = A(1+t)(1+2t)+B(1-t)(1+2t)$ $+ C(1-t^2)$

Putting $1 + t = 0 \Rightarrow t = -1$, we get

$$B = -\frac{1}{2}$$

Putting $1 - t = 0 \Rightarrow t = 1$, we get

$$A = \frac{1}{6}$$

Again putting $1 + 2t = 0 \Rightarrow t = -\frac{1}{2}$, we get

$$C = \frac{4}{3}$$

Putting the values of A, B and C in eq. (i),

$$\frac{1}{(1-t)(1+t)(1+2t)} = \frac{1}{6(1-t)}-\frac{1}{2(1+t)}+\frac{4}{3(1+2t)}$$

$$\therefore -\int\frac{dt}{(1-t)(1+t)(1+2t)}$$

$$= -\left[\frac{1}{6}.\frac{\log(1-t)}{-1}-\frac{1}{2}\log(1+t)+\frac{4}{3}\frac{\log(1+2t)}{2}\right]+C$$

$$= \frac{1}{6}\log(1-\cos x)+\frac{1}{2}\log(1+\cos x)$$

$$-\frac{2}{3}\log(1+2\cos x)+C.$$

OR

OSWAAL BOOKS LEARNING MADE SIMPLE

Let $$I = \int_0^{\pi/4} \log(1+\tan x)dx \quad ...(i)$$

Also, $$I = \int_0^{\pi/4} \log[1+\tan\left(\frac{\pi}{4}-x\right)]dx$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx \quad \left[\because \tan\frac{\pi}{4}=1\right]$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx \quad ...(ii)$$

On adding eq. (i) and (ii), we get

$$2I = \int_0^{\pi/4} \log\,(1+\tan x)dx + \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \log 2\int_0^{\pi/4} 1dx$$

$$= \log 2[x]_0^{\pi/4}$$

$$\Rightarrow \quad 2I = \log 2\left[\frac{\pi}{4}-0\right]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2.$$

Section-B

9. (i) Option (A) is correct.

Explanation:

Given equation = $2x - 3y + z = 0$

which is parallel to $2x - 3y + z = k$ ...(i)

It passes through $(1, -1, 2)$

Then, $2 \times 1 - 3(-1) + 2 = k$

$\Rightarrow \quad k = 7$

Putting the value of k in eq. (i)

$$2x - 3y + z = 7$$

$$\Rightarrow \quad 2x - 3y + z - 7 = 0.$$

(ii) Option (B) is correct.

Explanation:

$$2x + y - 2z = 3$$

$$\Rightarrow \quad \frac{2x}{3}+\frac{y}{3}-\frac{2z}{3} = 1$$

(Dividing by 3 on both sides)

$$\Rightarrow \quad \frac{x}{3/2}+\frac{y}{3}+\frac{z}{-3/2} = 1$$

$\therefore$ Intercepts are:

$a = \frac{3}{2}$, $b = 3$ and $c = \frac{-3}{2}$.

10. Given, line $\frac{x-1}{3} = \frac{y-2}{0} = \frac{z-3}{4}$

and equation of line:

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$$

So, $l = 3$, $m = 0$ and $n = 4$

Equation of the plane:

$$(x-a)l + (y-b)m + (z-c)n = 0$$

$$\Rightarrow \quad 3(x-a) + 0(y-b) + 4(z-c) = 0$$

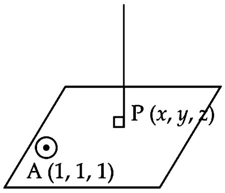
It passes through the point (1, 1, 1) *i.e.,*

$a = 1$, $b = 1$ and $c = 1$

$$\therefore \quad 3(x-1) + 4(z-1) = 0$$

$$\Rightarrow \quad 3x - 3 + 4z - 4 = 0$$

$$\Rightarrow \quad 3x + 4z - 7 = 0$$



which is an equation of the plane.

If $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ are perpendicular to each other, then $a.b = 0$

i.e., $$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$\therefore$ Distance of this plane from origin

$$= \left|\frac{a_1a_2+b_1b_2+c_1c_2+(-7)}{\sqrt{(3)^2+(4)^2}}\right|$$

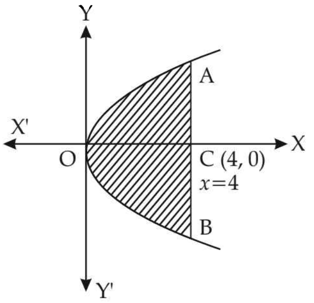
$$= \left|\frac{0-7}{\sqrt{9+16}}\right| = \left|\frac{-7}{5}\right|$$

$$= \frac{7}{5}.$$

11. Given, $y^2 = 16x$ represents a parabola and $x = 4$ is a line parallel to y-axis.

$\because$ $y^2 = 16x$ is symmetrical about x-axis.

$\therefore$ Required area = 2 (*ar.* of *OCAO*)



i.e., $$A = 2\int_0^4 y\,dx$$

$$= 2\int_0^4 \sqrt{16x}\,dx = 8\int_0^4 \sqrt{x}\,dx$$

$$= 8\left[\frac{x^{3/2}}{3/2}\right]_0^4 = 8\times\frac{2}{3}[x^{3/2}]_0^4$$

$$= \frac{16}{3}[(4)^{3/2}-(0)^{3/2}] = \frac{16}{3}\times 8$$

$$= \frac{128}{3} \text{ sq. units.}$$

Section-C

12. (i) Option (C) is correct.

Explanation: Given,

$$9x + 3y - 46 = 0$$

Regression line of x on y:

$$9x = -3y + 46$$

$$x = -\frac{1}{3}y + \frac{46}{9}$$

i.e., $$b_{xy} = -\frac{1}{3}$$

and $3x + 12y - 7 = 0$

Regression line of y on x:

$$12y = -3x + 7$$

$$y = -\frac{1}{4}x + \frac{7}{12}$$

i.e., $$b_{yx} = -\frac{1}{4}$$

$\therefore$ correlation coefficient,

$$r = -\sqrt{b_{yx}b_{xy}}$$

$$= -\sqrt{\left(-\frac{1}{3}\right)\times\left(-\frac{1}{4}\right)}$$

$$= -\sqrt{\frac{1}{12}}$$

$$= -\frac{1}{2\sqrt{3}}.$$

(ii) Option (A) is correct.

Explanation: Given, $\bar{x} = 40$, $\bar{y} = 6$, $\sigma_x = 10$, $\sigma_y = 1.5$ and $r = 0.9$

and $b_{xy} = r\,.\,\dfrac{\sigma_x}{\sigma_y} = 0.9\times\dfrac{10}{1.5} = 6$

Regression line of x on y:

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$\Rightarrow \quad x - 40 = 6(y - 6)$$

$$\Rightarrow \quad x - 40 = 6y - 36$$

$$\Rightarrow \quad x - 6y - 4 = 0.$$

13. Given, $\Sigma x = 15$, $\Sigma y = 25$, $\Sigma x^2 = 55$, $\Sigma y^2 = 135$, $\Sigma xy = 83$ and $n = 5$

$$\therefore \quad b_{xy} = \frac{\Sigma xy - \dfrac{\Sigma x.\Sigma y}{n}}{\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}}$$

$$= \frac{83 - \dfrac{15\times 25}{5}}{135 - \dfrac{(25)^2}{5}}$$

$$= \frac{4}{5}$$

$$= 0.8.$$

and $$b_{yx} = \frac{\Sigma xy - \dfrac{\Sigma x.\Sigma y}{n}}{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}}$$

$$= \frac{83 - \dfrac{15\times 25}{5}}{55 - \dfrac{(15)^2}{5}}$$

$$= \frac{83-75}{55-45}$$

$$= \frac{8}{10}$$

$$= 0.8.$$

14. Let x be the no. of units of commodity A produced and y be the no. of units of commodity B produced, then

$x \geq 0, y \geq 0,$

$$x + 2y \leq 8 \quad \text{...(i)}$$

$$3x + 2y \leq 12 \quad \text{...(ii)}$$

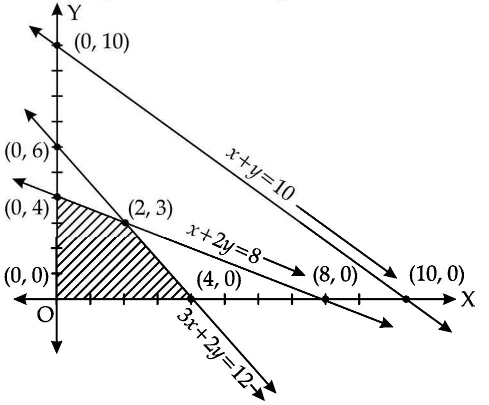
and $$x + y \leq 10 \quad \text{...(iii)}$$

The solution set of above three equations are:

(0, 4), (4, 0), (0, 6), (0, 10), (10, 0), (0, 0), (8, 0) and (2, 3)

Now, $z = 2x + 3y$ (Let Z be profit)

Draw the diagram with these points:



Now, we can determine the maximum value of Z by evaluating the value of Z at the vertices.

Points	Z = 2x + 3y
(0, 0)	0
(4, 0)	8
(2, 3)	13 (maximum)
(0, 4)	12

Hence, the maximum profit is ₹ 13.

■■

OSWAAL BOOKS LEARNING MADE SIMPLE

ISC Semester-1 Examination

MATHEMATICS

Solved Paper - 2021-22

Class–12th

Maximum Marks : 80

Time allowed: One and a half hours

*(Candidates are all allowed additional **10 minutes** for only reading the paper.)*

The Question Paper consists of three sections A, B and C.

Candidates are required to attempt all questions from Section A and all questions either from Section B OR Section C.

Each question/subpart of a question carries two mark.

Select and write the correct option for each of the following question.

Section-A

64 Marks

(Answer all Questions)

1. Let 'R' be a relation on N, set of all natural given by R = {(*a*, *b*) : *a* – *b* = 2}. Then :

(a) $(2, 4) \in R$ **(b)** $(10, 8) \in R$

(c) $(6, 8) \in R$ **(d)** $(8, 7) \in R$

2. If A = $\begin{vmatrix} zy & x & yz \\ xz & y & zx \\ yx & z & xy \end{vmatrix}$, then the value of A is equal to:

(a) 0 **(b)** xyz

(c) 1 **(d)** $\frac{1}{xyz}$

3. The function $f(x) = \frac{x^3}{3} - x$ is decreasing in the interval:

(a) $(-1, 1)$ **(b)** $(-\infty, -1)$

(c) $(1, \infty)$ **(d)** $(-\infty, -1) \cup (1, \infty)$

4. If $\cot^{-1}\frac{1}{5} + \tan^{-1}x = \frac{\pi}{2}$, than value of *x* is:

(a) $\frac{1}{5}$ **(b)** 1

(c) 0 **(d)** $\frac{-1}{5}$

5. Let the two functions *f*(*x*) and *g*(*x*) be defined as $f(x) = x^2 - 1$ and $g(x) = \sqrt{2}$ then (*fog*) (6) is:

(a) 5 **(b)** 7

(c) 35 **(b)** –35

6. If $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, then the matrix A^2 is equal to:

(a) $A^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{pmatrix}$ **(b)** $A^2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 9 & 0 \\ 16 & 0 & 0 \end{pmatrix}$

(c) $A^2 = \begin{pmatrix} 4 & 0 & 16 \\ 0 & 0 & 0 \\ 0 & 9 & 0 \end{pmatrix}$ **(d)** $A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 9 & 16 \\ 0 & 0 & 0 \end{pmatrix}$

7. The value of $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ is equal to:

(a) $\frac{1}{2}$ **(b)** 1

(c) –1 **(d)** $\frac{-1}{2}$

8. The expression for cos ($\tan^{-1} x$) is equal to:

(a) $\frac{1}{\sqrt{1-x^2}}$ **(b)** $\frac{1}{\sqrt{1+x^2}}$

(c) $\frac{\sqrt{1-x^2}}{2}$ **(d)** $\sqrt{1-x^2}$

9. If $2\begin{pmatrix} a & 9 \\ 6 & d \end{pmatrix} + 3\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 3\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$, then the values of *a* and *d* respectively are:

(a) 3, 6 **(b)** 3, 5

(c) 3, 9 **(d)** 3, 7

10. Differentiation of log $(1 + x^2)$ with respect to $\tan^{-1} x$ is:

(a) $\frac{1}{1+x^2}$ **(b)** 2x

(c) $\frac{-1}{1+x^2}$ **(d)** –x

11. The relation R = {(*a*, *a*), (*b*, *b*), (*c*, *c*)} on the set {*a*, *b*, *c*} is:

(a) symmetric only

(b) reflexive only

(c) transitive only

(d) an equivalence relation

12. If the function $f(x) = \begin{cases} 3x - 1, & x < 2 \\ k, & x = 2 \\ 2x + 1, & x > 2 \end{cases}$ is continuous at *x* = 2, then the value of '*k*' is:

(a) $k = 2$ **(b)** $k = 3$

(c) $k = 5$ **(d)** $k = 1$

13. If $x^2 + y^3 = 42$, then $\frac{dy}{dx}$ is:

(a) $\frac{dy}{dx} = \frac{-3y^2}{2x}$ **(b)** $\frac{dy}{dx} = \frac{3y^2}{2x}$

(c) $\frac{dy}{dx} = \frac{2x}{3y^2}$ **(d)** $\frac{dy}{dx} = \frac{-2x}{3y^2}$

14. If the matrix $A = \begin{pmatrix} 1 & x & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix}$ is singular, then the value of 'x' is:

(a) $x = \frac{8}{5}$ **(b)** $x = \frac{-8}{5}$

(c) $x = \frac{5}{8}$ **(d)** $x = 1$

15. The value of $\tan^{-1} 1 + \cos^{-1}\frac{1}{2}$ is:

(a) $\frac{5\pi}{12}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) $\frac{7\pi}{12}$

16. The slope of the tangent to the curve $\sqrt{x} + \sqrt{y} = a$ at $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$ is:

(a) 1 **(b)** –1

(c) $\frac{a}{4}$ **(d)** $\frac{a}{2}$

17. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then the matrix $A + A^T$ is:

(a) Symmetric matrix

(b) Skew-symmetric matrix

(c) Diagonal matrix

(d) Identity matrix

18. If $x = a\cos\theta$, $y = a\sin\theta$, then $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$ will be:

(a) $\frac{dy}{dx} = -1$ (b) $\frac{dy}{dx} = 1$

(c) $\frac{dy}{dx} = 0$ (d) $\frac{dy}{dx} = 2$

19. If $y = \tan^{-1} x$, then:

(a) $(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

(b) $\sqrt{(1-x^2)}\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

(c) $(1-x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

(d) $\sqrt{(1+x^2)}\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

20. If $A = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$, then $A(\text{adj } A)$ is equal to:

(a) $\begin{pmatrix} 4 & 0 \\ 10 & 24 \end{pmatrix}$ **(b)** $\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$ **(d)** $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

21. Consider the two functions $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x - 2$ and $g : R \rightarrow R$ given by $g(x) = x^2$

(i) The function $f(x)$ is

(a) One to one but not onto

(b) Onto but not one to one

(c) Neither one to one nor onto

(d) Bijective

(ii) The value of $f(1) + g(1)$ is:

(a) $\frac{1}{2}$ **(b)** 0

(c) 1 **(d)** $\frac{-1}{2}$

(iii) The expression for $(gof)(x)$ is:

(a) $x - 2$ **(b)** $(x-2)^2$

(c) $x^2 - 2$ **(d)** $x^2 - 3$

(iv) If $(gof)(x) = 0$, then the value of x will be:

(a) $x = \pm 2$ (b) $x = 2$

(c) $x = \pm\sqrt{2}$ (d) $x = 3$

22. A school wants to award its students for their achievement in Sports, Music and Literature with a total cash prize of ₹ 6000.

Three times the prize money for Literature added to the prize money given for Sports is equal to ₹ 11000.

The prize money given for Sports and Literature together is equal to two times of the prize money given for Music.

If x, y and z represent the prize money given for Sports, Music and Literature respectively, then:

(i) The set of linear equations representing the above information will be:

(a) $x + y + z = 6000$, $x + 3z = 11000$ and $x - 2y + z = 0$

(b) $x + y + z = 6000$, $x + 3z = 11000$ and $x + y + 2z = 0$

(c) $x + y + z = 6000$, $3x + z = 11000$ and $x + y - 2z = 0$

(d) $x + y + z = 6000$, $x + 3z = 11000$ and $2x + 2y - z = 0$

(ii) Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{pmatrix}$

Then the value of $|A|$ is:

(a) 3 **(b)** 6

(c) 4 **(d)** –1

(iii) The adj $A = \begin{pmatrix} 6 & p & 3 \\ q & 0 & r \\ -2 & 3 & -1 \end{pmatrix}$, the values of p, q and r respectively will be:

(a) 3, 2, –2 **(b)** 3, –2, –2

(c) –3, 2, –2 **(d)** 3, 2, 2

(iv) Using $|A|$ and adj A, calculate the prize money for Sports(x).

(a) $x = 1500$ **(b)** $x = 500$

(c) $x = 1000$ **(d)** $x = 2000$

OSWAAL BOOKS LEARNING MADE SIMPLE

23. A person has manufactures a water tank in the shape of a closed right circular cylinder. The volume of the cylinder is $\frac{539}{2}$ cubic units. If the height and radius of the cylinder be h and r, then:

(i) The height h in terms of radius r and the given volume will be:

(a) $h=\frac{539}{\pi r^2}$ **(b)** $h=\frac{539}{2\pi r^2}$

(c) $h=\frac{539}{2\pi r}$ **(d)** $h=\frac{539}{\pi r}$

(ii) Let the total surface area of the closed cylindrical tank be S, given by $S=\frac{539}{r}+2\pi r^2$

If the total surface area of the tank minimum, then the value of r will be:

(a) $r = 7$ cm **(b)** $r = 14$ cm

(c) $r = 49$ cm **(d)** $r=\frac{7}{2}$cm

(iii) The height of the tank h is equal to:

(a) $h = 7$ cm **(b)** $h = 14$ cm

(c) $h = 28$ cm **(d)** $h = 2$ cm

(iv) The minimum total surface area of the tank S will be:

(a) 231 sq. cm **(b)** 321 sq. cm

(c) 230 sq. cm **(d)** 221 sq. cm

Section-B

16 Marks

(Answer all Questions)

24. If $\vec{a}=2\hat{i}+4\hat{j}-\hat{k}$ and $3\hat{i}-2\hat{j}-\lambda\hat{k}$ such that they are perpendicular to each other, then the value of λ will be:

(a) 2 **(b)** –2

(c) 3 **(d)** –3

25. The equation of the line passing through the points (0, 1, 2) and (1, 3, 5) is:

(a) $\frac{x-1}{0}=\frac{y-2}{1}=\frac{z-3}{2}$ **(b)** $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

(c) $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ **(d)** $\frac{x-1}{1}=\frac{y-3}{3}=\frac{z-5}{5}$

26. The direction cosines of a line parallel to $\frac{x-1}{2}=\frac{y+3}{3}=\frac{z-6}{-6}$ are:

(a) $\left(\frac{-2}{7},\frac{-3}{7},\frac{-6}{7}\right)$ **(b)** $\left(\frac{2}{7},\frac{-3}{7},\frac{6}{7}\right)$

(c) $\left(\frac{2}{7},\frac{3}{7},\frac{6}{7}\right)$ **(d)** $\left(\frac{2}{7},\frac{3}{7},\frac{-6}{7}\right)$

27. The angle between the pairs of lines $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(\hat{i}+2\hat{j}+2\hat{k}\right)$ and $\vec{r}=5\hat{i}-2\hat{k}+\mu\left(3\hat{i}+2\hat{j}+3\hat{k}\right)$ is:

(a) $\theta=\sin^{-1}\frac{19}{21}$ **(b)** $\theta=\cos^{-1}\frac{22}{21}$

(c) $\theta=\cos^{-1}\frac{19}{20}$ **(d)** $\theta=\cos^{-1}\frac{19}{21}$

28. Consider the two vectors $\vec{a}=3\hat{i}+2\hat{j}+4\hat{k}$ and $\vec{b}=\hat{i}-3\hat{j}+\hat{k}$

(i) The vector perpendicular to both $\vec{a}$ and $\vec{b}$ will be:

(a) $14\hat{i}+\hat{j}-12\hat{k}$ **(b)** $14\hat{i}-\hat{j}+11\hat{k}$

(c) $14\hat{i}+\hat{j}-11\hat{k}$ **(d)** $14\hat{i}+\hat{j}+11\hat{k}$

(ii) The unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ are:

(a) $\frac{14\hat{i}+\hat{j}-12\hat{k}}{\sqrt{308}}$ **(b)** $\frac{14\hat{i}-\hat{j}+11\hat{k}}{\sqrt{318}}$

(c) $\frac{14\hat{i}+\hat{j}-11\hat{k}}{\sqrt{318}}$ **(d)** $\frac{14\hat{i}+\hat{j}+11\hat{k}}{\sqrt{318}}$

(iii) The value of $\left|2\vec{a}+\vec{b}\right|$ will be:

(a) $\sqrt{130}$ **(b)** $\sqrt{131}$

(c) $\sqrt{141}$ **(d)** $\sqrt{140}$

(iv) The area of the parallelogram formed by $\vec{a}$ and $\vec{b}$ as its diagonals will be

(a) $\frac{1}{2}\sqrt{318}$ **(b)** $2\sqrt{318}$

(c) $\frac{1}{2}\sqrt{308}$ **(d)** $2\sqrt{308}$

Section-C

16 Marks

(Answer all Questions)

29. The cost function of a firm is given by C(x) $=1500+25x+\frac{x^2}{10}$. Then the marginal cost of the firm MC(x) will be:

(a) $1500+\frac{x}{5}$ **(b)** $\frac{-1500}{x^2}+\frac{1}{10}$

(c) $25-\frac{x}{5}$ **(d)** $25+\frac{x}{5}$

30. The revenue of a monopolist is given by $R(x) = 120x^2 + 300 - x$. Then, the average revenue function $AR(x)$ at $x = 10$ will be:

(a) 1229 **(b)** 1500

(c) 1210 **(d)** 12310

31. A company sells its product at the rate of ₹10 per unit. The variable costs are estimated to be 25% of the total revenue received. If the fixed costs for the product are ₹4500, then the cost function will be:

(a) $\frac{15}{2}-4500x$ **(b)** $\frac{15}{x}-4500$

(c) $\frac{5x}{2}+4500$ **(d)** $\frac{25x}{2}-4500$

32. Let the total cost function be C(x) = $5x + 350$ and the total revenue function be R(x) = $50x - x^2$ for a company.

Then, the break-even points will be:

(a) –35 and 10 **(b)** 35 and 10

(c) 35 and –10 **(d)** –35 and –10

33. The demand function of a firm producing x units is given by $p = 200 - 5x$.

(i) The revenue function at $x = 20$ will be:

(a) 4000 **(b)** 2000

(c) 100 **(d)** –100

(ii) The marginal revenue MR(x) will be:

(a) $200 - 10x^2$ **(b)** $200 - 5x$

(c) $200 - 10x$ **(d)** $-5x^2$

(iii) The value of x, for which revenue increases, will be:

(a) $x < 20$ **(b)** $x > 20$

(c) $x = 20$ **(d)** $x = 200$

(iv) The slope of the marginal revenue will be:

(a) –45 **(b)** 45

(c) 10 **(d)** –10

■■

ANSWERS

Section-A

1. **Option (b) is correct.**

Explanation: The given relation is, $R = \{(a, b): a - b = 2\}$

As the relation is on set of all natural numbers therefore, 'a' should be greater than 'b'.

In the given options, in option (b) $10 > 8$ and $10 - 8 = 2$.

Thus, $(10, 8) \in R$.

2. **Option (a) is correct.**

Explanation: Given determinant

$$A = \begin{vmatrix} zy & x & yz \\ xz & y & zx \\ yz & z & xy \end{vmatrix}$$

In A, C_1 and C_3 are identical.

According to the property of determinants, if two rows or columns are identical, then its value is zero.

Therefore, $A = 0$.

3. **Option (a) is correct.**

Explanation: Given function

$$f(x) = \frac{x^3}{3} - x$$

Differentiating both sides w.r.t. x

$$f(x) = \frac{3x^2}{3} - 1$$

or $f'(x) = x^2 - 1$

Put $f'(x) = 0$

or $x^2 - 1 = 0$

or $(x - 1)(x + 1) = 0$

or $x = -1, 1$

In the interval $(-1, 1)$ $f(x)$ is decreasing.

4. **Option (a) is correct.**

Explanation:

$$\cot^{-1}\frac{1}{5} + \tan^{-1} x = \frac{\pi}{2} \quad \text{...(i)}$$

Also, $$\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \quad \text{...(ii)}$$

From (i) & (ii), we get $x = \frac{1}{5}$

5. **Option (a) is correct.**

Explanation: The given function are

$f(x) = x^2 - 1$ and $g(x) = \sqrt{x}$

$$(fog)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 - 1$$

$$= x - 1$$

$$(fog)(6) = 6 - 1 = 5$$

6. **Option (a) is correct.**

Explanation:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+9+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+16 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

7. **Option (a) is correct.**

Explanation:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{x^2}$$

$$\left[\text{Using, } 1 - \cos x = 2\sin^2 \frac{x}{2}\right]$$

$$= \lim_{x \to 0} 2\left(\frac{\sin \frac{x}{2}}{x}\right)^2$$

$$= \lim_{x \to 0} 2\left(\frac{\sin \frac{x}{2}}{\frac{2x}{2}}\right)^2 \quad \left[\text{Using, } \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= 2 \times \frac{1}{4} = \frac{1}{2}$$

8. **Option (b) is correct.**

Explanation: Let $\theta = \tan^{-1} x$

$\Rightarrow$ $x = \tan \theta$

Also, $$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1 + x^2}}$$

$$\therefore \quad \cos\left(\tan^{-1} x\right) = \frac{1}{\sqrt{1 + x^2}}$$

OSWAL BOOKS LEARNING MADE SIMPLE

9. Option (a) is correct.

Explanation:

$$2\begin{pmatrix} a & 9 \\ 6 & d \end{pmatrix} + 3\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 3\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2a & 18 \\ 12 & 2d \end{pmatrix} + \begin{pmatrix} 3 & -3 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 15 \\ 12 & 18 \end{pmatrix}$$

$$\begin{pmatrix} 2a+3 & 15 \\ 12 & 2d+6 \end{pmatrix} = \begin{pmatrix} 9 & 15 \\ 12 & 18 \end{pmatrix}$$

On comparing the corresponding elements, we get

$$2a+3 = 9 \text{ and } 2d + 6 = 18$$

or $\quad 2a = 6$ and $2d = 12$

or $\quad a = 3$ and $d = 6$

10. Option (b) is correct.

Explanation: Let $u = \log(1 + x^2)$ and $v = \tan^{-1} x$

$$\frac{du}{dx} = \frac{d}{dx}\left[\log\left(1+x^2\right)\right] = \frac{1}{1+x^2}(2x)$$

$$= \frac{2x}{1+x^2}$$

$$\frac{dv}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv} = \frac{2x}{1+x^2} \times \frac{1+x^2}{1} = 2x$$

11. Option (d) is correct.

Explanation: The given relation R = {(*a*, *a*), (*b*, *b*), (*c*, *c*) is an identity relation.

We know that, Identity relation is always an equivalence relation, therefore the given relation is an equivalence relation.

12. Option (c) is correct.

Explanation: Given the function $f(x)$ is continuous at $x = 2$.

Then, $\quad \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$

Now, $\quad \lim\limits_{x \to 2^-} (3x - 1) = 3(2) - 1 = 5$

and $\quad \lim\limits_{x \to 2^+} (2x + 1) = 2(2) + 1 = 5$

Hence, $\quad k = 5$

13. Option (d) is correct.

Explanation: The given equation is

$$x^2 + y^3 = 42$$

Differentiating both sides w.r.t. *x*, we get

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^3\right) = \frac{d}{dx}(42)$$

$$\Rightarrow \quad 2x + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{3y^2}$$

14. Option (b) is correct.

Explanation: When det(*A*) is zero , then *A* is a singular matrix.

Therefore, $\quad |A| = 0$

$$\Rightarrow \quad \begin{vmatrix} 1 & x & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - x\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1 + 5x + 7 = 0$$

$$\Rightarrow \quad 5x = -8$$

$$\Rightarrow \quad x = -\frac{8}{5}$$

15. Option (d) is correct.

Explanation:

$$\tan^{-1} 1 = \frac{\pi}{4} \text{ and } \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

$$\Rightarrow \quad \tan^{-1} 1 + \cos^{-1}\frac{1}{2} = \frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$$

16. Option (b) is correct.

Explanation: The given equation is, $\sqrt{x} + \sqrt{y} = a$

Differentiating both sides w.r.t. *x*, we get

$$\frac{d}{dx}\left(\sqrt{x}\right) + \frac{d}{dx}\left(\sqrt{y}\right) = \frac{d}{dx}(a)$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

At $\left(\frac{a^2}{4}, \frac{a^2}{4}\right) \quad \frac{dy}{dx} = -1$

17. Option (a) is correct.

Explanation:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\Rightarrow \quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

and $\quad A + A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix} \quad$ ...(i)

Now, $\quad \left(A + A^T\right)^T = \begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix} \quad$ ...(ii)

From (i) and (ii),

$$A + A^T = (A+A^T)^T$$

Therefore, $A + A^T$ is a symmetric matrix.

18. Option (c) is correct.

Explanation: Given, $x = a\cos\theta$ & $y = a\sin\theta$

$\therefore \quad \frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta) = -a\sin\theta$

and $\quad \frac{dy}{d\theta} = \frac{dy}{d\theta} = \frac{d}{d\theta}(a\sin\theta) = a\cos\theta$

Now, $\quad \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$

At $\theta = \frac{\pi}{2}, \quad \frac{dy}{dx} = \cot\frac{\pi}{2} = 0$

19. Option (a) is correct.

Explanation: Given, $y = \tan^{-1}x$

Differentiating w.r.t. *x*, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

or $$\left(1+x^2\right)\frac{dy}{dx} = 1$$

Again differentiating w.r.t. x, we get

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}(2x) = 11$$

20. Option (b) is correct.

Explanation:

Given, $$A = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$$

Minors of A are: $M_{11} = 3$, $M_{12} = 2$, $M_{21} = 1$, $M_{22} = -1$

Cofactors of A are: $C_{11} = 3$, $C_{12} = -2$, $C_{21} = -1$, $C_{22} = -1$

Therefore, $$\text{Adj}(A) = \begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 & -1 \\ -2 & -1 \end{pmatrix}$$

Now, $$A(\text{adj } A) = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3-2 & 1-1 \\ 6-6 & -2-3 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$$

21. (i) Option (d) is correct.

Explanation: Let $x_1, x_2 \in R$

Now, $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 - 2 = x_2 - 2$

$\Rightarrow$ $x_1 = x_2$

$\Rightarrow$ $f(x)$ is one-one.

Let $y \in R$

Let $y = f(x_0)$

Then, $x_0 - 2 = y$

$\Rightarrow$ $x_0 = y + 2$

Now, $y \in R$

$\Rightarrow$ $y + 2 \in R$

$\Rightarrow$ $x_0 \in R$

$f(x_0) = x_0 + 2 = y$

Therefore, for each $y \in R$, there exists $x_0 \in R$ such that $f(x_0)=y$

So, $f(x)$ is onto.

Thus, $f(x)$ is one-one and onto or bijective.

(ii) Option (b) is correct.

Explanation:

$f(1) = 1 - 2 = -1$ and $g(1) = (1)^2 = 1$

Now, $f(1) + g(1) = -1 + 1 = 0$

(iii) Option (b) is correct.

Explanation:

$$gof(x) = g(f(x)) = g(x-2) = (x-2)^2$$

(iv) Option (b) is correct.

Explanation:

$gof(x) = 0$

$\Rightarrow$ $(x-2)^2 = 0$

$\Rightarrow$ $x - 2 = 0$

$\Rightarrow$ $x = 2$

22. (i) Option (a) is correct.

Explanation:

Given,

Cash prize for achievement in sports is ₹x.

Cash prize for achievement in music is ₹y.

Cash prize for achievement in literature is ₹z.

Three times the prize money for literature is added to the money given for sports is equal to ₹11000.

$\Rightarrow$ $x + 3z = 11000$

The prize money given for sports and literature is equal to two times of prize money for music.

$\Rightarrow$ $x + z = 2y$ or $x - 2y + z = 0$

Total cash prize for all the three subjects is ₹6000

$x + y + z = 6000$

(ii) Option (b) is correct

Explanation:

$$|A| = 1\begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix}$$

$$= 1[0+6] - 1[1-3] + 1[-2-0]$$

$$= 6 + 2 - 2 = 6$$

(iii) Option (c) is correct.

Explanation: The above system of equations can be written in matrix form as AX = B.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

Here, $|A| = 6$ [From above part (ii)]

Which means A is non-singular. Hence, it is invertible.

Now, cofactors of A are:

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix} = 0+6 = 6$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -(1-3) = 2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} = -2-0 = -2$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -(1+2) = -3$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1-1 = 0$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -(-2-1) = 3$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3-0 = 3$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0-1 = -1$$

$$\therefore \quad adj(A) = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}^T$$

Therefore,

OSWAAL BOOKS LEARNING MADE SIMPLE

$$\text{adj}(A) = \begin{bmatrix} 6 & -3 & 3 \\ 22 & 00 & -22 \\ -2 & 3 & -1 \end{bmatrix}$$

On comparing, we get $p = -3, q = 2$ and $r = -2$

(iv) Option (b) is correct.

Explanation: $X = A^{-1} B$

or $X = \dfrac{adj(A)}{|A|}.B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 36000 - 33000 + 0 \\ 12000 + 0 - 0 \\ -12000 + 33000 - 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3000}{6} \\ \frac{12000}{6} \\ \frac{21000}{6} \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

Therefore, the prize money for sports (x) is ₹ 500.

23. (i) Option (b) is correct.

Explanation: It is mentioned in the question that radius $=r$ units & height $=h$ units

Also, volume of cylinder $= \dfrac{539}{2}$ cubic units

$$\Rightarrow \quad \pi r^2 h = \frac{539}{2}$$

$$\Rightarrow \quad h = \frac{539}{2\pi r^2}$$

(ii) Option (d) is correct.

Explanation: Total surface area,

$$S = 2\pi rh + 2\pi r^2$$

$$= 2\pi r \frac{539}{2\pi r^2} + 2\pi r^3$$

[From above part (i)]

$$= \frac{539}{r} + 2\pi r^2$$

Now, $\dfrac{ds}{dr} = -\dfrac{539}{r^2} + 4\pi r$ and $\dfrac{d^2s}{dr^2} = \dfrac{1078}{r^3} + 4\pi$

Put $\dfrac{ds}{dr} = 0$

$$\Rightarrow \quad -\frac{539}{r^2} + 4\pi r = 0$$

$$\Rightarrow \quad r^3 = \frac{539 \times 7}{4 \times 22}$$

$$\Rightarrow \quad r^3 = \frac{7 \times 7 \times 7}{2 \times 2 \times 2}$$

$$\Rightarrow \quad r = \frac{7}{2}$$

Here, at $r = \dfrac{7}{2}, \dfrac{d^2s}{dr^2} > 0.$ Hence, minimum.

(iii) Option (a) is correct.

Explanation:

As, $h = \dfrac{539}{2\pi r^2} = \dfrac{539}{2 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}} = 7$ cm

(iv) Option (a) is correct.

Explanation: Surface area is minimum at $r = \dfrac{7}{2}$ cm

Therefore, $S = \dfrac{539}{r} + 2\pi r^2 = \dfrac{539}{\frac{7}{2}} + 2 \times \dfrac{22}{7} \times \dfrac{7}{2} \times \dfrac{7}{2}$

$= 154 + 77 = 231$ sq. cm

Section-B

24. Option (a) is correct.

Explanation: When two vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a}.\vec{b} = 0$

Therefore, $(2\hat{i} + 4\hat{j} - \hat{k}).(3\hat{i} - 2\hat{j} - \gamma\hat{k}) = 0$

$\Rightarrow \quad 6 - 8 + \gamma = 0$

$\Rightarrow \quad \gamma = 2$

25. Option (c) is correct.

Explanation: The equation of a line passing through two points (x_1, y_1, z_1) (x_2, y_2, z_2) is given by,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Therefore, The line is passing through (0, 1, 2) & (1, 3, 5) is

$$\frac{x - 0}{1 - 0} = \frac{y - 1}{3 - 1} = \frac{z - 2}{5 - 2}$$

$$\Rightarrow \quad \frac{x}{1} = \frac{y - 1}{2} = \frac{z - 2}{3}$$

26. Option (d) is correct.

Explanation: Equation of the line given in the question is, $\dfrac{x - 1}{2} = \dfrac{y + 3}{3} = \dfrac{z + 6}{6}$

Direction ratios of this line is <2, 3, –6>

Direction ratios of the line parallel to the given line are also same, <2, 3, –6>

Let the direction cosines of the second line are $<l, m, n>$

$$l = \frac{2}{\sqrt{2^2 + 3^2 + (-6)^2}} = \frac{2}{\sqrt{4 + 9 + 36}} = \frac{2}{\sqrt{49}} = \frac{2}{7}$$

Similarly, $m = \dfrac{3}{7}$ and $n = \dfrac{-6}{7}$

27. Option (d) is correct.

Explanation: The angle between two line $\bar{r}_1 = \bar{a}_1 + \gamma\bar{b}_1$ and $\bar{r}_2 = \bar{a}_2 + \mu\bar{b}_2$ is

$$\cos\theta = \frac{\bar{b}_1, \bar{b}_2}{|\bar{b}_1||\bar{b}_2|}$$

Therefore, $\cos\theta = \dfrac{(\hat{i} + 2\hat{j} + 2\hat{k})(3\hat{i} + 2\hat{j} + 6\hat{k})}{\sqrt{1^2 + 2^2 + 2^2}\sqrt{3^2 + 2^2 + 6^2}}$

$$= \frac{3 + 4 + 12}{\sqrt{9} \times \sqrt{49}} = \frac{19}{21}$$

$$\theta = \cos^{-1}\frac{19}{21}$$

28. (i) Option (c) is correct.

Explanation: The vector which is perpendicular to both $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b}$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 4 \\ 1 & -3 & 1 \end{vmatrix}$$

$$= \hat{i}(2+12) - \hat{j}(3-4) + \hat{k}(-9-2)$$

$$\vec{a}\times\vec{b} = 14\hat{i} + \hat{j} - 11\hat{k}$$

(ii) Option (c) is correct.

Explanation: Let unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\hat{C}$

$$\hat{C} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{14\hat{i}+\hat{j}-11\hat{k}}{\sqrt{14^2+1^2+11^2}} = \quad 14 \qquad 11$$

$$= \frac{14\hat{i}+\hat{j}-11\hat{k}}{\sqrt{318}}$$

(iii) Option (b) is correct.

Explanation:

$$|2\vec{a}+\vec{b}| = \left|2\left(3\hat{i}+2\hat{j}+4\hat{k}\right)+\left(\hat{i}-3\hat{j}+\hat{k}\right)\right|$$

$$= |6\hat{i}+4\hat{j}+8\hat{k}+\hat{i}-3\hat{j}+\hat{k}|$$

$$= |7\hat{i}+\hat{j}+9\hat{k}| = \sqrt{7^2+1^2+9^2} \qquad 49 \quad 1 \quad 81 \qquad 131$$

$$= \sqrt{49+1+81} = \sqrt{131}$$

(iv) Option (a) is correct.

Explanation: Here,

Area of parallelogram

$$= \frac{1}{2}\left|\vec{a}\times\vec{b}\right| = \frac{1}{2}\sqrt{318} \text{ sq. units}$$

[Given $\vec{a}$ and $\vec{b}$ are diagonals]

Section-B

29. Option (d) is correct.

Explanation:

Given, $C(x) = 1500 + 25x + \frac{x^2}{10}$

Therefore, $MC(x) = \frac{d}{dx}\left(1500 + 25x + \frac{x^2}{10}\right)$

$$= 25 + \frac{2x}{10} = 25 + \frac{x}{5}$$

30. Option (a) is correct.

Explanation:

Given, $R(x) = 120x^2 + 300 - x$

Therefore, $AR(x) = \frac{R(x)}{x} = \frac{120x^2+300-x}{x}$

At $x = 10$, $AR(10) = \frac{120(10)^2+300-10}{10}$

$$= \frac{12000+300-10}{10}$$

$$= \frac{12290}{10} = 1229$$

31. Option (c) is correct.

Explanation: Let x be the number of units sold.

Price of 1 unit ₹10.

Total revenue = ₹$10x$

Cost function, $C(x) = 4500 + 25\%$ of ₹$10x$

$$= 4500 + \frac{25}{100}\times 10x = 4500 + \frac{5}{2}x$$

32. Option (b) is correct.

Explanation: For break-even values, $C(x) = R(x)$

Therefore, $5x + 350 = 50x - x^2$

or $x^2 - 45x + 350 = 0$

Which is a quadratic equation in x, using quadratic formula

$D = b^2 - 4ac$ and $x = \frac{-b \pm \sqrt{D}}{2a}$

Here, $D = (-45)^2 - 4(1)(350)$

$= 2025 - 1400 = 625$

Therefore, $x = \frac{-(-45)\pm\sqrt{625}}{2(1)} = \frac{45+25}{2}, \frac{45-25}{2}$

or, $x = \frac{70}{2}, \frac{20}{2} = 35, 10$

33. (i) Option (b) is correct.

Explanation:

Demand function

$$p = 200 - 5x$$

Revenue function,

$$R(x) = px = 200x - 5x^2$$

At $x = 20$, $R(x) = 200(20) - 5(20)^2$

$= 4000 - 2000$

$= 2000$

(ii) Option (c) is correct.

Explanation:

Here, $R(x) = 200x - 5x^2$

Therefore, $MR(x) = \frac{d}{dx}(200x - 5x^2)$

$= 200 - 10x$

(iii) Option (a) is correct.

Explanation: $MR > 0$

$\Rightarrow \quad 200 - 10x > 0$

$\Rightarrow \quad -10x > -200$

$\Rightarrow \quad x < \frac{200}{10}$

$\Rightarrow \quad x < 20$

(iv) Option (d) is correct.

Explanation: Slope of Marginal revenue,

$$\frac{d}{dx}(MR) = \frac{d}{dx}(200-10x) = -10$$

OSWAAL BOOKS LEARNING MADE SIMPLE

Semester – 1

OMR SHEET

Booklet Series

A

Use English Numbers / Letters only. Use Blue / Black Ball Point Pen to write in box.

Booklet Series

☐

Ⓐ
Ⓑ
Ⓒ
Ⓓ

Subject

☐

Roll Number

☐☐☐☐☐☐☐☐☐

⓪ ⓪ ⓪ ⓪ ⓪ ⓪ ⓪ ⓪ ⓪
① ① ① ① ① ① ① ① ①
② ② ② ② ② ② ② ② ②
③ ③ ③ ③ ③ ③ ③ ③ ③
④ ④ ④ ④ ④ ④ ④ ④ ④
⑤ ⑤ ⑤ ⑤ ⑤ ⑤ ⑤ ⑤ ⑤
⑥ ⑥ ⑥ ⑥ ⑥ ⑥ ⑥ ⑥ ⑥
⑦ ⑦ ⑦ ⑦ ⑦ ⑦ ⑦ ⑦ ⑦
⑧ ⑧ ⑧ ⑧ ⑧ ⑧ ⑧ ⑧ ⑧
⑨ ⑨ ⑨ ⑨ ⑨ ⑨ ⑨ ⑨ ⑨

Name ☐☐☐☐☐☐☐☐☐☐☐☐☐☐
☐☐☐☐☐☐☐☐☐☐☐☐☐☐

Test Date ☐☐☐☐☐☐

Student's Signature ☐

Invigilator's Signature ☐

Certified that all the entries In this section have been properly filled by the student

Proper Marking

The OMR Sheet will be computer checked Fill the circles completely and dark enough for proper detection, Use ballpen (black or blue) for marking.

Ⓐ Ⓑ ● Ⓓ

Avoid Improper Marking

◒ ◉ ▨

Partially Filled

◍

Lightly Filled

✓ ⊗

Test Center Code

⓪ ⓪
① ①
② ②
③ ③
④ ④
⑤ ⑤
⑥ ⑥
⑦ ⑦
⑧ ⑧
⑨ ⑨

IMPORTANT

The candidate should check that the Test Book Series printed on the OMR Sheet is the same as printed on the Test Booklet. In case of discrepancy, the candidate should immediately report the matter to the invigilator for replacement of both the Test Booklet and the Answer Sheet.

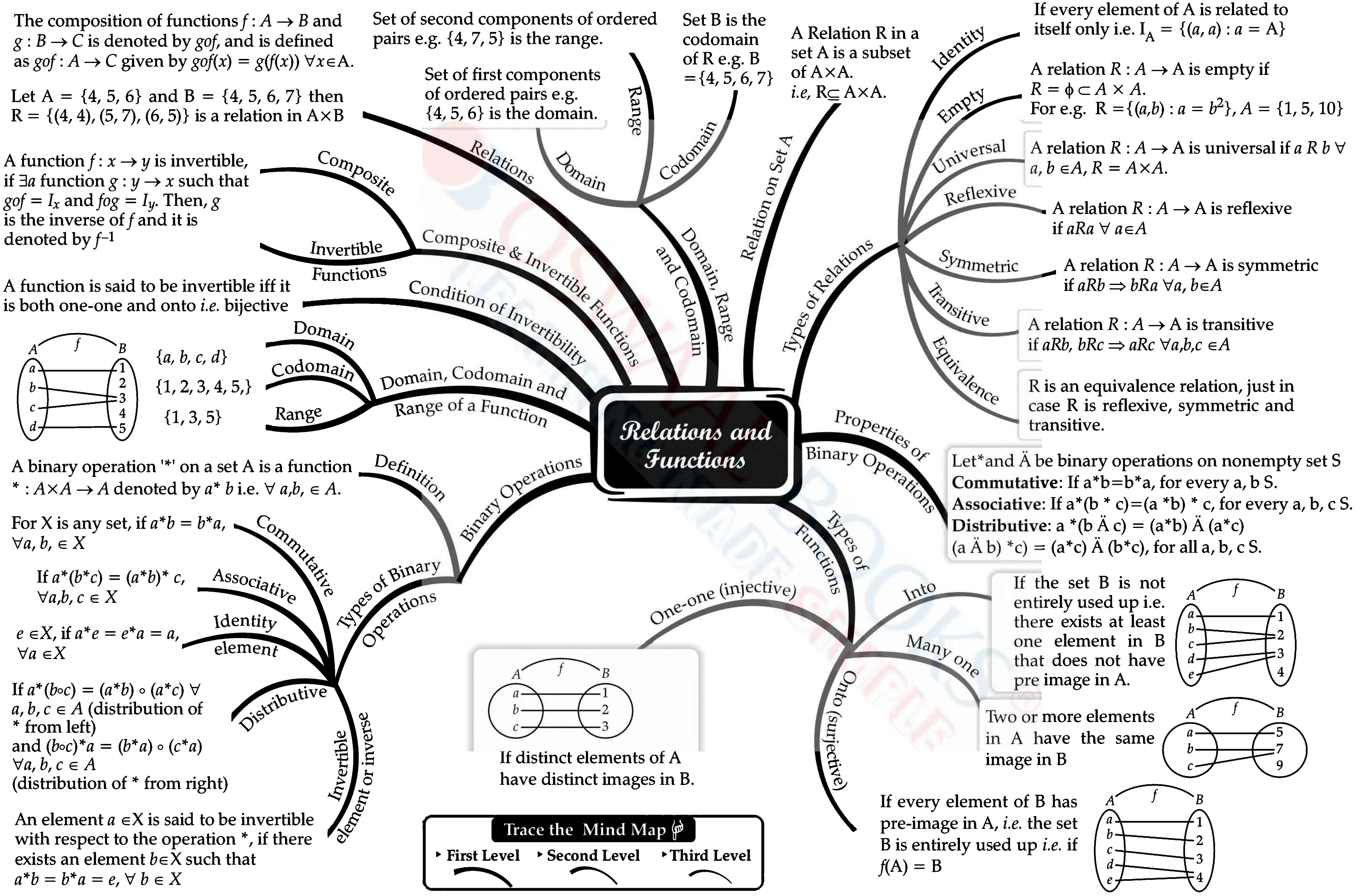
Darken the circle for each question.

Q.No.	Response
	Section-I
1.	
a.	① ② ③ ④
b.	① ② ③ ④
c.	① ② ③ ④
d.	① ② ③ ④
e.	① ② ③ ④
2.	
a.	① ② ③ ④
b.	① ② ③ ④
c.	① ② ③ ④

Q.No.	Response
d.	① ② ③ ④
e.	① ② ③ ④
3.	
a.	① ② ③ ④
b.	① ② ③ ④
c.	① ② ③ ④
d.	① ② ③ ④
e.	① ② ③ ④
	Section-II
4.	
a.	① ② ③ ④
b.	① ② ③ ④

Q.No.	Response
c.	① ② ③ ④
d.	① ② ③ ④
e.	① ② ③ ④
5.	
a.	① ② ③ ④
b.	① ② ③ ④
c.	① ② ③ ④
d.	① ② ③ ④
e.	① ② ③ ④
6.	
a.	① ② ③ ④
b.	① ② ③ ④
c.	① ② ③ ④

Q.No.	Response
d.	① ② ③ ④
e.	① ② ③ ④
	Section-III
7.	
a.	① ② ③ ④
b.	① ② ③ ④
c.	① ② ③ ④
d.	① ② ③ ④
e.	① ② ③ ④
8.	
a.	① ② ③ ④
b.	① ② ③ ④
c.	① ② ③ ④
d.	① ② ③ ④
e.	① ② ③ ④



Relations and Functions

- Relations
  - Let A = {4, 5, 6} and B = {4, 5, 6, 7} then R = {(4, 4), (5, 7), (6, 5)} is a relation in A×B
- Domain, Range and Codomain
  - Domain: Set of first components of ordered pairs e.g. {4, 5, 6} is the domain.
  - Range: Set of second components of ordered pairs e.g. {4, 7, 5} is the range.
  - Codomain: Set B is the codomain of R e.g. B ={4, 5, 6, 7}
- Relation on Set A
  - A Relation R in a set A is a subset of A×A. i.e, $R \subseteq A \times A$.
- Types of Relations
  - Identity: If every element of A is related to itself only i.e. $I_A = \{(a, a) : a = A\}$
  - Empty: A relation $R : A \to A$ is empty if $R = \phi \subset A \times A$. For e.g. $R = \{(a,b) : a = b^2\}, A = \{1, 5, 10\}$
  - Universal: A relation $R : A \to A$ is universal if $a\,R\,b\ \forall\ a, b \in A, R = A \times A$.
  - Reflexive: A relation $R : A \to A$ is reflexive if $aRa\ \forall\ a \in A$
  - Symmetric: A relation $R : A \to A$ is symmetric if $aRb \Rightarrow bRa\ \forall a, b \in A$
  - Transitive: A relation $R : A \to A$ is transitive if $aRb,\ bRc \Rightarrow aRc\ \forall a,b,c \in A$
  - Equivalence: R is an equivalence relation, just in case R is reflexive, symmetric and transitive.
- Properties of Binary Operations
  - Let* and Ä be binary operations on nonempty set S
  - **Commutative**: If a*b=b*a, for every a, b S.
  - **Associative**: If a*(b * c)=(a *b) * c, for every a, b, c S.
  - **Distributive**: a *(b Ä c) = (a*b) Ä (a*c) (a Ä b) *c) = (a*c) Ä (b*c), for all a, b, c S.
- Types of Functions
  - Into: If the set B is not entirely used up i.e. there exists at least one element in B that does not have pre image in A.
  - Many one: Two or more elements in A have the same image in B
  - Onto (surjective): If every element of B has pre-image in A, *i.e.* the set B is entirely used up *i.e.* if f(A) = B
  - One-one (injective): If distinct elements of A have distinct images in B.
- Binary Operations
  - Definition: A binary operation '*' on a set A is a function $* : A \times A \to A$ denoted by $a * b$ i.e. $\forall\ a,b, \in A$.
  - Types of Binary Operations
    - Commutative: For X is any set, if $a*b = b*a$, $\forall a, b, \in X$
    - Associative: If $a*(b*c) = (a*b)*c$, $\forall a,b, c \in X$
    - Identity element: $e \in X$, if $a*e = e*a = a$, $\forall a \in X$
    - Distributive: If $a*(b \circ c) = (a*b) \circ (a*c)\ \forall\ a, b, c \in A$ (distribution of * from left) and $(b \circ c)*a = (b*a) \circ (c*a)\ \forall a, b, c \in A$ (distribution of * from right)
    - Invertible element or inverse: An element $a \in X$ is said to be invertible with respect to the operation *, if there exists an element $b \in X$ such that $a*b = b*a = e, \forall\ b \in X$
- Domain, Codomain and Range of a Function
  - Domain: $\{a, b, c, d\}$
  - Codomain: $\{1, 2, 3, 4, 5,\}$
  - Range: $\{1, 3, 5\}$
- Condition of Invertibility
  - A function is said to be invertible iff it is both one-one and onto *i.e.* bijective
- Composite & Invertible Functions
  - Composite: The composition of functions $f : A \to B$ and $g : B \to C$ is denoted by gof, and is defined as $gof : A \to C$ given by $gof(x) = g(f(x))\ \forall x \in A$.
  - Invertible Functions: A function $f : x \to y$ is invertible, if $\exists a$ function $g : y \to x$ such that $gof = I_x$ and $fog = I_y$. Then, g is the inverse of f and it is denoted by f^{-1}

Trace the Mind Map

▸ First Level ▸ Second Level ▸ Third Level

UNIT-1

Relations and Functions

CHAPTER 1

RELATIONS AND FUNCTIONS

SYLLABUS

Types of relations: *reflexive, symmetric, transitive and equivalence relations. One to one and onto functions, composite functions, inverse of a function. Binary operations.*

- **Relations as:**
  - *Relation on a set A*
  - *Identity relation, empty relation, universal relation.*
  - *Types of Relations: reflexive, symmetric, transitive and equivalence relation.*
- **Binary Operation:** all axioms and properties
- **Functions:**
  - *As special relations, concept of writing "y is a function of x" as y = f(x).*
  - *Types: one to one, many to one, into, onto.*
  - *Real valued function.*
  - *Domain and range of a function.*
  - *Conditions of invertibility.*
  - *Composite functions and invertible functions (algebraic functions only)*

Trend Analysis

List of Concepts	2018	2019	2020
Functions	–	1 Q (4 marks)	1 Q (2 marks)
Composite functions	1 Q (4 marks)	1 Q (2 marks)	1 Q (4 marks)
Binary operations	1 Q (2 marks)		

TOPIC-1

Relations

TOPIC - 1
Relations .... P. 01
TOPIC - 2
Functions .... P. 08
TOPIC - 3
Composite Functions .... P. 17
TOPIC - 4
Binary Operations .... P. 21

Revision Notes

➢ **Definition**

A relation R, from a non-empty set A to another non-empty set B is mathematically defined as an arbitrary subset of $A \times B$. Equivalently, any subset of $A \times$ B is a relation from A to B.

Thus, R is a relation from A to B $\Leftrightarrow$ $R \subseteq A \times B$

$$\Leftrightarrow R \subseteq \{(a, b) : a \in A, b \in B\}$$

Illustrations:

(a) Let $A = \{1, 2, 4\}$, $B = \{4, 6\}$ and let $R = \{(1, 4), (1, 6), (2, 4), (2, 6), (4, 6)\}$, Here, $R \subseteq A \times B$ and therefore R is a relation from A to B.

(b) Let $A = \{1, 2, 3\}$, $B = \{2, 3, 5, 7\}$ and let $R = \{(2, 3), (3, 5), (5, 7)\}$ Here, $R \not\subset AB$ and therefore R is not a relation from A to B. Since, $(5, 7) \in R$ but $(5, 7) \notin A \times B$.

Scan to know more about this topic

Relation

(c) Let $A = \{-1, 1, 2\}$, $B = \{1, 4, 9, 10\}$; let $a \in A$ and $b \in B$ and $a\,R\,b$ means $a^2 = b$ then, $R = \{(-1, 1), (1, 1), (2, 4)\}$.

- **Types of Relations**

(a) Empty relation: A relation R from A to B is called an empty relation or a void relation from A to B if $R = \phi$.

For example, let $A = \{2, 4, 6\}$, $B = \{7, 11\}$

Let $R = \{(a, b): a \in A, b \in B \text{ and } a - b \text{ is even}\}$.

Here, R is an empty relation.

Important Facts

- **A relation from A to B is also called a relation from A into B.**
- $(a, b) \in R$ is also written as aRb (read as a is related to b).
- Let A and B be two non-empty finite sets having p and q elements respectively. Then $n(A \times B) = n(A)\cdot n(B) = pq$. Then total number of subsets of $A \times B = 2^{pq}$. Since, each subset of $A \times B$ is a relation from A to B, therefore total number of relations from A to B is given as 2^{pq}.

(b) Universal relation: A relation R from A to B is said to be a universal relation, if $R = A \times B$.

For example, let $A = \{1, 2\}$, $B = \{1, 3\}$

Let $R = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$

Here, $R = A \times B$,

So, relation R is a universal relation.

(c) Identity relation : A relation R defined on a set A is said to be the identity relation on A, if

$$R = \{(a, b) : a \in A, b \in A \text{ and } a = b\}$$

Thus, identity relation $R = \{(a, a) : \forall\, a \in A\}$

The identity relation on set A is also denoted by I_A.

For example, let $A = \{1, 2, 3, 4\}$

Then, $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

But the relation given by $R = \{(1, 1), (2, 2), (1, 3), (4, 4)\}$ is not an identity relation because element 1 is related to elements 1 and 3.

Important Facts

- The void relation, *i.e.,* ϕ and universal relation, *i.e.,* $A \times A$ on A are respectively the smallest and largest relations defined on the set A. Also these are sometimes called **Trivial Relations.** Any other relation is called a **non-trivial relation.**
- The relation $R = \phi$ and $R = A \times A$ are two extreme relations.

(d) Reflexive relation : A relation R defined on a set A is said to be reflexive, if $aRa\ \forall\, a \in A$, *i.e.,* $(a, a) \in R\ \forall\, a \in A$.

For example, let $A = \{1, 2, 3\}$ and R_1, R_2, R_3 be the relation given as

$$R_1 = \{(1, 1), (2, 2), (3, 3)\},$$
$$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3)\}$$
$$R_3 = \{(2, 2), (2, 3), (3, 2), (1, 1)\}$$

Here, R_1 and R_2 are reflexive relations on A but R_3 is not reflexive as $3 \in A$ but $(3, 3) \notin R_3$.

Important Fact

- In an identity relation on A every element of A should be related to itself only.

(e) Symmetric relation : A relation R defined on a set A is symmetric, if

$$(a, b) \in R \Rightarrow (b, a) \in R\ \forall\ a, b \in A, \text{ i.e., } aRb \Rightarrow bRa \text{ (i.e., whenever } aRb \text{ then } bRa).$$

For example, let $A = \{1, 2, 3\}$,

$$R_1 = \{(1, 2), (2, 1)\}$$
$$R_2 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$$
$$R_3 = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$$
$$R_4 = \{(1, 3), (3, 1), (2, 3)\}$$

Here, R_1, R_2 and R_3 are symmetric relations on A. But R_4 is not symmetric because $(2, 3) \in R_4$ but $(3, 2) \notin R_4$.

Important Facts

- The universal relation on non-void set A is reflexive.
- The identity relation is always a reflexive relation but the converse may or may not be true. As shown in the example, R_1 is both identity as well as reflexive relation on A but R_2 is only reflexive relation on A.

(f) Transitive relation : A relation R on a set A is transitive, if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

i.e., aRb and $bRc \Rightarrow aRc$.

For example, let $A = \{1, 2, 3\}$,

$$R_1 = \{(1, 2), (2, 3), (1, 3), (3, 2)\}$$

and $R_2 = \{(1, 3), (3, 2), (1, 2)\}$

Here, R_2 is transitive relation whereas R_1 is not transitive because $(2, 3) \in R_1$ and $(3, 2) \in R_1$ but $(2, 2) \notin R_1$.

(g) Equivalence relation : Let A be a non-empty set, then a relation R on A is said to be an equivalence relation, if

(i) R is reflexive, *i.e.,* $(a, a) \in R\ \forall\, a \in A$, *i.e.,* aRa.

(ii) R is symmetric, *i.e.,* $(a, b) \in R \Rightarrow (b, a) \in R\ \forall\ a, b \in A$, *i.e.,* $aRb \Rightarrow bRa$.

(iii) R is transitive, *i.e.,* $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R\ \forall\ a, b, c \in A$, *i.e.,* aRb and $bRc \Rightarrow aRc$.

For example, let $A = \{1, 2, 3\}$

$$R = \{(1, 2), (1, 1), (2, 1), (2, 2), (3, 3)\}$$

Here, R is reflexive, symmetric and transitive. So, R is an equivalence relation on A.

Scan to know more about this topic

Equivalence Relation

OSWAAL BOOKS LEARNING MADE SIMPLE

- **Equivalence classes :** Let A be an equivalence relation in a set A and let $a \in A$. Then, the set of all those elements of A which are related to a, is called equivalence class determined by a and it is denoted by $[a]$. Thus, $[a] = \{b \in A : (a, b) \in R\}$

- **DOMAIN & RANGE OF A RELATION**

  (a) Domain of a relation : Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in A$ such that $(a, b) \in R \ \forall \ b \in B$.

  Thus, $\text{Dom.}(R) = \{a \in A : (a, b) \in R \ \forall \ b \in B\}$.

  That is, the domain of R is the set of first elements of all the ordered pairs which belong to R.

  (b) Range of a relation : Let R be a relation from A to B. The range of relation R is the set of all those elements $b \in B$ such that $(a, b) \in R \ \forall \ a \in A$.

  Thus, Range of $R = \{b \in B : (a, b) \in R \ \forall \ a \in A\}$.

  That is, the range of R is the set of second elements of all the ordered pairs which belong to R.

Important Facts

- An important property of an equivalence relation is that it divides the set into pair-wise disjoint subsets called equivalence classes whose collection is called a partition of the set.
- Note that the union of all equivalence classes give the whole set.
  e.g., Let R denotes the equivalence relation in the set Z of integers given by $R = \{(a, b) : 2$ divides $a - b\}$. Then the equivalence class $[0]$ is $[0] = [0, \pm 2, \pm 4, \pm 6, ...]$.

  (c) Co-domain of a relation : Let R be a relation from A to B. Then B is called the co-domain of the relation R. So we can observe that co-domain of a relation R from A into B is the set B as a whole.

  Illustrations : Let $a \in A$ and $b \in B$ and

  (a) Let $A = \{1, 2, 3, 7\}$, $B = \{3, 6\}$. If aRb means $a < b$.

  Then we have $R = \{(1, 3), (1, 6), (2, 3), (2, 6), (3, 6)\}$.

  Here, $\text{Dom}(R) = \{1, 2, 3\}$, Range of $R = \{3, 6\}$, Co-domain of $R = B = \{3, 6\}$

  (b) Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$.

  If $R_1 = \{(1, 2), (2, 4), (3, 6)\}$, and $R_2 = \{(2, 4\}, (2, 6), (3, 8), (1, 6)\}$

  Then both R_1 and R_2 are related from A to B because

  $$R_1 \subseteq A \times B, R_2 \subseteq A \times B$$

  Here, $\text{Dom}(R_1) = \{1, 2, 3\}$, Range of $R_1 = \{2, 4, 6\}$;

  $\text{Dom}(R_2) = \{2, 3, 1\}$, Range of $R_2 = \{4, 6, 8\}$

- **INVERSE RELATION**

  Let $R \subseteq A \times B$ be a relation from A to B. Then, the inverse relation of R, to be denoted by R^{-1}, is a relation from B to A defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

  Thus $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1} \ \forall \ a \in A, b \in B$.
  Clearly, $\text{Dom}(R^{-1}) = $ Range of R, Range of $R^{-1} = \text{Dom}(R)$
  Also, $(R^{-1})^{-1} = R$.

  For example, Let $A = \{1, 2, 4\}$, $B = \{3, 0\}$ and let $R = \{(1, 3), (4, 0), (2, 3)\}$ be a relation from A to B, then $R^{-1} = \{(3, 1), (0, 4), (3, 2)\}$.

Know the Terms

- **(a)** A relation R from A to B is an empty relation or void relation if $R = \phi$.

  (b) A relation R on a set A is an empty relation or void if $R = \phi$.
- **(a)** A relation R from A to B is a universal relation if $R = A \times B$.

  (b) A relation R on a set A is a universal relation if $R = A \times A$.
- A relation R on a set A is reflexive if $aRa, \ \forall \ a \in A$.
- A relation R on a set A is symmetric if whenever aRb then bRa for all $a, b \in A$.
- A relation R on a set A is transitive if whenever aRb and bRc then aRc for all $a, b, c \in A$.
- A relation R on A is identity relation if $R = \{(a, a), \forall \ a \in A\}$, *i.e.*, R contains only elements of the type $(a, a) \ \forall \ a \in A$ and it contains no other element.
- A relation R on a non-empty set A is an equivalence relation if it includes reflexive, symmetric and transitive relations.

Mnemonics

RUSTEE

R	U	S	T	E	E
↓	↓	↓	↓	↓	↓
REFLEXIVE Relation	UNIVERSAL Relation	SYMMETRIC Relation	TRANSITIVE Relation	EMPTY Relation	EQUIVALENCE Relation

Interpretation:

Reflexive Relation: A relation R on set A is said to be a reflexive if $(a, a) \in R$ for every $a \in A$

Symmetric Relation: A relation R on set A is said to be symmetric if $(a, b) \in R \Leftrightarrow (b, a) \in R$.

Transitive Relations: A Relation R on set A is said to be transitive if $(a, b) \in R$ and $(b, c) \in R \Leftrightarrow (a, c) \in R$.

Mnemonic for Equivalence Relations

Ram	Sells	Tickets
↓	↓	↓
Reflexive	Symmetric	Transitive

OSWAAL BOOKS LEARNING MADE SIMPLE

1 Mark Questions

AI Q. 1. Let R be the relation in the set N, given by R = $\{(x, y): x = y + 3, y > 5\}$. Choose the correct answer from the following :

(a) $(7, 4) \in R$ **(b) $(9, 6) \in R$**

(c) $(4, 1) \in R$ **(d) $(8, 5) \in R$**

[ISC Specimen Paper, 2021]

Sol. Correct option : (b).

Explanation : Given $R = \{x,y) : x = y + 3, y > 5\}$ is relation in the set N.

$9 = 6 + 3 \Rightarrow (9, 6) \in R$

Q 2. Consider the non-empty set consisting of children in a family and a relation R defined as aRb if a is brother of b. Then R is

(a) symmetric but not transitive.

(b) transitive but not symmetric.

(c) neither symmetric nor transitive.

(d) both symmetric and transitive.

Sol. Correct option : (b)

Explanation : $aRb \Rightarrow a$ is brother of b.

This does not mean b is also a brother of a as b can be a sister of a.

Hence, R is not symmetric.

$aRb \Rightarrow a$ is brother of b

and $bRc \Rightarrow b$ is a brother of c.

So, a is brother of c.

Hence, R is transitive.

Q 3. The maximum number of equivalence relations on the set A = {1, 2, 3} are

(a) 1 **(b) 2**

(c) 3 **(d) 5**

Sol. Correct option : (d)

Explanation : Given that, $A = \{1, 2, 3\}$

Now, number of equivalence relations are as follows:

$R_1 = \{(1,1),(2,2),(3,3)\}$

$R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

$R_3 = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$

$R_4 = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$

$R_5 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$

$\therefore$ Maximum number of equivalence relation on the set $A = \{1, 2, 3\} = 5$

Q4. If a relation R on the set {1, 2, 3} be defined by $R = \{(1, 2)\}$, then R is

(a) reflexive. **(b) transitive.**

(c) symmetric. **(d) None of these**

Sol. Correct option : (b)

Explanation : R on the set {1, 2, 3} is defined by $R = \{(1, 2)\}$

It is clear that R is transitive.

Q 5. Let us define a relation R in R as aRb if $a \geq b$. Then R is

(a) an equivalence relation.

(b) reflexive, transitive but not symmetric.

(c) symmetric, transitive but not reflexive.

(d) neither transitive nor reflexive but symmetric.

Sol. Correct option : (b)

Explanation : Given that, aRb if $a \geq b$

$\Rightarrow aRa \Rightarrow a \geq a$ which is true

Let aRb, $a \geq b$, then $b \geq a$ which is not true so R is not symmetric.

But aRb and bRc
$\Rightarrow a \geq b$ and $b \geq c$
$\Rightarrow a \geq c$
Hence, R is transitive.

Q 6. If $R = \{(x, y) : x + 2y = 8\}$ is a relation on N, write the range of R.

Sol. $R = \{(2, 3), (4, 2), (6, 1)\}$
Range $= \{3, 2, 1\}$.

Q 7. Let $R = \{(a, a^3)$: a is a prime number less than 5$\}$ be a relation. Find the range of R.

Sol. Given, $R = \{(a, a^3)$: a is a prime number less than 5$\}$
$\Rightarrow$ $R = \{(2, 8), (3, 27)\}$
$\therefore$ Range $= \{8, 27\}$.

Q 8. State the reason why the relation $R = \{(a, b) : a \leq b^2\}$ on the set R of real numbers is not reflexive.

Sol. $\frac{1}{2} > \left(\frac{1}{2}\right)^2$ hence $\left(\frac{1}{2}, \frac{1}{2}\right)$ is not an element of R.
Hence, R is not reflexive.

Q 9. Let R be an equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : 2$ divides $(a - b)\}$. Write the equivalence class [0].

Sol. Given, $R = \{(a, b) : 2$ divides $(a - b)\}$
$\forall\, a, b \in A = \{0, 1, 2, 3, 4, 5\}$
Equivalence class $[0] = \{0, 2, 4\}$

Q 10. How many equivalence relations on the set {1, 2, 3} containing (1, 2) and (2, 1) are there in all ? Justify your answer.

Sol. Equivalence relations could be the following :
$\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ and $\{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$
So, only two equivalence relations.

Q 11. State the reason for the relation R in the set {1, 2, 3} given by $R = \{(1, 2), (2, 1)\}$ not to be transitive.

Sol. We know that, for a relation to be transitive,
$(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.
Here, $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$.
$\therefore R$ is not transitive.

2 Marks Questions

Q 1. Define reflexive relation. Give one example.

Sol. Reflexive relation : A relation R on a set A is called reflexive relation if aRa, for every $a \in A$; if $(a, a) \in R$, for every $a \in A$.
Example, let
$A = \{1, 2, 3\}$
$A \times A = \{(1, 1)\ (1, 2)\ (1, 3)\ (2, 1)\ (2, 2)\ (2, 3)\ (3, 1)\ (3, 2)\ (3, 3)\} \in R.$
Since, $(a, a) \in R$, for every $a \in A$.

AI Q. 2. Define symmetric relation. Give one example.

Sol. Symmetric relation : A relation R on a set A is called symmetric relation if aRb implies bRa, for every $a, b \in A$ *i.e.*, if $(a, b) \in R \Rightarrow (b, a) \in R, \forall\, a, b \in A$.
Example
$A = (1, 2, 3)$
$A \times A = \{(1, 2)\ (2, 1)\ (1, 1)\ (2, 2)\ (3, 3)\ (1, 3)\ (2, 3)\ (3, 1)\ (3, 2)\} \in R$
Since, $(a, b) \Rightarrow R \Rightarrow (b, a) \Rightarrow R$, for every $a, b \in A$.
Relation is said to be symmetric.

Q 3. Define transitive relation. Give one example.

Sol. A relation R on a non-empty set A is called a transitive relation, if $(a, b), (b, c) \in R$, then $(a, c) \in R$, *i.e.*, aRb, bRc implies aRc.
Thus, a relation R on a non-empty set A is said to be transitive if there exist $a, b, c \in A$ such that $(a, b)\ (b, c) \in R \Rightarrow (a, c) \in R$.
Example
Let $A = \{1, 2, 3, 6\}$
$R = \{(3, 6), (6, 1), (3, 1)\}$
Since, $3R6$ and $6R1 \Rightarrow (3, 1) \in R$
$\therefore A$ is transitive.

Q 4. Let $f : X \rightarrow Y$ be a function. Define a relation R on X given by $R = \{(a, b) : f(a) = f(b)\}$. Show that R is an equivalence relation ?

Sol. (i) Let $a \in x$, then $f(a) = f(a) \Rightarrow (a, a) \in R$. Relation is reflexive.
(ii) Let $(a, b) \in R$ then $f(a) = f(b) \Rightarrow f(b) = f(a)$
$\Rightarrow (b, a) \in R$. Relation is symmetric.
(iii) Let $(a, b), (b, c) \in R, f(a) = f(b), f(b) = f(c) \Rightarrow f(a) = f(c), (a, c) \in R$. Relation is transitive.
All the three relations are satisfied it is an equivalence relation.

Q 5. Given an example of a relation which is
(i) Reflexive, symmetric and transitive.
(ii) Reflexive, symmetric and not transitive.

Sol. Let $A = (1, 2, 3)$
$A \times A = \{(1, 1)(2, 2)(3, 3)(1, 2)(2, 1)(3, 2)(2, 3)(1, 3), (3, 1)\}$
(i) $R = \{(1, 1)\ (1, 2)\ (2, 1)\ (2, 2)\ (2, 3)\ (1, 3)\ (3, 3)\ (3, 2)\ (3, 1)\}$
(ii) $R = \{(1, 1)\ (1, 2)\ (2, 1)\ (2, 2)\ (1, 3), (3, 3)\ (3, 1)\}$

Q 6. Let A be the set of all human beings in a town at a particular time. Determine whether the relation $R = \{(x, y) : x$ is wife of y ; $x, y \in A\}$ is reflexive, symmetric and transitive.

Sol. Given, A = Set of all human beings in a town at a particular time and $R = \{(x, y); x$ is the wife of $y; x, y \in A\}$
(i) Since, x is wife of x, is not true $(x, x) \notin R$
So, R is not reflexive.
(ii) x is wife of y, but y is not the wife of x, $(y, x) \notin R$.
So, R is not symmetric.
(iii) x is wife of y, but y cannot be the wife of z.
Thus, x is not the wife of z.
So, R is not transitive.

Q 7. Consider the relation of perpendicular lines on a set of lines in a plane. Show that this relation is symmetric and neither reflexive and nor transitive.

Sol. Let the relation of perpendicular lines on a set of lines in a plane is denoted by R.
For Symmetry : $l_1 R l_2 \Rightarrow l_1 \perp l_2 \Rightarrow l_2 \perp l_1 \Rightarrow l_2 R l_1$
$\therefore$ The relation R is symmetric.

For Reflexive : $l_1 \, R \, l_1$

No line can be perpendicular to itself.

$\therefore$ The relation R is not reflexive

For Transitive : $l_1 \, R \, l_2$, and $l_2 \, R \, l_3$

$\Rightarrow l_1 \perp l_2$ and $l_2 \perp l_3$ but

l_1 is not $\perp$ to l_3 or $l_1 \not R \, l_3$

$\therefore$ Relation is not transitive.

4 Marks Questions

AI **Q. 1. Show that the relation R in the set $N \times N$ defined by $(a, b) \, R \, (c, d)$ if $a^2 + d^2 = b^2 + c^2 \; \forall \; a, b, c, d \in N$, is an equivalence relation.**

Sol. Let $(a, b) \in N \times N$

Then,

$\because \quad a^2 + b^2 = a^2 + b^2$

$\therefore \quad (a, b) \, R \, (a, b)$

Hence, R is reflexive.

Let $(a, b), (c, d) \in N \times N$ such that

$(a, b) \, R \, (c, d)$

$\Rightarrow \quad a^2 + d^2 = b^2 + c^2$

$\Rightarrow \quad c^2 + b^2 = d^2 + a^2$

$\Rightarrow \quad (c, d) \, R \, (a, b)$

Hence, R is symmetric.

Let $(a, b), (c, d), (e, f) \in N \times N$ such that

$(a, b) \, R \, (c, d)$; $(c, d) \, R \, (e, f)$.

$\Rightarrow \quad (a, b) \, R \, (c, d) \Rightarrow a^2 + d^2 = b^2 + c^2 \quad \ldots\text{(i)}$

and $\quad (c, d) \, R \, (e, f) \Rightarrow c^2 + f^2 = d^2 + e^2 \quad \ldots\text{(ii)}$

Adding (i) and (ii),

$\Rightarrow \quad a^2 + d^2 + c^2 + f^2 = b^2 + c^2 + d^2 + e^2$

$\Rightarrow \quad a^2 + f^2 = b^2 + e^2$

$\Rightarrow \quad (a, b) \, R \, (e, f)$

Hence, R is transitive.

Since, R is reflexive, symmetric and transitive. Therefore, R is an equivalence relation.

Q. 2. Let R be a relation defined on the set of natural numbers N as follow:

$R = \{(x, y) : x \in N, y \in N$ and $2x + y = 24\}$

Find the domain and range of the relation R. Also, find if R is an equivalence relation or not.

Sol. $R = \{(1, 22), (2, 20), (3, 18), (4, 16), (5, 14), (6, 12), (7, 10), (8, 8), (9, 6), (10, 4), (11, 2)\}$

Domain $= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Range $= \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$

$(1, 22) \in R$ but $(22, 1) \notin R$

$\Rightarrow R$ is not symmetric.

$\therefore R$ is not an equivalence relation.

Q.3. Let $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation in $A \times A$ defined by $(a, b) \, R \, (c, d)$ if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation. Also, obtain the equivalence class $[(2, 5)]$.

Sol. For $(a, b) \in A \times A$

$a + b = b + a$

$\therefore \quad (a, b) \, R \, (a, b)$

Hence, R is reflexive.

For $(a, b), (c, d) \in A \times A$

$(a, b) \, R \, (c, d)$, *i.e.*, $a + d = b + c$

$\Rightarrow \quad c + b = d + a$

$\therefore \quad (c, d) \, R \, (a, b)$

Hence, R is symmetric

For $(a, b), (c, d), (e, f) \in A \times A$

If $(a, b) \, R \, (c, d)$ and $(c, d) \, R \, (e, f)$

i.e., $a + d = b + c$ and $c + f = d + e$

Adding, $a + d + c + f = b + c + d + e$

$\Rightarrow \quad a + f = b + e$

$\therefore (a, b) \, R \, (e, f)$

Hence, R is transitive.

R is reflexive, symmetric, and transitive.

Hence, R is an equivalence relation.

$[2, 5] = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

Q. 4. Show that the relation R on R defined as $R = \{(a, b) : a \leq b\}$, is reflexive, and transitive but not symmetric.

Sol. We have, $R = \{(a, b) : a \leq b\}$, where $a, b \in R$

Reflexivity : For any $a \in R$

$a \leq a$

$\Rightarrow (a, a) \in R$ for all $a \in R$

$\Rightarrow R$ is reflexive.

Symmetry : We observe that $(2, 3) \in R$, but $(3, 2) \notin R$.

So, R in not symmetric.

Transitivity : Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$\Rightarrow \quad a \leq b$ and $b \leq c$

$\Rightarrow \quad a \leq c$

$\Rightarrow \quad (a, c) \in R$

So, R is transitive.

Q. 5. Show that the relation S in the set R of real numbers defined as $S = \{(a, b) : a, b \in R$ and $a \leq b^3\}$ is neither reflexive nor symmetric nor transitive.

Sol. $S = \{(a, b) : a, b \in R$ and $a \leq b^3\}$.

Reflexive As $\frac{1}{2} \leq \left(\frac{1}{2}\right)^3$, where $\frac{1}{2} \in R$, is not true.

$\therefore \quad \left(\frac{1}{2}, \frac{1}{2}\right) \notin S$

Thus, S is not reflexive.

Symmetric As $-2 \leq (3)^3$, where $-2, 3 \in R$, is true but $3 \leq (-2)^3$ is not true.

i.e. $(-2, 3) \in S$ but $(3, -2) \notin S$.

Therefore, S is not symmetric.

Transitive As $3 \leq \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} \leq \left(\frac{4}{3}\right)^3$, where

OSWAAL BOOKS LEARNING MADE SIMPLE

$3, \frac{3}{2}, \frac{4}{3} \in R$, are true but $3 \le \left(\frac{4}{3}\right)^3$ is not true.

i.e. $\left(3, \frac{3}{2}\right) \in S$ and $\left(\frac{3}{2}, \frac{4}{3}\right) \in S$ but $\left(3, \frac{4}{3}\right) \notin S$.

Therefore, S is not transitive.

Hence, S is neither reflexive nor symmetric nor transitive.

6 Marks Questions

AI **Q. 1. Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by (a, b) R (c, d) if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation.**

Sol. Given, $(a, b)\ R\ (c, d)$ as $ad(b + c) = bc(a + d)$

For $\quad a, b \in N$

$\Rightarrow \quad ab(b + a) = ba(a + b)$

$\Rightarrow \quad (a, b)\ R\ (a, b)$

$\therefore$ R is reflexive ...(i)

Let $(a, b)\ R\ (c, d)\ \forall\ a, b, c, d \in N$

$\therefore \quad ad(b + c) = bc(a + d)$ ...(ii)

Also, $\quad (c, d)\ R\ (a, b)$

$\because \quad cb(d + a) = da(c + b)$

or, $\quad da\ (c + b) = cb\ (d + a)$

(By commutation of addition and multiplication only)

Hence, $\quad cb\ (d + a) = da\ (c + b)$

or, $\quad (c, d)\ R\ (a, b)$

$\therefore$ R is symmetric ...(iii)

Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)\ \forall\ a, b, c, d, e, f \in N$

$(a, b)\ R\ (c, d) \Rightarrow ad(b + c) = bc(a + d)$ ...(iv)

and $\quad (c, d)\ R\ (e, f) \Rightarrow cf(d + e) = de(c + f)$ ...(v)

$$\Rightarrow \quad \frac{ab}{a-b} = \frac{cd}{c-d} \text{ and } \frac{cd}{c-d} = \frac{ef}{e-f}$$

$$\Rightarrow \quad \frac{ab}{a-b} = \frac{ef}{e-f}$$

$\Rightarrow \quad af(b + e) = be(a + f)$

Hence, $\quad (a, b)\ R\ (e, f)$

$\therefore$ R is transitive ...(vi)

From equations (i), (iii) and (vi), R is an equivalence relation.

Q. 2. Show that the relation R in the Set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b|$ is divisible by 2$\}$ is an equivalence relation. Write all the equivalence classes of R.

Sol. Given, $R = \{(a, b): |a - b|$ is divisible by 2$\}$

and $A = \{1, 2, 3, 4, 5\}$

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$
$(1, 3), (1, 5), (2, 4), (3, 5), (3, 1),$
$(5, 1), (4, 2), (5, 3)\}$

(i) $\forall\ a \in A, (a, a) \in R,$

$\therefore$ R is reflexive

[As $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) \in R]$

(ii) $\forall\ (a, b) \in A, (b, a) \in R,$

$\therefore$ R is symmetric

[As $\{(1, 3), (1, 5), (2, 4), (3, 5), (3, 1), (5, 1), (4, 2), (5, 3) \in R\}]$

(iii) $\forall\ (a, b), (b, c) \in R, (a, c) \in R$

$\therefore$ R is transitive

[As $\{(1, 3), (3, 1) \in R \Rightarrow (1, 1) \in R$ and similarly others]

$\therefore$ R is an equivalence relation.

Equivalence classes are

$[1] = \{1, 3, 5\}$

and $\quad [2] = \{2, 4\}$

TOPIC-2
Functions

Revision Notes

Scan to know more about this topic

Functions

- **Function :**

  Consider A and B be two non-empty sets, then rule f which associates each element of A with a unique element of B is called a function or the mapping from A to B or f maps A to B. If f is mapping from A to B, then we write $f : A \rightarrow B$ which is read as is mapping from A to B or f is a function from A to B.

  If f associates $a \in A$ to $b \in B$, then we say that b is the image of the element a under the function f or b is the f- image of a or the value of f at 'a' is denoted by $f(a)$ and we write $b = f(a)$. The element a is called the pre-image or inverse-image of b.

  Remember the following points :

  (a) Negative number should not occur under the square root (even root), *i.e.*, Expression under the square root sign must be always ≥ 0.

(b) Denominator should never be zero.

(c) For $\log_b a$ to be defined $a > 0, b > 0$ and $b \neq 1$. Also, $\log_b 1$ is equal to zero, *i.e.*, 0.

- **Difference between relation and function :** A relation from a set A to another set B is any subset of $A \times B$; while a function f from A to B is a subset of $A \times B$ satisfying following conditions:

(a) For every $x \in A$, there exists $y \in B$ such that $(x, y) \in f$.

(b) If $(x, y) \in f$ and $(x, z) \in f$ then, $y = z$.

Important Facts

- It is necessary that every f- image is in B; but there may be some elements in B which are not the f- images of any element of A *i.e.*, whose pre-image under f is not in A.
- Two or more elements of A may have same image in B.
- $f: x \rightarrow y$ means that under the function f from A to B, an elements x of A has image y in B.
- Usually we denote the function f by writing $y = f(x)$ and read it as y **is a function of** x.

S. No.	Function	Relation
(i)	Each elements of A is related to some element of B.	There may be some elements of A which are not related to any element of B.
(ii)	An element of A is not be related to more than one element of B.	An element of A may be related to more than one element of B.

- **Domain of a function :** If a function is expressed in the form $y = f(x)$, then domain of f means set of all those real values of x for which y is real (*i.e.*, y is well-defined).

Remember the following points for domain :

(a) First find the domain of the given function.

(b) If the domain does not contain an interval, then find the values of y putting these values of x from the domain. The set of all these values of y obtained will be the range.

(c) If domain is the set of all real numbers R or set of all real numbers except a few points, then express x in terms of y and from this find the real values of y for which x is real and belongs to the domain.

- **Range of the function :** If a function is expressed in the form $y = f(x)$, then range of f means set of all possible real values of y corresponding to every value of x in its domain.
- **Real function :** If the domain and range of a function f are subsets of R (the set of real numbers), then f is said to be a real function.
- **Some important real functions and their Domain and Range :**

S. No.	Function	Representation	Domain	Range
(i)	Identity function	$I(x) = x \, \forall \, x \in R$	R	R
(ii)	Modulus function or Absolute value function	$f(x) = \lvert x \rvert = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$	R	$[0, \infty]$
(iii)	Greatest integer function or integral function or step function.	$f(x) = [x] \, \forall \, x \in R$	R	Z
(iv)	Signum function	$f(x) = \begin{cases} \frac{\lvert x \rvert}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ (*i.e.*,) $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$	R	$\{-1, 0, 1\}$
(v)	Exponential function	$f(x) = a^x, \forall \, a \neq 1, a > 0$	R	$(0, \infty)$
(vi)	Logarithmic function	$f(x) = \log_a x, \forall \, a \neq 1, a > 0$ and $x > 0$	$(0, \infty)$	R
(vii)	Irrational function	$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
(viii)	Rational function	$f(x) = \frac{1}{x}$	$R - \{0\}$	$R - \{0\}$

- **Types of Function :**

(a) One-one function (Injective function or Injection) : A function $f: A \rightarrow B$ is one-one function or injective function if distinct element of A have distinct images in B.

Thus, $f: A \rightarrow B$ is one-one $\Leftrightarrow f(a) = f(b) \Rightarrow a = b, \forall \, a, b \in A$

$$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b) \; \forall \, a, b \in A.$$

Scan to know more about this topic

Types of functions

- If A and B are two sets having m and n elements respectively such that $m \leq n$, then total number of one-one functions from set A to set B is ${}^nC_m \times m\,!$ *i.e.,* ${}^nP_m$.
- If $n(A) = m$, then the number if injective functions defined from A onto itself is $m!$.

Algorithm to check the Injective of a Function :

Step 1 : Take any two arbitrary elements a, b in the domain of f.

Step 2 : Put $f(a) = f(b)$

Step 3 : Solve $f(a) = f(b)$. If it gives $a = b$ only then f is one-one function.

(b) Many-one function : If any two or more elements of set A are connected with a single element of set B, then we call this function as many-one function.

Thus, $f : A \rightarrow B$ is many-one $\Leftrightarrow f(a) = f(b) \Leftrightarrow a \neq b,\ \forall\ a, b \in A$

(c) Onto function (Surjective function or Surjection) : A function $f : A \rightarrow B$ is onto function or a surjective function if every element of B is the f-image of some element of A. That implies $f(A) = B$ or range of f is the co-domain of f.

Thus, $f : A \rightarrow B$ is onto $\Leftrightarrow f(A) = B$ *i.e.,* range of f = co-domain of f.

Algorithm to check the Subjectivity of a Function :

Step 1: Take an element $b \in B$, where B is the co-domain of the function.

Step 2: Put $f(x) = b$.

Step 3: Solve the equation $f(x) = b$ for x and obtain x in terms of b. Let $x = g(b)$.

Step 4: If for all values of $b \in B$, the values of x obtained from $x = g(b)$ are in A, then f is onto. If there are some $b \in B$ for which values of x, given by $x = g(b)$, is not in A. Then, f is not onto.

(d) Into function : Function 'f' from set A to set B is into function, if set B has atleast an element which is not connected with any of the element of set A.

(e) One-one onto function (Bijective function or Bijection) : A function $f : A \rightarrow B$ is said to be one-one onto or bijective if it is both one-one and onto *i.e.,* if the distinct elements of A have distinct images in B and each element of B is the image of some element of A.

Also, note that a bijective function is also called a one-to-one function or one-to-one correspondence.

(f) Equal function : Two functions f and g having the same domain D are said to be equal if $f(x) = g(x)$ for all $x \in D$.

➢ For an ordinary finite set A, a one-one function $f : A \rightarrow A$ is necessarily onto and an onto function $f : A \rightarrow A$ is necessarily one-one for every finite set A.

Thus, for a bijective function from A to B,

(a) A and B should be non-empty.

(b) Each element of A should have image in B.

(c) No element of A should have more than one image in B.

(d) If A and B have respectively m and n number of elements then the number of functions defined from A to B is n^m.

➢ **Inverse of a Function :**

Let $f : A \rightarrow B$ be a bijection. Then, a function $g : B \rightarrow A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called the inverse of f *i.e.,* $f(x) = y \Leftrightarrow g(y) = x$.

The inverse of f is generally denoted by f^{-1},

Thus, if $f : A \rightarrow B$ is a bijection, then a function $f^{-1} : B \rightarrow A$ is such that $f^{-1}(y) = x$.

Scan to know more about this topic

Inverse of a function

➢ **Algorithm to Find the Inverse of a Function :**

Step 1: Put $f(x) = y$ where $y \in B$ and $x \in A$.

Step 2: Solve $f(x) = y$ to obtain x in terms of y.

Step 3: Replace x by $f^{-1}(y)$ in the relation obtained in Step 2.

Step 4: In order to get the required inverse of f *i.e.,* $f^{-1}(x)$, replace y by x in the expression obtained in Step 3 *i.e.,* in the expression $f^{-1}(y)$.

Mnemonics

Types of Functions

Interpretation:

Moving	**Immense**	**Organs**	**Insert**	**Outdoor**	**Extreme**	**Operations**
↓	↓	↓	↓	↓	↓	↓
Many-One Function	Into Function	One-One Function	Identity Function	Onto Function	Equal Function	One-One Onto Function

1 Mark Questions

Q 1. If $A = \{1, 2, 3\}$, $B = \{x, y\}$, then the number of functions from A to B is :

(a) 3 **(b) 6**

(c) 8 **(d) 12**

[ISC Specimen Paper, 2021]

Sol. Correct option : (c).

Explanation:

Given $A = \{1, 2, 3\}$

$\therefore$ $n(A) = 3$

and $B = \{x, y\} \Rightarrow n(B) = 2$

$\therefore$ The no. of functions from A to B is $= 2^3 = 8$

Q 2. If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

(a) 720 **(b) 120**

(c) 0 **(d) None of these**

Sol. Correct option : (c)

Explanation : We know that, if A and B are two non-empty finite sets containing m and n elements, respectively, then the number of one-one and onto mapping from A to B is

$n!$ if $m = n$

0, if $m \neq n$

Given that, $m = 5$ and $n = 6$

$\therefore$ $m \neq n$

Number of one-one and onto mapping = 0

AI Q. 3. Let $A = \{1, 2, 3, ...n\}$ and $B = \{a, b\}$. Then the number of surjections from A into B is

(a) $^nP_2$ **(b) 2^n-2**

(c) 2^n-1 **(d) None of these**

Sol. Correct option : (a)

Explanation : Given that $A = \{1, 2, 3, ...n\}$ and $B = \{a, b\}$

We know that, if A and B are two non-empty finite sets containing m and n elements, respectively, then the number of surjection from A into B is

$^nC_m \times m!$ *if* $n \geq m$

0, *if* $m < n$

Here, $m = 2$

$$^nC_2 \times 2! = \frac{n!}{2!(n-2)!} \times 2!$$

$$= \frac{n!}{(n-2)!}$$

Q 4. Let $f : R \rightarrow R$ be defined by $f(x) = \frac{1}{x}, \forall\ x \in R$. Then f is

(a) one-one **(b) onto**

(c) bijective **(d) not defined**

Sol. Correct option : (d)

Explanation :

We have, $f(x) = \frac{1}{x}$, $\forall x \in R$

For $x = 0$, $f(x)$ is not defined.

Q 5. Which of the following functions from Z into Z are bijections?

(a) $f(x) = x^3$ **(b) $f(x) = x + 2$**

(c) $f(x) = 2x + 1$ (d) **$f(x) = x^2 + 1$**

Sol. Correct option : (b)

Explanation : For bijection on Z, $f(x)$ must be one-one and onto.

Function $f(x) = x^2 + 1$ is many-one as $f(1) = f(-1)$

Range of $f(x) = x^3$ is not Z for $x \in Z$.

Also $f(x) = 2x + 1$ takes only values of type $= 2k + 1$ for $x \in k \in Z$

But $f(x) = x + 2$ takes all integral values for $x \in Z$

Hence $f(x) = x + 2$ is bijection of Z.

Q. 6. Let $f : R \rightarrow R$ be defined as $f(x) = 3x$. Choose the correct answer.

(a) f is one- one onto

(b) f is many-one onto

(c) f is one-one but not onto

(d) f is neither one-one nor onto

Sol. Correct option : (a)

Explanation : $f : R \rightarrow R$ is defined as $f(x) = 3x$.

Let $a, b \in R$ such that $f(x) = f(y)$.

$\Rightarrow$ $3x = 3y$

$\Rightarrow$ $x = y$

$\therefore f$ is one-one.

OSWAAL BOOKS® LEARNING MADE SIMPLE

Also, for any real number (y) in co-domain R, there exists $\frac{y}{3}$ in R such that $f\left(\frac{y}{3}\right)=3\left(\frac{y}{3}\right)=y$.

$\therefore$ f is onto.

Hence, function f is one-one and onto.

Q. 7. If A = {1, 2, 3}, B = {4, 5, 6, 7} and f = {(1, 4), (2, 5), (3, 6)} is a function from A to B. State whether f is one-one or not.

Sol. Given, $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and $f: A \to B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$ *i.e.* $f(1) = 4$, $f(2) = 5$ and $f(3) = 6$.

Hence, f is one-one.

AI Q. 8. If $f : R \to R$ defined as $f(x)=\frac{2x-7}{4}$ is an invertible function, write $f^{-1}(x)$.

Sol. Given, $f(x)=\frac{2x-7}{4}$

Let $f(x)=y \Rightarrow \frac{2x-7}{4}=y$

$\therefore$ $x=\frac{4y+7}{2} \in R$

$\therefore$ $f^{-1}(x)=\frac{4x+7}{2}$

Q. 9. What is meant by one-one function.

Sol. A function $f: A \to B$ is said to be one-one if

$a \neq b$

$\Rightarrow$ $f(a) \neq f(b)$ for all $a, b \in A$

or $f(a) = f(b)$

$\Rightarrow$ $a = b$ for all $a, b \in A$

In other words, f is one-one, if no two elements of A have same image, *i.e.*, no two elements of A are mapped to same elements of B.

Q. 10. $f(x) = x^2, x \in R$. Find $\frac{f(1{\cdot}1)-f(1)}{1{\cdot}1-1}$.

Sol.

$$f(x) = x^2, x \in R$$

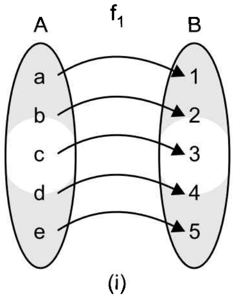
$$f(1{\cdot}1) = (1{\cdot}1)^2 = 1{\cdot}21,$$

$$f(1) = (1)^2 = 1$$

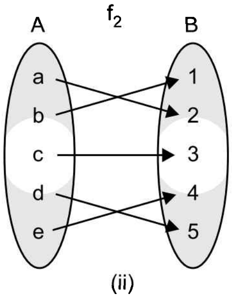
$$\therefore \frac{f(1{\cdot}1)-f(1)}{1{\cdot}1-1} = \frac{1{\cdot}21-1}{1{\cdot}1-1} = \frac{0{\cdot}21}{0{\cdot}1} = 2{\cdot}1$$

2 Marks Questions

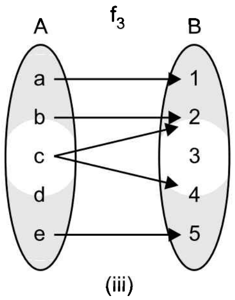
Q 1. Check whether functions in the following diagrams are onto :



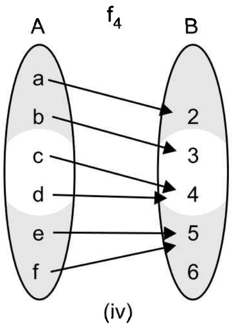
(i)



(ii)



(iii)



(iv)

Sol. (i) Since, every element in B has pre-image in A. So, f_1 is onto function.

(ii) Since, every element in B has pre-image in A. So, f_2 is onto function.

(iii) Since c has more than one image. So, f_3 is not a function.

(iv) Since, 6 ∈ B does not have pre-image of A. So, f_3 is not onto function.

Q 2. Show that the function f in $A = R - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one.

Sol. Given $f(x) = \frac{4x+3}{6x-4}$

Let, $f(x_1) = f(x_2)$,

then $\frac{4x_1+3}{6x_1-4} = \frac{4x_2+3}{6x_2-4}$

or $(4x_1 + 3)(6x_2 - 4) = (6x_1 - 4)(4x_2 + 3)$

or $24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$

or $-16x_1 + 18x_2 = 18x_1 - 16x_2$

or $-16x_1 - 18x_1 = -18x_2 - 16x_2$

or $-34x_1 = -34x_2$

or $x_1 = x_2$

or f is one-one.

AI Q. 3. Show that the function f in $A = R - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is onto.

Sol. Let, $y = f(x)$

or $y = \frac{4x+3}{6x-4}$

or $y(6x - 4) = 4x + 3$

or $6xy - 4y = 4x+3$

or $6xy - 4x = 4y + 3$

or $\quad x(6y-4) = 4y+3$

or $\quad x = \dfrac{4y+3}{6y-4} \in B = R - \left\{\dfrac{2}{3}\right\}$

or For every value of y except $y = \dfrac{2}{3}$, there is a pre-image $x = \dfrac{4y+3}{6y-4} = g(y)$.

$\therefore$ f is onto.

Q 4. Let $f(x)$ be a function on R defined as $f(x) = 2x + 3$. Find x when $f^{-1}(x) = 0$.

Sol. $\quad f(x) = 2x + 3$

Let $\quad y = 2x + 3$

$\Rightarrow \quad x = \dfrac{y-3}{2}$

$f^{-1}(x) = \dfrac{x-3}{2}$

$f^{1}(x) = 0 \Rightarrow \dfrac{x-3}{2} = 0$

$\Rightarrow \quad x = 3$

4 Marks Questions

Q 1. If : $A \rightarrow A$ and $A = R - \left\{\dfrac{8}{5}\right\}$, show that the function $f(x) = \dfrac{8x+3}{5x-8}$ is one - one onto. Hence, find f^{-1}.

[ISC, 2019]

Sol. Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$

$\therefore \quad \dfrac{8x_1+3}{5x_1-8} = \dfrac{8x_2+3}{5x_2-8} \Rightarrow 40x_1x_2 - 64x_1 + 15x_2 - 24$

$= 40x_1x_2 + 15x_1 - 64x_2 - 24$

$\Rightarrow 79x_1 = 79x_2$

$\therefore x_1 = x_2$

$\therefore$ the function f is one one.

Let $y = f(x)$

$$y = \frac{8x+3}{5x-8}$$

$5xy - 8y = 8x + 3$

$5xy - 8x = 8y + 3$

$x(5y - 8) = 8y + 3$

$x = \dfrac{8y+3}{5y-8}$ when $y \in A$

$\Rightarrow 5y - 8 \neq 0$

$\Rightarrow y = \dfrac{8}{5}$

$\therefore$ Range of $f = R - \left\{\dfrac{8}{5}\right\}$

$\Rightarrow f$ is onto

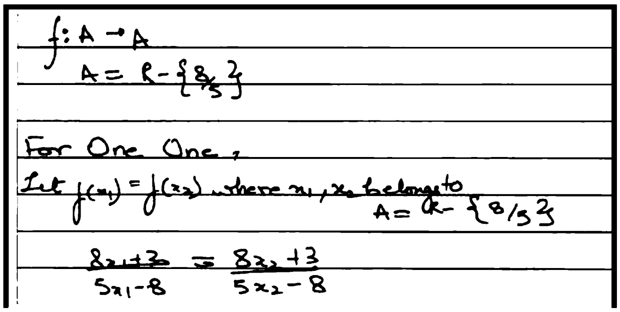
$\therefore$ the function of f is one one onto $\Rightarrow f$ is invertible.

$x = \dfrac{8x+3}{5x-8}$

$f^{-1}(y) = \dfrac{8y+3}{5y-8}$ i.e. $f^{-1}(x) = \dfrac{8x+3}{5x-8}$

[ISC Marking Scheme, 2019]

OR



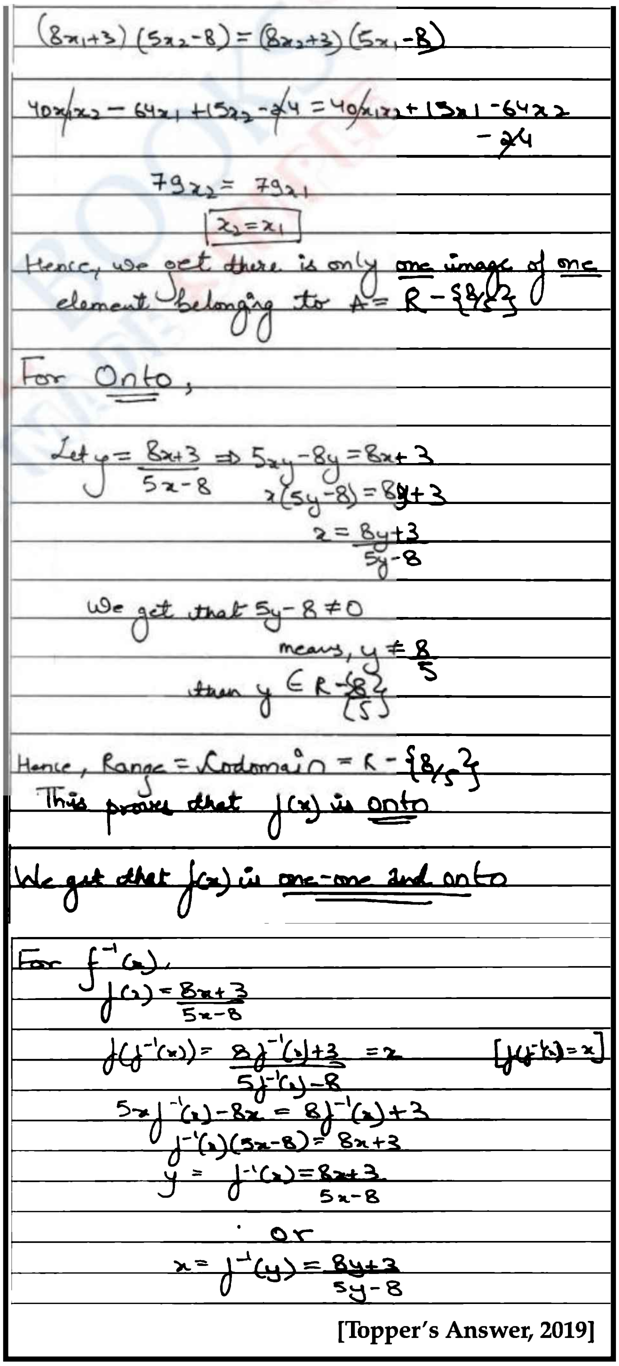
$f : A \rightarrow A$

$A = R - \{8/5\}$

For One One,

Let $f(x_1) = f(x_2)$ where x_1, x_2 belongs to $A = R - \{8/5\}$

$\dfrac{8x_1+3}{5x_1-8} = \dfrac{8x_2+3}{5x_2-8}$



$(8x_1+3)(5x_2-8) = (8x_2+3)(5x_1-8)$

$\cancel{40x_1x_2} - 64x_1 + 15x_2 - \cancel{24} = \cancel{40x_1x_2} + 15x_1 - 64x_2 - \cancel{24}$

$79x_2 = 79x_1$

$\boxed{x_2 = x_1}$

Hence, we get there is only one image of one element belonging to $A = R - \{8/5\}$

For Onto,

Let $y = \dfrac{8x+3}{5x-8} \Rightarrow 5xy - 8y = 8x + 3$

$x(5y-8) = 8y+3$

$x = \dfrac{8y+3}{5y-8}$

We get that $5y - 8 \neq 0$

means, $y \neq \dfrac{8}{5}$

then $y \in R - \left\{\dfrac{8}{5}\right\}$

Hence, Range = Codomain = $R - \{8/5\}$

This proves that $f(x)$ is onto

We get that $f(x)$ is one-one and onto

For $f^{-1}(x)$,

$f(x) = \dfrac{8x+3}{5x-8}$

$f(f^{-1}(x)) = \dfrac{8f^{-1}(x)+3}{5f^{-1}(x)-8} = x \qquad [f(f^{-1}(x)) = x]$

$5xf^{-1}(x) - 8x = 8f^{-1}(x) + 3$

$f^{-1}(x)(5x-8) = 8x+3$

$y = f^{-1}(x) = \dfrac{8x+3}{5x-8}$

or

$x = f^{-1}(y) = \dfrac{8y+3}{5y-8}$

[Topper's Answer, 2019]

Examiner's Comment

- Most of the candidates proved that the given function is one-one function, but some candidates could not give any justification to say that it is onto function also.

Answering Tip

- In such type of questions give justification for both one-one and onto functions.

AI **Q. 2. Let R^+ be the set of all positive real number and $f : R^+ \to [4, \infty)$; $f(x) = x^2 + 4$. Show that inverse of f exist and find f^{-1}. [ISC Specimen Paper, 2018]**

Sol. Given, $f : R_+ \to [4, \infty)$; $f(x) = x^2 + 4$

To show f is one-one :

Let $f(x_1) = f(x_2)$,

then $x_1^2 + 4 = x_2^2 + 4$

$\Rightarrow \quad x_1^2 = x_2^2$

$\Rightarrow \quad x_1 = x_2$ or $x_1 = -x_2$

$\because \quad x \in R_+$

$\therefore \quad x_1 \neq -x_2$

$\Rightarrow$ *f is one-one.*

To show f is onto :

Let $y \in [4, \infty)$

$\therefore \quad y = f(x)$

$\Rightarrow \quad y = x^2 + 4$

$\Rightarrow \quad x = \sqrt{y-4} \in R^+$

$\Rightarrow$ For every value of y, there is a pre-image.

$\Rightarrow f$ is onto.

$x = \sqrt{y-4} = f^{-1}(y)$.

Since, f is one-one and onto, therefore f is invertible.

Thus, $\quad f^{-1}(x) = \sqrt{x-4}$

Q. 3. Let $A = R - \{3\}$, $B = R - \{1\}$. Let $f : A \to B$ be defined by $f(x) = \left(\dfrac{x-2}{x-3}\right) \forall\, x \in A$. Then show that f is bijective. Hence find $f^{-1}(x)$.

Sol. Given, $\quad f(x) = \dfrac{x-2}{x-3}$

To show f is one-one :

Let, $f(x_1) = f(x_2)$, then $\dfrac{x_1 - 2}{x_1 - 3} = \dfrac{x_2 - 2}{x_2 - 3}$

$\Rightarrow \quad (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$

$\Rightarrow \quad x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow \quad -3x_1 - 2x_2 = -2x_1 - 3x_2$

$\Rightarrow \quad x_1 = x_2$

$\therefore f$ is one-one function.

To show f is onto : Let $y \in B$

$\therefore \quad y = f(x)$

$\Rightarrow \quad y = \dfrac{x-2}{x-3} \Rightarrow y(x-3) = x - 2$

$\Rightarrow \quad xy - 3y = x - 2$

$\Rightarrow \quad xy - x = 3y - 2$

$\Rightarrow \quad x(y-1) = 3y - 2$

$\Rightarrow \quad x = \dfrac{3y-2}{y-1} \in B\ \forall\ R - \{1\}$

For every value of y except $y = 1$, there is a pre-image

$x = \dfrac{3y-2}{y-1} = g(y)$

$\Rightarrow \quad x \in A$

$\therefore\ f$ in onto function.

To find f^{-1} :

Since, f is one-one and onto, therefore f is invertible.

Thus, $\ f^{-1}(x) = \dfrac{3x-2}{x-1} = g(x)$

Q. 4. Let $W \to W$ be defined as $f(n) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}$

Show that f is invertible. Find the inverse of f, where W is the set of all whole numbers.

Sol. For one-one suppose $f(x_1) = f(x_2)$

Note that if x_1 is odd and x_2 is even,

then we will have $x_1 - 1 = x_2 + 1 \Rightarrow x_1 - x_2 = 2$

which is impossible.

Similarly, the possibility of x_1 being even and x_2 being odd can also be ruled out, using the similar argument.

Therefore, both x_1 and x_2 must be either odd or even. Suppose both x_1 and x_2 are odd.

Then, $\quad f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$.

Similarly, if both x_1 and x_2 are even,

Then, also $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$.

Thus, f is one-one.

For onto

$$f(x) = \begin{cases} n-1, \text{if } n \text{ is odd} \\ n+1, \text{if } n \text{ is even} \end{cases}$$

Let $f(n) = y$ such that $y \in$ W

If n is odd	**If n is even**
$f(n) = n - 1$	$f(n) = n + 1$
$y = n - 1$	$y = n + 1$
$n = y + 1$	$n = y - 1$
If n is odd, y is even	If n is even, y is odd

$$n = \begin{cases} y-1, \text{if } y \text{ is odd} \\ y+1, \text{if } y \text{ is even} \end{cases}$$

Hence, if y is whole number, n will also be a whole number *i.e.*, $n \in W$

Thus, f is onto.

For f^{-1}

$$f(n) = \begin{cases} n+1, \text{if } n \text{ is even} \\ n-1, \text{if } n \text{ is odd} \end{cases}$$

For finding inverse, we put $f(n) = y$ and find n in term of y. We have done that while proving onto

$$n = \begin{cases} y-1, \text{if } y \text{ is odd} \\ y+1, \text{if } y \text{ is even} \end{cases}$$

$$\therefore \text{ Inverse of } f = \quad g(y) = \begin{cases} y-1, \text{if } y \text{ is odd} \\ y+1, \text{if } y \text{ is even} \end{cases}$$

where $g : W \to W$

Now, replacing y with n

$$g(n) = \begin{cases} n+1, \text{if } n \text{ is even} \\ n-1, \text{if } n \text{ is odd} \end{cases}$$

$$= f(n)$$

AI Q. 5. Let $f : N \to N$ be defined as

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{when } n \text{ is odd} \\ \dfrac{n}{2}, & \text{when } n \text{ is even} \end{cases}$$

for all $n \in N$. State whether the function f is bijective. Justify your answer.

Sol. We have $f : N \to N$

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{when } n \text{ is odd} \\ \dfrac{n}{2}, & \text{when } n \text{ is even} \end{cases} \quad \forall\, n \in N.$$

For one-one

$$f(1) = \frac{1+1}{2} = 1 \text{ and } f(2) = \frac{2}{2} = 1$$

$\therefore \quad f(1) = f(2)$, where $1 \neq 2$.

Therefore, f is not one-one

For onto

Consider a natural number n in co-domain N.

Case I. When n is odd

There exists $n = 2r + 1$ for some $r \in N$

Then, there exists $4r + 1 \in N$ such that

$$f(4r+1) = \frac{4r+1+1}{2} = 2r+1$$

Case II. When n is even

There exists $n = 2r$ for some $r \in N$

Then, there exists $4r \in N$ such that

$$f(4r) = \frac{4r}{2} = 2r$$

Therefore, f is onto.

But since f is not one-one, so it is not bijective (as a bijective function it is necessary to be both one-one as well as onto function).

6 Marks Questions

Q. 1. Let $f : N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : N \to S$, where S is the range of f. Find inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$.

Sol. Given, $f(x) = 9x^2 + 6x - 5$ and $f : N \to S$

For one-one

Let $x_1, x_2 \in N$ such that $f(x_1) = f(x_2)$

Given,

$$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$\Rightarrow \quad 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2) \times \{9(x_1 + x_2) + 6\} = 0$$

$$\Rightarrow \quad (x_1 - x_2) = 0, \text{ as } 9(x_1 + x_2) + 6 \neq 0$$

$$(\because x_1, x_2 \in N)$$

$$\therefore \quad x_1 = x_2$$

Hence, f is one-one

For onto

Now, let $\quad y = 9x^2 + 6x - 5$

$$\therefore \quad 9x^2 + 6x - (5 + y) = 0$$

$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{36 + 4 \times 9(5+y)}}{18}$$

$$= \frac{-6 \pm 6\sqrt{1+5+y}}{18}$$

$$= \frac{-1 \pm \sqrt{6+y}}{3}$$

For all $y \in S$, there exist $x \in N$, such that

$f^{-1}(x)$

$\therefore$ f is onto. Hence, f is invertible and

$$f^{-1}(x) = \frac{-1 + \sqrt{6+x}}{3}$$

$$f^{-1}(43) = \frac{-1 + \sqrt{6+43}}{3} = 2$$

$$\text{and } f^{-1}(163) = \frac{-1 + \sqrt{6+163}}{3} = 4$$

Q. 2. Let $f : N \to R$ be a function defined as $f(x) = x^2 + 4x + 7$. Show that $f : N \to S$, where S is the range of f, and f is invertible. Find the inverse of f.

Sol. Assume, $\quad f(x) = y = x^2 + 4x + 7$

$$= (x+2)^2 + 3$$

$$\Rightarrow \quad y - 3 = (x+2)^2$$

$$\Rightarrow \quad x + 2 = \pm\sqrt{y-3}$$

$$\Rightarrow \quad x = \sqrt{y-3} - 2 \text{ as } x \in N$$

Let us define $g : S \to N$ by

$$g(y) = \sqrt{y-3} - 2$$

Now, $\quad gof(x) = g[f(x)] = g(x^2 + 4x + 7)$

$$= g[(x+2)^2 + 3]$$

$$= \sqrt{(x+2)^2 + 3 - 3} - 2$$

$$= x + 2 - 2 = x$$

OSWAAL BOOKS LEARNING MADE SIMPLE

Now, $fog(y) = f[g(y)]$

$$= f(\sqrt{y-3}-2)$$

$$= (\sqrt{y-3}-2)^2+4(\sqrt{y-3}-2)+7$$

$$= y - 3 + 3 = y$$

$\Rightarrow$ $gof = I_N$

and $= I_S$

$\Rightarrow f$ is invertible and $f^{-1} = g = \sqrt{x-3}-2$

Q. 3. Show that the function $f : R\ \{x \in R : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in R$ is one-one and onto function. Hence find $f^{-1}(x)$.

Sol. For One-One : Let $x_1, x_2 \in R$

Such that $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|}$$

Case (i) : If $x_1, x_2 > 0$ then

$$\frac{x_1}{1+x_1} = \frac{x_2}{1+x_2}$$

$$\Rightarrow \quad x_1 + x_1x_2 = x_2 + x_1x_2$$

$$\Rightarrow \quad x_1 = x_2$$

Case (ii) : If $x_1, x_2 < 0$ then

$$\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$$

$$x_1 - x_1x_2 = x_2 - x_1x_2$$

$$\Rightarrow \quad x_1 = x_2$$

Case (iii) : If $x_1 > 0, x_2 < 0$ (Similar for $x_1 < 0, x_2 > 0$)

$$x_1 \neq x_2$$

$$\Rightarrow \quad \frac{x_1}{1+|x_1|} \neq \frac{x_2}{1-|x_2|}$$

$$\Rightarrow \quad f(x_1) \neq f(x_2)$$

$\therefore$ From cases (i), (ii) & (iii), f is a one-one function.

For onto : Let any $y \in \{x \in R : -1 < x < 1\}$

$(\because -1 < y < 1)$

Such that $y = f(x)$

$$\Rightarrow \quad y = \frac{x}{1+|x|}$$

$$\Rightarrow \quad y = \frac{x}{1\pm x} \Rightarrow x = \frac{y}{1\pm y}$$

As $y \neq \pm 1$

$$\therefore \quad x = \frac{y}{1\pm y} \in R$$

$\therefore f$ is an onto function.

$$f^{-1}(x) = \begin{cases} \dfrac{x}{1+x}, & \text{if } x < 0 \\ \dfrac{x}{1-x}, & \text{if } x \geq 0 \end{cases}$$

Q. 4. Consider $f : R_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$, where R_+ is the set of all non-negative real numbers. Show that f is not invertible. Modify the co-domain of f to make f invertible and then find the inverse of f.

Sol. $f : R^+ \rightarrow [-5, \infty]$ given by $f(x) = 9x^2 + 6x - 5$

To show: f is one-one and onto

Let us assume that f is not one-one.

$\therefore$ There exists 2 as more numbers for which images are same

For x_1, x_2 R$^+$

Let $f(x_1) = f(x_2)$

$$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_1^2 + 6x_1 - 5$$

$$\Rightarrow \quad 9x_1^2 + 6x_2 = 9x_1^2 + 6x_2$$

$$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[9(x_1 - x_2) + 6] = 0$$

Since x_1 and x_2 are positive

$9(x_1 - x_2) + 6 > 0$

$\therefore x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

Therefore, it contradicts are assumption.

Hence the function is one-one

Now, let as prove that f is onto

A function f: $X \rightarrow Y$ is onto it every $y \in Y$, these exists an image in X.

$$f(x) = 9x^2 + 6x - 5$$

$$= 9x^2 + 6x + 1 - 6$$

$$= (3x + 1)^2 - 6$$

Now, for all $x \in R^+$ as $[0, \infty)$, $f(x) \in [-5, \infty)$

$\therefore$ Range = co-domain

Hence, f is onto.

We have proved, that function is one-one & onto. Hence in turn it is proved that f is invertible.

Let $y = f(x)$

or $y = 9x^2 + 6x - 5$

$$(3x + 1)^2 = y + 6$$

$$x = \frac{\sqrt{y+6}-1}{3}$$

$f^{-1} : [-5, \infty) \rightarrow [0, \infty)$, $f^{-1}(y)$

$$= \frac{\sqrt{y+6}-1}{3}$$

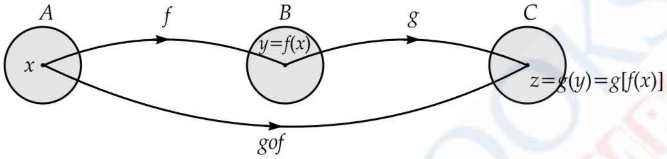
TOPIC-3

Composite Functions

Revision Notes

- Let $f : A \to B$ and $g : B \to C$ be any two functions. Then f maps an element $x \in A$ to an element $f(x) = y \in B$; and this y is mapped by g to an element $z \in C$. Thus, $z = g(y) = g[f(x)]$. Therefore, we have a rule which associates with each $x \in A$, a unique element $z = g[f(x)]$ of C. This rule is therefore a mapping from A to C. We denote this mapping by gof (read as g composition f) and call it the **'composite mapping of f and g'.**
- **Definition :**

  If $f : A \to B$ and $g : B \to C$ be any two mappings (functions), then the composite mapping gof of f and g is defined by $gof : A \to C$ such that $(gof)\ (x) = g(f(x))$ for all $x \in A$



Scan to know more about this topic

Composite functions

- The composite of two functions is also called the resultant of two functions or the function of a function.
- Remember that gof is defined only if co-domain of (f) = domain of (g).
- Observe that the order of events occur from right to left *i.e.*, gof reads composites of f and g and it means that we have to first apply f and then, follow it up with g.
- Domain of (gof) = Domain of (f) and Co-domain of (gof) = co-domain of (g).
- Note that for the composite function gof to exist, it is essential that range of must be a subset of the domain of g.
- If gof is defined then, it is not necessary that fog is also defined.
- If gof is one-one, then f is also one-one function. Similarly, if gof is onto then g is onto function.

1 Mark Questions

Q. 1. If $f(x) = 27x^3$ and $g(x) = x^{1/3}$, then $gof(x)$ is

(a) $3x$ (b) x (c) $3x^2$ (d) $6x$

Sol. Correct option : (a)

***Explanation* :**

Given, $f(x) = 27x^3$ and $g(x) = x^{1/3}$

Now,
$$\begin{aligned} gof(x) &= g[f(x)] = g(27x^3) \\ &= (27x^3)^{1/3} = (27)^{1/3}.(x^3)^{1/3} \\ &= (3^3)^{1/3} \,.\, (x^3)^{1/3} = 3x \end{aligned}$$

$\therefore$ $gof(x) = 3x$

Q. 2. If $f : R \to R$ is defined by $f(x) = (3 - x^3)^{1/3}$, then $fof(x)$ = ...

(a) $3x$ (b) x (c) $3x^2$ (d) $6x$

Sol. Correct option : (b)

Explanation : Given function is $f : R \to R$ such that $f(x) = (3 - x^3)^{1/3}$

Now,
$$\begin{aligned} fof(x) &= f[f(x)] = f[(3 - x^3)x^{1/3}] \\ &= [3 - \{(3 - x^3)^{1/3}\}^3]^{1/3} \\ &= [3 - (3 - x^3)]^{1/3} = (x^3)^{1/3} \\ &= x \end{aligned}$$

Q. 3. If f be the greatest integer function defined as $f(x)$ = [x] and g be the modulus function defined as $g(x) = |x|$, then the value of $gof\left(-\frac{5}{4}\right)$ is ....

(a) 1 (b) 2 (c) 3 (d) 4

Sol. Correct option : (b)

Explanation :

$$\begin{aligned} g\{[f]\} &= gf\{[x]\} \\ &= g\{[x]\} = |(x)| \end{aligned}$$

$$\begin{aligned} gof\left(\frac{-5}{4}\right) &= \left|\left[\frac{-5}{4}\right]\right| \\ &= |-2| = 2 \end{aligned}$$

Q. 4. Let $f : R \to R$ be defined by $f(x) = 3x^2 - 5$ and $g : R \to R$ be defined by $g(x) = \dfrac{x}{x^2+1}$. find gof.

Sol.
$$\begin{aligned} gof(x) &= g\{f(x)\} = g(3x^2 - 5) \\ &= \frac{3x^2 - 5}{(3x^2 - 5)^2 + 1} \\ &= \frac{3x^2 - 5}{9x^4 - 30x^2 + 26} \end{aligned}$$

OSWAAL BOOKS LEARNING MADE SIMPLE

Q. 5. If $f : R \to R$ is defined by $f(x) = 3x + 2$, then define $f\,[f(x)]$.

Sol. Given $f(x) = 3x + 2$

$\therefore$ $f\,[f(x)] = f(3x + 2) = 3(3x + 2) + 2$

$= 9x + 6 + 2 = 9x + 8$

Q. 6. Write fog, if $f : R \to R$ and $g : R \to R$ are given by $f(x) = 8x^3$ and $g(x) = x^{1/3}$.

Sol. Given $f(x) = 8x^3$ and $g(x) = x^{1/3}$

$\therefore fog(x) = f[g(x)] = f(x^{1/3}) = 8(x^{1/3})^3 = 8x$

Q. 7. If $f : R \to R$ and $g : R \to R$ are given by $f(x) = \sin x$ and $g(x) = 5x^2$, then find $gof(x)$

Sol. Given $f(x) = \sin x$ and $g(x) = 5x^2$.

$\therefore gof(x) = g[f(x)] = g(\sin x)$

$= 5(\sin x)^2$ or $5\sin^2 x$

2 Marks Questions

Q 1. Let $f : R \to R$, $f(x) = x^3$ and $g : R \to R$, $g(x) = 2x^2 + 1$ and R is the set of real numbers, then find $fog(x)$ and $gof(x)$.

[ISC, 2019] [ISC Specimen Paper, 2021]

Sol. $fog(x) = f\,[g(x)] = f(2x^2 + 1)$

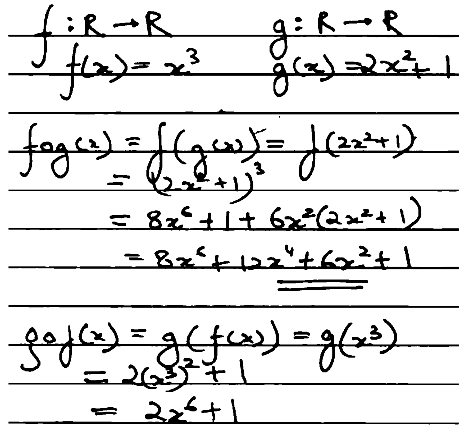
$= (2x^2 + 1)^3$

$gof(x) = g[f(x)] = g(x^3)$

$= 2(x^3)^2 + 1 = 2x^6 + 1$

[ISC Marking Scheme, 2019]

OR



$f : R \to R$ $\quad$ $g : R \to R$

$f(x) = x^3$ $\quad$ $g(x) = 2x^2 + 1$

$fog(x) = f(g(x)) = f(2x^2+1)$

$= (2x^2+1)^3$

$= 8x^6 + 1 + 6x^2(2x^2+1)$

$= 8x^6 + 12x^4 + 6x^2 + 1$

$gof(x) = g(f(x)) = g(x^3)$

$= 2(x^3)^2 + 1$

$= 2x^6 + 1$

[Topper's Answer 2019]

Q. 2. Let $f : \{1, 3, 4\} \to \{1, 2, 5\}$ and $g : \{1, 2, 5\} \to \{1, 3\}$ given by $f = \{(1, 2)\ (3, 5)\ (4, 1)$ and $g = \{(1, 3)\}, (2, 3), (5, 1)\}$ Write down gof.

Sol. $(gof)\,(x) = g(f(x))$ *i.e.*, Domain $= \{1, 3, 4\}$

Put $x = 1 \Rightarrow (gof)\,(1) = g(f(1)) = g(2) = 3$

Put $x = 3 \Rightarrow (gof)\,(3) = g(f(3)) = g(5) = 1$

Put $x = 4 \Rightarrow (gof)\,(4) = g(f(4)) = g(1) = 3$

$\therefore gof = \{(1, 3), (3, 1), (4, 3)\}$

Q. 3. Let f, g be real valued functions defined as $f(x) = x^2 + 1, x \in R$ and $g(x) = 2x + 1, x \in R$. Find the value of fog and gof.

Sol. $f(x) = x^2 + 1,\ g(x) = 2x + 1$

$fog = f\{g(x)\} = f(2x + 1)$

$fog = (2x + 1)^2 + 1$

$= 4x^2 + 4x + 1 + 1$

$= 4x^2 + 4x + 2$

$gof = g(f(x)) = g(x^2 + 1)$

$= 2(x^2 + 1) + 1$

$= 2x^2 + 2 + 1$

$= 2x^2 + 3$

Q. 4. Write $f\,o\,g$, if $f : R \to R$ and $g : R \to R$ are given by $f(x) = |x|$ and $g(x) = |5x - 2|$.

Sol. Given, $f(x) = |x|, g(x) = |5x - 2|$

$\therefore$ $fog(x) = f[g(x)] = f\{|5x - 2|$

$= ||5x - 2|| = |5x - 2|$

$[\because ||x|| = |x|]$

Q. 5. Let $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : (3, 4, 5, 9) \to \{7, 11\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = 5$, $f(5) = 9$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Sol. $gof(2) = g(3) = 7$

$gof(3) = g(4) = 7$

$gof(4) = g(5) = 11$

$gof(5) = g(9) = 11$

$\therefore gof = \{(2, 7), (3, 7), (4, 11), (5, 11)\}$

Q. 6. Let $f : R \to R$ be defined by $f(x) = 3x^2 - 5$ and $g : R \to R$ be defined as $g(x) = 3x$. Find $gof(-2)$.

Sol. $f(x) = 3x^2 - 5$

$g(x) = 3x$

$gof(x) = 3(3x^2 - 5) = 9x^2 - 15$

$gof(-2) = 36 - 15 = 21$

4 Marks Questions

Q 1. If the function

$f : R \to R$ be defined as $f(x) = \dfrac{3x+4}{5x-7}, \left(x \neq \dfrac{7}{5}\right)$ and $g : R \to R$ be defined as $g(x) = \dfrac{7x+4}{5x-3}, \left(x \neq \dfrac{3}{5}\right)$ show that $(gof)(x) = (fog)\,(x)$.

[ISC, 2020]

Sol. $f(x) = \dfrac{3x+4}{5x-7} \qquad \left(x \neq \dfrac{7}{5}\right)$

$g(x) = \dfrac{7x+4}{5x-3} \qquad \left(x \neq \dfrac{3}{5}\right)$

$$(gof)x = g(f(x))$$

$$= \frac{7 \times \frac{3x+4}{5x-7} + 4}{5 \times \frac{3x+4}{5x-7} - 3}$$

$$= \frac{\frac{21x+28+20x-28}{5x-7}}{\frac{15x+20-15x+21}{5x-7}}$$

$$(gof)x = \frac{41x}{41} = x.$$

$$(fog)x = f(g(x))$$

$$= \frac{3 \times \frac{7x+4}{5x-3} + 4}{5 \times \frac{7x+4}{5x-3} - 7}$$

$$= \frac{21x+12+20x-12}{35x+20-35x+21}$$

$$= \frac{41x}{41} = x$$

$$(fog)x = (gof)x$$

$$= x.$$

Hence proved.

[ISC Marking Scheme, 2020]

Examiner's Comment

- Few students are confused with fog and gof. Some of them made mistakes in solving such type of questions.

Answering Tip

- While substituting the values of one function to another, write one by one step to avoid calculation errors.

AI **Q. 2. If the function $f(x) = \sqrt{2x-3}$ is invertible then find its inverse. Hence prove that $(fof^{-1})(x) = x$.**

[ISC, 2018]

Sol. Let $y = f(x) = \sqrt{2x-3}$

$\Rightarrow \quad y^2 = 2x - 3$

$\Rightarrow \quad x = \dfrac{y^2+3}{2}$

$\Rightarrow \quad f^{-1}(y) = \dfrac{y^2+3}{2}$

$\therefore \quad f^{-1}(x) = \dfrac{x^2+3}{2}$

$(fof^{-1})(x) = f[f^{-1}(x)]$

$$= f\left(\frac{x^2+3}{2}\right) = \sqrt{2.\frac{x^2+3}{2} - 3} = x$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- A number of candidates did not read the question carefully and made an attempt to prove that the given function is invertible instead of proving $(fof^{-1})(x) = x$.

Answering Tip

- Understand the concept of composite function.

Q. 3. Let $f, g : R \rightarrow R$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$ for all $x \in R$. Then find fog and $g\,of$.

Sol. $f(x) = |x| + x$ and $g(x) = |x| - x$

$$f(x) = \begin{cases} x + x, & \text{if } x \geq 0 \\ -x + x, & \text{if } x < 0 \end{cases}$$

$$f(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

and

$$g(x) = \begin{cases} x - x, & \text{if } x \geq 0 \\ -x - x, & \text{if } x < 0 \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases}$$

Thus, for $x \geq 0$, $gof(x) = g(2x) = 0$

and for $x < 0$, $gof(x) = g(0) = 0$

$\Rightarrow gof(x) = 0 \ \forall \ x \in R$

Thus, for $x \geq 0$, $fog(x) = f(0) = 2(0) = 0$

and for $x < 0$, $fog(x) = f(-2x) = -4x$

$$\Rightarrow \quad fog(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ -4x, & \text{if } x < 0 \end{cases}$$

Q. 4. If the function $f : R \rightarrow R$ be given by $f(x) = x^2 + 2$ and $g : R \rightarrow R$ be given by $g(x) = \dfrac{x}{x-1}, x \neq 1$, find fog and gof and hence find $fog(2)$ and $gof(-3)$.

Sol. $fog(x) = f\left(\dfrac{x}{x-1}\right) = \left(\dfrac{x}{x-1}\right)^2 + 2$

$$fog(2) = \left(\frac{2}{2-1}\right)^2 + 2 = 4 + 2 = 6$$

$$gof(x) = g(x^2+2) = \frac{x^2+2}{x^2+1}$$

$$gof(-3) = \frac{(-3)^2+2}{(-3)^2+1} = \frac{9+2}{9+1} = \frac{11}{10}$$

Q. 5. If $f(x) = \dfrac{4x+3}{6x-4}, x \neq \dfrac{2}{3}$, then show that $fof(x) = x$ for all $x \neq \dfrac{2}{3}$. What is the inverse of f?

Sol. $fof(x) = f[f(x)]$

$$= f\left(\frac{4x+3}{6x-4}\right)$$

OSWAAL BOOKS

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} = \frac{\left(\frac{16x+12}{6x-4}\right)+3}{\left(\frac{24x+18}{6x-4}\right)-4}$$

$$= \frac{\left(\frac{16x+12+3(6x-4)}{6x-4}\right)}{\left(\frac{24x+18-4(6x-4)}{6x-4}\right)}$$

$$= \frac{\left(\frac{16x+12+18x-12)}{6x-4}\right)}{\left(\frac{24x+18-24x+16}{6x-4}\right)} = \frac{34x}{34}$$

$fof\,(x) = x$

$$\therefore\ f^{-1}(x) = f(x) = \frac{4x+3}{6x-4},\ x \neq \frac{2}{3}$$

6 Marks Questions

AI **Q. 1. Let $f : N \to R$ be a function defined as $f(x) = 4x^2+12x+15$. Then show that $f : N \to S$, where S is range of f, is invertible. Also find the inverse of f, fof^{-1} and $f^{-1}of$.**

Sol. Let y be an arbitrary element *of* range f, then $y = 4x^2 + 12x + 15$, for some x in N, which implies that $y - 15 = 4x^2 + 12x$

$$\Rightarrow \quad 4(x^2+3x) = y-15$$

$$\Rightarrow \quad 4\left[\left(x+\frac{3}{2}\right)^2 - \frac{9}{4}\right] = y-15$$

$$\Rightarrow \quad \left[\left(x+\frac{3}{2}\right)^2 - \frac{9}{4}\right] = \frac{y-15}{4}$$

$$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2 = \frac{y-15}{4}+\frac{9}{4}$$

$$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2 = \frac{y-15+9}{4}$$

$$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2 = \frac{y-6}{4}$$

$$\Rightarrow \quad x+\frac{3}{2} = \pm\frac{\sqrt{y-6}}{2}$$

$$\Rightarrow \quad x = \frac{-3+\sqrt{y-6}}{2} = \frac{-3-\sqrt{y-6}}{2}$$

But $x \in N$ $\Rightarrow$ $$x = \frac{-3+\sqrt{y-6}}{2}$$

$$\therefore \quad f^{-1}(x) = \frac{-3+\sqrt{x-6x}}{2} \in S = \text{range of } f$$

$$= f\left(\frac{-3+\sqrt{x-6}}{2}\right)$$

$$= 4\left(\frac{-3+\sqrt{x-6}}{2}\right)^2 + 12\left(\frac{-3+\sqrt{x-6}}{2}\right) + 15$$

$$= 4\left(\frac{9+x-6-6\sqrt{x-6}}{4}\right) + 6(-3+\sqrt{x-6}) + 15$$

$$= 3+x-6\sqrt{x-6}-18+6\sqrt{x-6}+15 = x$$

$$= x$$

$$= g\,[f(x)]$$

$$= g(4x^2+12x+15)$$

$$= \frac{-3+\sqrt{4x^2+12x+15-6}}{2}$$

$$= \frac{-3+\sqrt{4x^2+12x+9}}{2}$$

$$= \frac{-3+\sqrt{(2x+3)^2}}{2}$$

$$= \frac{-3+2x+3}{2} = \frac{2x}{2} = x$$

$$\Rightarrow \quad gof(x) = x$$

$$\therefore \quad fog\ (x) = gof\,(x) = x$$

Q. 2. If $f(x) = \sqrt{x}$, $(x \geq 0)$ and $g(x) = x^2 - 1$ are two real functions, then find *fog* and *gof* and check whether *fog = gof*.

Sol. The given functions are

$$f(x) = \sqrt{x},\ x \geq 0 \text{ and } g(x) = x^2 - 1$$

We have,

Domain of $f = [0, \infty)$ and range of $f = [0, \infty)$

Domain of $g = R$ and range of $g = [-1, \infty)$

$$\begin{bmatrix} \because\ x^2 \geq 0 \text{ for all } x \in R \\ \therefore\ x^2 - 1 \geq -1 \text{ for all } x \in R \end{bmatrix}$$

Computation of *gof* : We observe that range of

$f = [0, \infty) \subseteq$ Domain of g

$\therefore$ *gof* exists and

Also, $\quad gof\,(x) = g[f(x)]$

$$= g(\sqrt{x}) = (\sqrt{x})^2 - 1$$

$$= x - 1$$

Thus, $gof : (0\ \infty) \to R$ is defined as $gof\,(x) = x - 1$

Computation of *fog* : We observe that range of $g = (-1, \infty)$ Domain of f

$\therefore$ Domain of fog = $[x : x \in$ Domain of g and $g(x) \in$ Domain of $f]$,

$$= [x : x \in R \text{ and } g(x) \in [0, \infty)]$$

$$= [x : x \in R \text{ and } x^2 - 1 \in [0, \infty)]$$

$= [x : x \in R \text{ and } x^2 - 1 \geq 0]$

$= [x : x \in R \text{ and } x \leq -1 \text{ or } x \geq 1]$

$\therefore$ Domain of *fog* $= x : x \in (-\infty - 1] \cup [1, \infty)$

Also, $fog(x) = f[g(x)] = f(x^2 - 1) = \sqrt{x^2 - 1}$

Thus, $fog : (-\infty, -1] \cup [1, \infty) \to R$ is defined as $fog(x) = \sqrt{x^2 - 1}$.

We find that *fog* and *gof* have distinct domains. Also, their expressions are not same.

Hence, $fog \neq gof$.

Q. 3. If the function $f : R \to R$ be defined by $f(x) = 2x - 3$ and $g : R \to R$ by $g(x) = x^3 + 5$, then find *fog* and show that *fog* is invertible. Also, find $(fog)^{-1}$, hence find $(fog)^{-1}(9)$.

Sol. $f\,o\,g : R \to R$ defined by $f\,o\,g(x) = f(g(x)) = f(x^3 + 5) = 2(x^3 + 5) - 3 = 2x^3 + 7$

Let $x_1, x_2 \in R(D_{f\,o\,g})$ such that $f\,o\,g(x_1) = fog(x_2)$ or $2x_1^3 + 7 = 2x_2^3 + 7$ or $x_1^3 = x_2^3$ or $x_1 = x_2$.

Hence, $f\,o\,g$ is one-one.

Let $y \in R(Codomain_{f\,o\,g})$.

Then for any x, $fog(x) = y$ if $2x^3 + 7 = y$, *i.e.*, if, $2x^3 = y - 7$, *i.e.*, $x = \sqrt[3]{\frac{y-7}{2}}$, which $\in R(D_{fog})$.

fog is onto.

Since, *fog* is both one-one and onto, it is invertible.

$(fog)^{-1} : R \to R$ defined by $(fog)^{-1}(x) = \sqrt[3]{\frac{x-7}{2}}$

and $(fog)^{-1}(9) = \sqrt[3]{\frac{9-7}{2}} = 1$

Examiner's Comment

- Generally students do not read the question carefully and only prove the given function is invertible but skip to find $(fog)^{-1}$ (9).

Answering Tip

- Read the questions carefully and all the instructions given in the question paper before attempting the questions.

OSWAAL BOOKS LEARNING MADE SIMPLE

TOPIC-4
Binary Operations

Revision Notes

Scan to know more about this topic

Binary operations

- An operation is a process which gives a new element from two given elements; *e.g.*, addition, subtraction, multiplication and division of numbers. If the new element belongs to the same set to which the two given elements belong, the operation is called a binary operation.

Definition :

A binary operation (or binary composition) $*$ on a non-empty set A is a mapping which associates with each ordered pair $(a, b) \in A \times A$, a unique element of A. This unique element is denoted by ab.

Thus, a binary operation * on A is a mapping$*: A \times A \to A$ defined by $*\,(a, b) = a * a * b$.

Clearly, $a \in A, b \in A \Rightarrow b \in A$.

- **Terms related to binary operations :**
  - A **binary operation** $*$ on a set A is a function $*$ from $A \times A$ to A.
  - An operation $*$ on A is **commutative** if $a * b = b * a \;\forall\; a, b \in A$.
  - An operation $*$ on A is **associative** if $(a * b) * c = a * (b * c) \;\forall\; a, b, c \in A$.
  - An operation $*$ on A is **left distributive** if $a*(b \otimes c) = (a*b) \otimes (a*c), \forall a, b, c \in A$. Also, an operation $*$ on A is **right distributive** if $(b \otimes c) * a = (b * a) \otimes (c * a), \forall a, b, c \in A$.
  - An **identity element** of a binary operation $*$ on A is an element $e \in A$ such that $e * a = a * e = a \;\; \forall a \in A$. Identity element for a binary operation, if it exists is unique.
- **Inverse of an element :**

Let $*$ be a binary operation on a non-empty set A and e be the identity element for the binary operation $*$. An element $b \in A$ is called the inverse element or simply inverse of a for binary operation if $a * b = b * a = e$.

An element $a \in A$ is invertible if an only if, its inverse exists.

1 Mark Questions

Q. 1. Let binary operation '*' be defined on set A. The operation will be commutative if ....

(a) $a * b = b * a$

(b) $(a * b) * c = a * (b * c)$

(c) (b o c) * a = $(b * a)$ o $(c * a)$

(d) $a * b = a$

Sol. Correct option: (a)

Explanation: A binary operation '*' defined on set A is said to be commutative only if $a * b = b * a$, $\forall a, b \in A$.

If $(a * b) * c = a * (b * c)$, then the operation is said to associative $\forall\ a, c \in A$.

If $(b \text{ o } c) * a = (b * a) \text{ o } (c * a)$, then the operation is said to be distributive $\forall\ a, b, c \in A$.

AI **Q. 2. Let '*' be binary operation on N defined by $a * b = a - b + ab^2$, then find 4 * 5.**

(a) 9 (b) 88 (c) 98 (d) 99

Sol. Correct option: (d)

Explanation: The binary operation is defined by $a*b = a - b + ab^2$.

$\therefore\ 4 * 5 = 4 - 5 + 4(5^2) = -1 + 100 = 99.$

3. Let '*' be defined on the set N. Which of the following are both commutative and associative?

(a) $a * b = a + b$ (b) $a * b = a - b$

(c) $a * b = ab^2$ (d) $a * b = ab$

Sol. Correct option: (a)

Explanation: The binary operation '*' is both commutative and associative for $a * b = a + b$.

The operation is commutative on $a * b = a + b$ because $a + b = b + a$.

The operation is associative on $a * b = a + b$ because $(a + b) + c = a + (b + c)$.

4. Let '*' be a binary operation defined by $a * b = 4ab$. Find $(a * b) * a$.

(a) $4a^2 b$ (b) $16a^2 b$

(c) $16ab^2$ (d) $4ab^2$

Sol. Correct option: (b)

Explanation: Given that, $a*b=4ab$.

Then, $(a * b) * a = (4ab) * a = 4(4ab)(a) = 16a^2 b$.

Q 5 Let * be a binary operation by $a * b = \sqrt{a^2 + b^2}$. Find 3 * 4. [ISC Specimen Paper, 2021]

Sol. Given $a * b = \sqrt{a^2 + b^2}$

$$\therefore \quad 3 * 4 = \sqrt{3^2 + 4^2} = \pm 5$$

AI **Q. 6. State the reason for the following Binary operation*, defined on the set Z of integers, to be not commutative. $a * b = ab^3$.**

Sol. We have $1, 2 \in Z$ such that

$$1 * 2 = 8$$

and $$2 * 1 = 2$$

This implies that

$$1 * 2 \neq 2 * 1$$

Hence * is not commutative.

Q. 7. Let * : $R \times R \to R$, given by $(a, b) \to a + 4b^2$ is a binary operation. Compute $(-5) * (2 * 0)$

Sol. Given, $a * b = a + 4b^2$

$$\begin{aligned}(-5) * (2 * 0) &= -5 * [2 + 4(0)^2] \\ &= -5 * 2 \\ &= -5 + 4(2)^2 \\ &= -5 + 16 \\ \Rightarrow \quad (-5) * (2 * 0) &= 11.\end{aligned}$$

Q. 8. If the binary operation * on the set of integers Z is defined by $a * b = a + 3b^2$, then find the value of 8 * 3.

Sol. Given, $a * b = a + 3b^2$

$$\begin{aligned}(8 * 3) &= 8 + 3(3)^2 \\ &= 8 + 27 = 35 \\ \therefore \quad (8 * 3) &= 35\end{aligned}$$

Q. 9. Let * be a binary operation on N given by $a * b$ = LCM(a, b) for all $a, b \in N$. Find 5 * 7.

Sol. Given, $a * b = \text{LCM}(a, b)$

$$(5 * 7) = \text{LCM}(5, 7) = 35$$

Q. 10. Consider the binary operation * on the set {1, 2, 3, 4, 5} defined by $a * b$ = min $\{a, b\}$. Write the operation table of the operation *.

Sol.

$a*b$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

2 Marks Questions

Q. 1. Determine whether the binary operation * on R defined by $a * b = |a - b|$ is commutative. Also, find the value of (–3) * 2.

Sol.

$$\begin{aligned}a * b &= |a - b| \\ b * a &= |b - a| \\ &= |-1|\ |a - b| \\ a * b &= b * a \\ &= |a - b|\end{aligned}$$

$\therefore$ Operation * is commutative

$$\begin{aligned}(-3) * 2 &= |-3 - (+2)| \\ &= |-5| = 5\end{aligned}$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Many candidates used conventional multiplication operation instead of the operation as defined in the question.

Answering Tip

- Do adequate practice for the comprehension of well-defined mathematical operations.

AI **2. The binary operation $*: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined as $a * b = 2a + b$ find $(2 * 3) * 4$.** **[ISC, 2018]**

Sol. $a * b = 2a + b$

$\therefore$ $2 * 3 = 2.2 + 3 = 7$

Now, $(2 * 3) * 4 = 7 * 4$

$= 2.7 + 4 = 18$

(ISC Marking Scheme, 2018)

Q. 3. A binary operation $*$ defined on Q – {1} is given by $a * b = a + b - ab$. Find the identity element. **[ISC Specimen Paper, 2018]**

Sol. Given, $a * b = a + b - ab$

Let $e \in$ Q – {1} be the identity element.

Statement $a * e = e * a = a$

$\therefore$ $a * e = a + e - ae = a$ ...(i)

and $e * a = e + a - ea = a$ ...(ii)

From eq (i) $e\,(1 - a) = 0$ and From equations (ii) $e\,(1 - a) = 0$

$\Rightarrow$ $e = 0$ as $a \neq 1$

Hence, 0 is the identity element.

Q. 4. Let $*$ be a binary operation, on the set of all non-zero real number, given by $a * b = \dfrac{ab}{5}$, for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.

Sol. Given, $a * b = \dfrac{ab}{5}$, for all $a, b \in R - \{0\}$

$\Rightarrow$ $2 * (x * 5) = 10$

$\Rightarrow$ $2 * \left(\dfrac{5x}{5}\right) = 10$

$\Rightarrow$ $2 * x = 10$

$\Rightarrow$ $\dfrac{2x}{5} = 10$

$\Rightarrow$ $x = \dfrac{50}{2}$

$\Rightarrow$ $x = 25$

AI **Q. 5. If the binary operation $*$ on the set Z of integers is defined by $a * b = a + b - 5$, then write the identity element for the operation $*$ in Z.**

Sol. Let $e \in Z$, $a * e = e * a = a$

Now, $a * e = a$

$\Rightarrow$ $a + e - 5 = a$

$\Rightarrow$ $e - 5 = 0$

$\Rightarrow$ $e = 5$

Also, $e * a = a$

$\Rightarrow$ $e + a - 5 = a$

$\Rightarrow$ $e - 5 = 0$

$\Rightarrow$ $e = 5$

$\therefore$ The identity element of Z under binary operation = 5.

4 Marks Questions

Q. 1. Let S be the set of all rational numbers except 1 and $*$ be defined on S by $a * b = a + b - ab$, for all $a, b \in S$. Prove that (i) $*$ is a binary operation on S, (ii) $*$ is commutative as well as associative.

Sol. **(i)** Sum, difference and product of rational numbers is a unique rational number.

$\therefore$ For each $(a, b) \in S \times S$, there exists a unique image $(a + b - ab)$ in S.

$\Rightarrow$ $*$ is a function $\Rightarrow$ $*$ is a binary operation on S.

(ii) $a * b = a + b - ab = b + a - ba$

$= b * a$

$\therefore$ $*$ is commutative

$a * (b * c) = a * (b + c - bc)$

$= a + (b + c - bc) - a(b + c - bc)$

$= a + b + c - bc - ab - ac + abc$ ...(i)

$(a * b) * c = (a + b - ab) * c$

$= (a + b - ab) + c - (a + b - ab)c$

$= a + b - ab + c - ac - bc + abc$ ...(ii)

From (i) and (ii),

$a * (b * c) = (a * b) * c$

$\therefore$ $*$ is associative.

AI **Q. 2. Let $A = R \times R$ and $*$ be the binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that $*$ is commutative and associative. Find the identity element for $*$ on A.**

Sol. Given,

$(a, b) * (c, d) = (a+c, b+d)$

$= (c + a, d+b)$

(as addition of real numbers is commutative)

$= (c, d) * (a, b)$

$\therefore$ $*$ is commutative.

Now, take

$[(a, b) * (c, d)] * (e, f) = (a+c, b+d) * (e, f)$

$= (a + c + e, b + d + f)$

$= (a, b) * (c + e, d + f)$

$= (a, b) * [(c, d) * (e, f)]$

$\therefore$ $*$ is associative.

Let (e, e') be the identity element.

$(a, b) * (e, e') = (a, b)$

$(a + e, b + e') = (a, b)$

$e = 0, e' = 0$

$\Rightarrow$ Identity element is $*(0, 0)$

Q. 3. Determine whether the operation $*$ defined below on Q is binary operation or not.

$a * b = ab + 1$

If yes, check the commutative and the associative properties.

Sol. Given, $*$ on Q, defined by $a * b = ab + 1$

Let, $a \in Q, b \in Q$,

then $ab \in Q$

and $(ab + 1) \in Q$

$\Rightarrow \quad a * b = ab + 1$ is defined on Q

$\therefore$ $*$ is *a* binary operation on Q.

Commutative :

$$a * b = ab + 1$$
$$= ba + 1 \qquad (\because ba = ab \text{ in } Q)$$
$$\Rightarrow \quad a * b = b * a$$

So, $*$ is commutative on Q.

Associative :

$$(a * b) * c = (ab + 1) * c = (ab + 1)c + 1$$
$$= abc + c + 1$$
$$a * (b * c) = a * (bc + 1)$$
$$= a(bc + 1) + 1$$
$$= abc + a + 1$$

So $*$ is not associative on Q.

6 Marks Questions

Q. 1. Show that the binary operation * on $A = R-\{-1\}$ defined as $a * b = a + b + ab$ for all $a, b, c \in A$ is commutative and associative on A. Also, find the identity element of * on A is invertible.

Sol. Let $a, b \in A$, $a * b = a + b + ab$

Commutativity : $\forall\ a, b \in A$

$$a * b = a + b + ab$$
$$= b + a + ba$$
$$= b * a$$

Associativity: *Let* $a, b, c \in A$

Now, $(a * b) * c = (a + b + ab) * c$

$$=(a + b + ab) + c + (a + b + ab)\,c$$
$$(a * b) * c = a + b + c + ab + bc + ac + abc$$

Again, $a * (b * c) = a * (b + c + bc)$

$$= a + (b + c + bc) + a\,(b + c + bc)$$
$$= a + b + c + ab + bc + ac + abc$$

Clearly,

$$(a * b) * c = a * (b * c)\ \forall\ a, b, c \in A$$

$\therefore$ * is associative

Identity : Let $e \in A$ such that

$$a * e = e * a = a$$
$$a + e + ae = e + a + ea = a$$
$$e = 0$$

$\therefore$ Identity element of A is $e = 0$

Inverse : Let $b \in A$ such that

$a * b = b * a = 0$

$\Rightarrow \quad a + b + ab = 0$ and $b + a + ba = 0$

$$\Rightarrow \quad a = -b - ab$$
$$\Rightarrow \quad a = -b(1 + a)$$
$$\Rightarrow \quad b = \frac{-a}{1+a} \qquad a \in A \therefore a \neq -1$$

$\therefore$ Invertible element of A is $\dfrac{-a}{1+a}$ for all $a \in A$.

Q. 2. Let $*$ be a binary operation defined on $Q \times Q$ by $(a, b) * (c, d) = (ac, b + ad)$, where Q is the set of rational numbers. Determine, whether $*$ is commutative and associative. Find the identity element for $*$ and the invertible elements of $Q \times Q$.

Sol. Let $(a, b), (c, d) \in Q \times Q$. Then $b + ad$ may not be equal to $d + \dfrac{cb}{bc}$. We find that $(1,2) * (2,3) = (2,5)$, $(2, 3) * (1, 2) = (2, 7) \neq (2, 5)$. Hence, $*$ is not commutative.

Let, $(a, b), (c, d), (e, f) \in Q \times Q$,

Now,

$$\{(a, b) * (c, d)\} * (e, f) = (ac, b + ad) * (e, f)$$
$$= (ace, b + ad + acf) \qquad \text{...(i)}$$

and

$$(a, b) * \{(c, d) * (e, f)\} = (a, b) * (ce, d + cf)$$
$$= (ace, b + ad + acf) \qquad \text{...(ii)}$$

From eqs. (i) and (ii), we get

$$\{(a, b) * (c, d)\} * (e, f) = (a, b) * \{(c, d) * (e, f)\}$$

Hence, * is associative.

Let (x, y) be the identity element $Q \times Q$

Thus, $(a, b) * (x, y) = (x, y) * (a, b) = (a, b)$ for all $(a, b) \in Q \times Q)$

$$(a, b) * (x, y) = (a, b)$$
$$\Rightarrow \quad (ax, b + ay) = (a, b)$$
$$\Rightarrow \quad ax = a \text{ and } b + ay = b$$
$$\Rightarrow \quad x = 1 \text{ and } y = 0$$

Therefore, $(1, 0) \in Q \times Q$ is the identity element of $Q \times Q$ with respect to the operation $*$.

Let (p, q) is the inverse element of $(a, b) \in Q \times Q$.

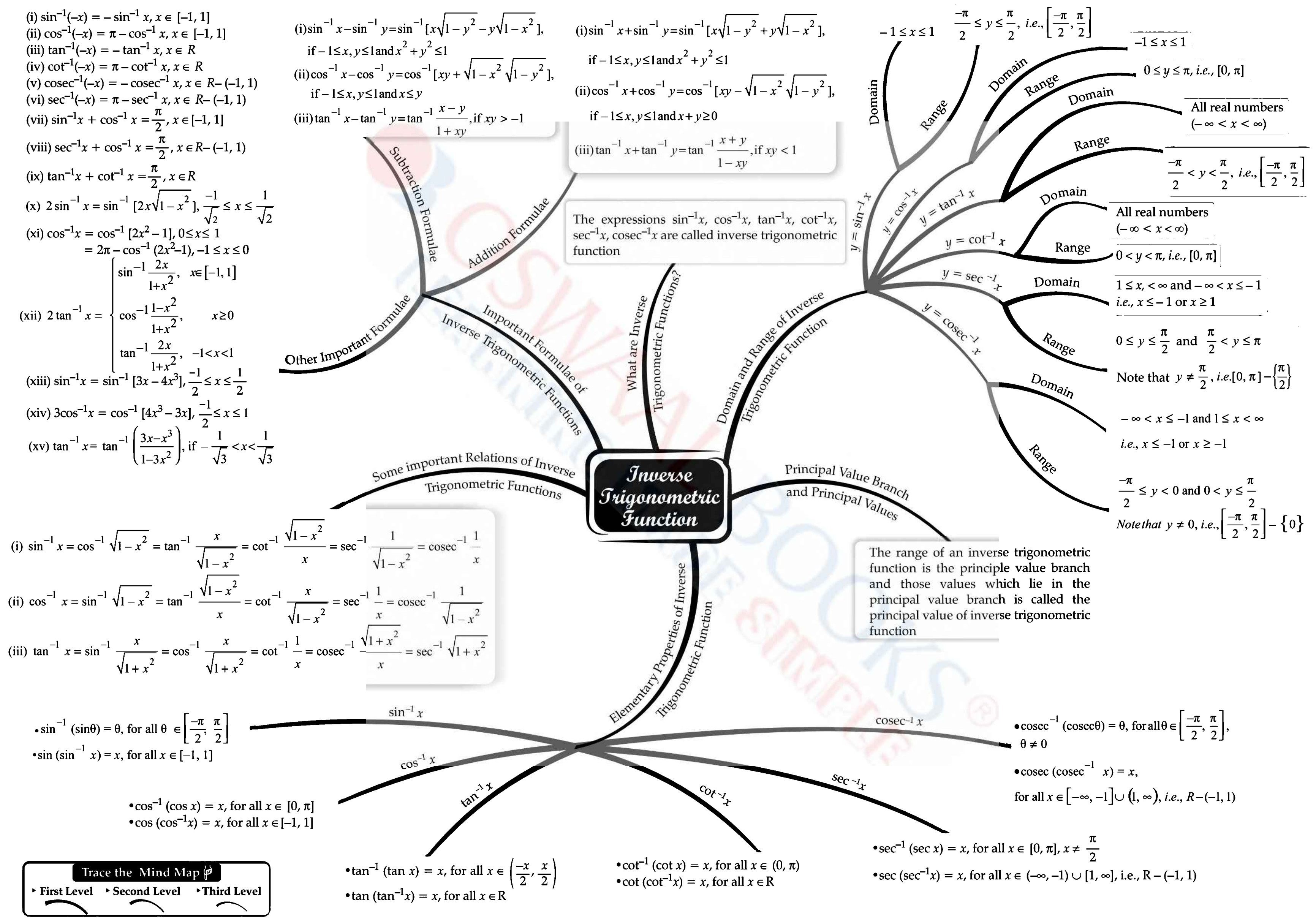
Thus,

$$(a, b) * (p, q) = (1, 0)$$

i.e., $\quad (ap, b + aq) = (1, 0)$

$$\Rightarrow \quad ap = 1 \text{ and } b + aq = 0$$
$$\Rightarrow \quad p = \frac{1}{a} \text{ and } q = \frac{-b}{a}$$

Therefore, the inverse of $(a, b) \in Q \times Q$ is $\left(\dfrac{1}{a}, \dfrac{-b}{a}\right)$, $a \neq 0$.



Inverse Trigonometric Function

What are Inverse Trigonometric Functions?

The expressions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$ are called inverse trigonometric function

Domain and Range of Inverse Trigonometric Function

$y = \sin^{-1} x$
- Domain: $-1 \le x \le 1$
- Range: $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, *i.e.*, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$y = \cos^{-1} x$
- Domain: $-1 \le x \le 1$
- Range: $0 \le y \le \pi$, *i.e.*, $[0, \pi]$

$y = \tan^{-1} x$
- Domain: All real numbers ($-\infty < x < \infty$)
- Range: $\frac{-\pi}{2} < y < \frac{\pi}{2}$, *i.e.*, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$y = \cot^{-1} x$
- Domain: All real numbers ($-\infty < x < \infty$)
- Range: $0 < y < \pi$, *i.e.*, $[0, \pi]$

$y = \sec^{-1} x$
- Domain: $1 \le x, < \infty$ and $-\infty < x \le -1$ *i.e.*, $x \le -1$ or $x \ge 1$
- Range: $0 \le y \le \frac{\pi}{2}$ and $\frac{\pi}{2} < y \le \pi$ Note that $y \ne \frac{\pi}{2}$, *i.e.*$[0, \pi] - \left\{\frac{\pi}{2}\right\}$

$y = \text{cosec}^{-1} x$
- Domain: $-\infty < x \le -1$ and $1 \le x < \infty$ *i.e.*, $x \le -1$ or $x \ge -1$
- Range: $\frac{-\pi}{2} \le y < 0$ and $0 < y \le \frac{\pi}{2}$ *Note that* $y \ne 0$, *i.e.*, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Principal Value Branch and Principal Values

The range of an inverse trigonometric function is the principle value branch and those values which lie in the principal value branch is called the principal value of inverse trigonometric function

Elementary Properties of Inverse Trigonometric Function

$\sin^{-1} x$
- $\sin^{-1}(\sin\theta) = \theta$, for all $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
- $\sin(\sin^{-1} x) = x$, for all $x \in [-1, 1]$

$\cos^{-1} x$
- $\cos^{-1}(\cos x) = x$, for all $x \in [0, \pi]$
- $\cos(\cos^{-1}x) = x$, for all $x \in [-1, 1]$

$\tan^{-1} x$
- $\tan^{-1}(\tan x) = x$, for all $x \in \left(\frac{-x}{2}, \frac{x}{2}\right)$
- $\tan(\tan^{-1}x) = x$, for all $x \in R$

$\cot^{-1} x$
- $\cot^{-1}(\cot x) = x$, for all $x \in (0, \pi)$
- $\cot(\cot^{-1}x) = x$, for all $x \in R$

$\sec^{-1} x$
- $\sec^{-1}(\sec x) = x$, for all $x \in [0, \pi]$, $x \ne \frac{\pi}{2}$
- $\sec(\sec^{-1}x) = x$, for all $x \in (-\infty, -1) \cup [1, \infty]$, i.e., $R - (-1, 1)$

$\text{cosec}^{-1} x$
- $\text{cosec}^{-1}(\text{cosec}\theta) = \theta$, for all $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\theta \ne 0$
- $\text{cosec}(\text{cosec}^{-1} x) = x$, for all $x \in [-\infty, -1] \cup (1, \infty)$, *i.e.*, $R - (-1, 1)$

Some important Relations of Inverse Trigonometric Functions

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \text{cosec}^{-1}\frac{1}{x}$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x} = \text{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}$

(iii) $\tan^{-1} x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{x}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \text{cosec}^{-1}\frac{\sqrt{1+x^2}}{x} = \sec^{-1}\sqrt{1+x^2}$

Important Formulae of Inverse Trigonometric Functions

Addition Formulae

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$,
if $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$,
if $-1 \le x, y \le 1$ and $x + y \ge 0$

(iii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, if $xy < 1$

Subtraction Formulae

(i) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$,
if $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$

(ii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}[xy + \sqrt{1-x^2}\sqrt{1-y^2}]$,
if $-1 \le x, y \le 1$ and $x \le y$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$, if $xy > -1$

Other Important Formulae

(i) $\sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]$
(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1, 1]$
(iii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$
(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$
(v) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x, x \in R - (-1, 1)$
(vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x, x \in R - (-1, 1)$
(vii) $\sin^{-1}x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1, 1]$
(viii) $\sec^{-1}x + \cos^{-1} x = \frac{\pi}{2}, x \in R - (-1, 1)$
(ix) $\tan^{-1}x + \cot^{-1} x = \frac{\pi}{2}, x \in R$
(x) $2\sin^{-1} x = \sin^{-1}[2x\sqrt{1-x^2}], \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$
(xi) $\cos^{-1}x = \cos^{-1}[2x^2 - 1], 0 \le x \le 1$
$= 2\pi - \cos^{-1}(2x^2 - 1), -1 \le x \le 0$

(xii) $2\tan^{-1} x = \begin{cases} \sin^{-1}\frac{2x}{1+x^2}, & x \in [-1, 1] \\ \cos^{-1}\frac{1-x^2}{1+x^2}, & x \ge 0 \\ \tan^{-1}\frac{2x}{1+x^2}, & -1 < x < 1 \end{cases}$

(xiii) $\sin^{-1}x = \sin^{-1}[3x - 4x^3], \frac{-1}{2} \le x \le \frac{1}{2}$
(xiv) $3\cos^{-1}x = \cos^{-1}[4x^3 - 3x], \frac{-1}{2} \le x \le 1$
(xv) $\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Trace the Mind Map
- First Level
- Second Level
- Third Level

CHAPTER

2 INVERSE TRIGONOMETRIC FUNCTIONS

SYLLABUS

- ***Definition, domain, range, principal value branch.***
- ***Graphs of inverse trigonometric functions.***
- ***Elementary properties of inverse trigonometric functions.***
- ***Principal Values***
- **$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ *etc. and their graphs.***
- $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$
- $\sin^{-1}x = \operatorname{cosec}^{-1}\dfrac{1}{x}$; $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ ***and similar relations for*** $\cot^{-1}x$, $\tan^{-1}x$, ***etc.***

  $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$

  $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-y^2}\sqrt{1-x^2}\right)$

  Similarly, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}, xy < 1$

  $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}, xy > -1$
- ***Formulae for*** $2\sin^{-1}x$, $2\cos^{-1}x$, $2\tan^{-1}x$, $3\tan^{-1}x$ ***etc. and application of these formulae.***

Trend Analysis

List of Concept	2016	2017	2018	2019	2020
Inverse trigonometric functions	1 Q (4 marks) 1 Q (6 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (2 marks) 1 Q (4 marks)

Revision Notes

- In mathematics, the inverse trigonometric functions are the inverse function of trigonometric functions. Specifically, they are inverse of the sine, cosine, tangent, cotangent, secant and cosecant functions, and are used to obtain an angle from any of the angle's trigonometric ratios.
- **Domains and Ranges of Inverse Trigonometric Functions :**

Inverse Trigonometric Functions *i.e.*, $f(x)$	Domain/Value of x	Range/Value of $f(x)$
$\sin^{-1}x$	$[-1, 1]$	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\operatorname{cosec}^{-1}x$	$R - (-1, 1)$ $(-\infty, -1) \cup [1, +\infty)$	$\left(-\pi, -\dfrac{\pi}{2}\right] \cup \left(0, \dfrac{\pi}{2}\right]$
$\sec^{-1}x$	$R - (-1, 1)$ $(-\infty, -1] \cup [1, +\infty)$	$\left[0, \dfrac{\pi}{2}\right) \cup \left[\pi, \dfrac{3\pi}{2}\right)$

$\tan^{-1}x$	$R\,(-\infty, +\infty)$	$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
$\cot^{-1}x$	$R\,(-\infty, +\infty)$	$(0, \pi)$

Note

- The symbol $\sin^{-1}x$ is used to denote the smallest angle whether positive or negative, such that the sine of this angle will give us x. Similarly, $\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$ and $\cot^{-1} x$ are defined.
- You should note that $\sin^{-1}x$ can be written as arcsin x. Similarly, other inverse trigonometric functions can also be written as arccos x, arctan x, arcsec x etc.
- Also, note that $\sin^{-1}x$ (and similarly other inverse trigonometric functions) is entirely different from $(\sin x)^{-1}$. In fact, $\sin^{-1}x$ is measure of an angle in Radians whose sine is x whereas $(\sin x)^{-1}$ is $\dfrac{1}{\sin x}$ (which is obvious as per the laws of exponents).
- Keep in mind that these inverse trigonometric relations are true only in their domains, *i.e.*, they are valid only for some values of 'x' for which inverse trigonometric functions are well defined.

OSWAAL BOOKS LEARNING MADE SIMPLE

- **Principal Value :**

Numerically smallest angle is known as the principal value.

For finding the principal value, following algorithm can be followed.

Step 1 : First, draw a trigonometric circle and mark the quadrant in which the angle may be lie.

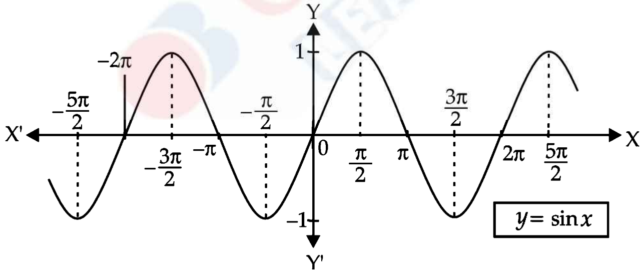
Step 2 : Select anti-clockwise direction for 1st and 2nd quadrant in which the angle may be lie.

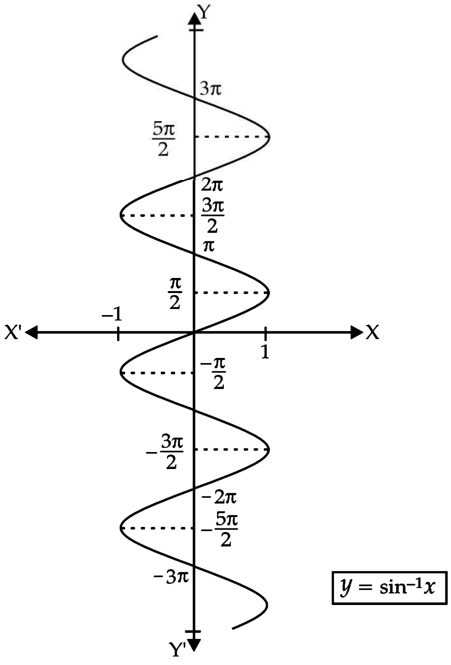
Step 3 : Find the angles in the first rotation.

Step 4 : Select the numerically least (magnitude wise) angle among these two values. The angle thus found will be the principal value.

Step 5 : In case, two angles one with positive sign and the other with the negative sign qualify for the numerically least angle then, it is the convention to select the angle with positive sign as principal value.

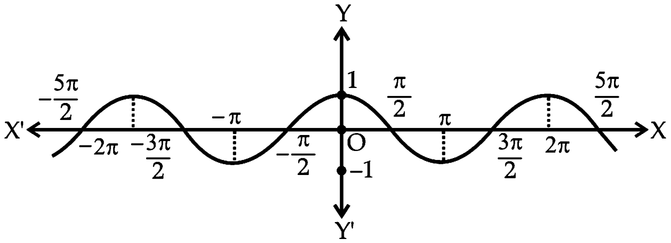
- **Principal value of branch function of $\sin^{-1}$:** It is a function with domain $[-1, 1]$ and range $\left[\dfrac{-3\pi}{2}, \dfrac{-\pi}{2}\right], \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ or $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$ and so on corresponding to each interval, we get a branch of the function $\sin^{-1}x$. The branch with range $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ is called the principal value branch. Thus, $\sin^{-1}[-1,1] \rightarrow \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$.



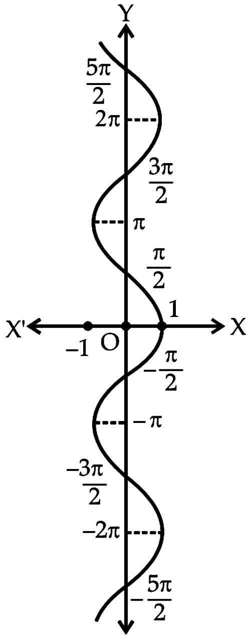


- **Principal value of branch function of $\cos^{-1}$:** The graph of the function $\cos^{-1}$ is as shown in figure. Domain of the function $\cos^{-1}$ is $[-1, 1]$. Its range in one of the intervals $[-\pi, 0]$, $[0, \pi]$, $[\pi, 2\pi]$, etc. is one-one and onto with the range $[-1, 1]$. The branch with range $(0, \pi)$ is called the principal value branch of the function $\cos^{-1}$.

  Thus, $\cos^{-1} [-1, 1] \to [0, \pi]$



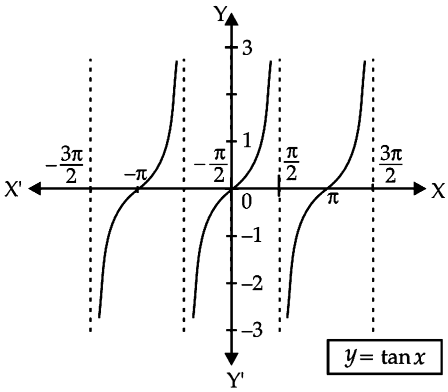
$y = \cos x$



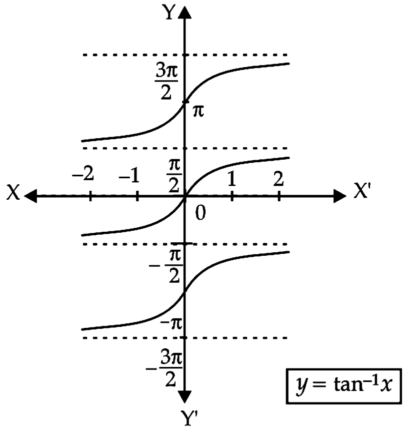
$y = \cos^{-1} x$

- **Principal value of branch function of $\tan^{-1}$:** The function $\tan^{-1}$ is defined whose domain is set of real numbers and range is one of the intervals

$$\left(\frac{-3\pi}{2}, \frac{\pi}{2}\right), \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$



$y = \tan x$

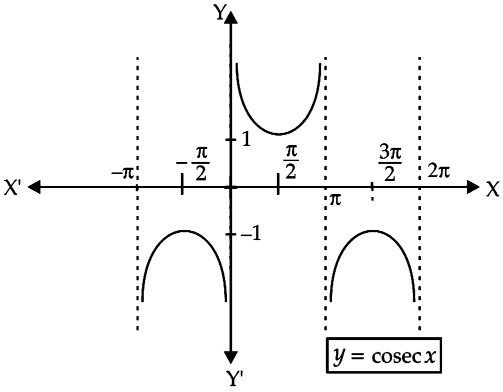


$y = \tan^{-1} x$

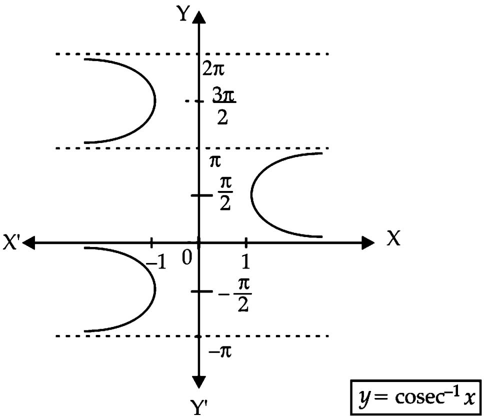
The branch with range $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is called the principal value branch of function $\tan^{-1}$. Thus, $\tan^{-1} : R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

- **Principal value of branch function of cosec^{-1} :** The graph of function cosec^{-1} is shown in figure. The cosec^{-1} is defined on a function whose domain is $R - (-1, 1)$ and the range is any one of the interval,

$$\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{\pi\}, \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}, \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}, \ldots$$



$y = \text{cosec}\,x$



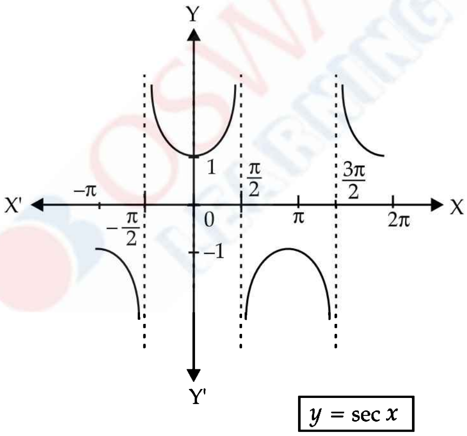
$y = \text{cosec}^{-1}x$

The function corresponding to the range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is called the principal value branch of cosec^{-1}.

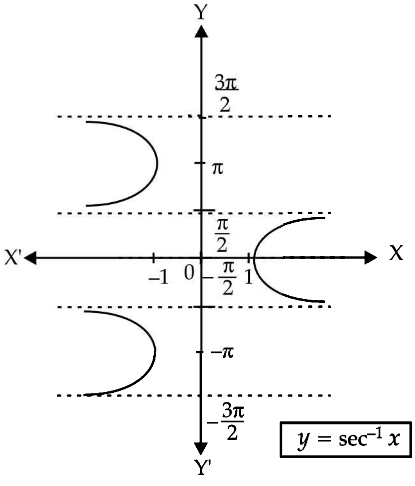
Thus, $\text{cosec}^{-1} : R - (-1, 1) \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

- **Principal value of branch function of sec^{-1} :** The graph of function sec^{-1} is shown in figure. The sec^{-1} is defined as a function whose domain $R - (-1, 1)$ and range is $[-\pi, 0] - \left\{\frac{-\pi}{2}\right\}, [0, \pi] - \left\{\frac{\pi}{2}\right\}, [\pi - 2\pi] - \left\{\frac{3\pi}{2}\right\}$, etc. Function corresponding to range $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ is known as the principal value branch of sec^{-1}.

Thus, $\sec^{-1} : R - (-1, 1) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$



$y = \sec x$



$y = \sec^{-1}x$

- **The principal value of branch function of cot^{-1} :**

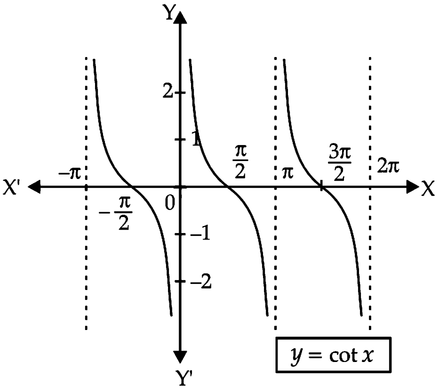
  The graph of function cot^{-1} is shown below.

  The cot^{-1} function is defined on function whose domain is R and the range is any of the intervals, $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$, ...

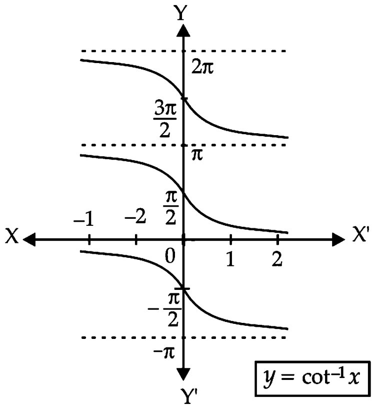
  The function corresponding to $(0, \pi)$ is called the principal value branch of the function cot^{-1}.

  Then, $\cot^{-1} : R \to (0, \pi)$

OSWAAL BOOKS LEARNING MADE SIMPLE



$y = \cot x$



$y = \cot^{-1} x$

Note

A function $f : A \rightarrow B$ is said to be invertible if f is bijective (*i.e.*, one-one and onto). The inverse of the function f is denoted by $f : B \rightarrow A$ such that $f^{-1}(y) = x$ if $f(x) = y$, $\forall\, x \in A,\ y \in B$. As trigonometric functions are many-one so, their inverse doesn't exist. But they become one-one onto by restricting their domains. Therefore, all the restrictions required so that the inverse of the concerned trigonometric functions do exist. If these restrictions are removed, the terms will represent **Inverse Trigonometric Relations** and not the functions. Note that the inverse trigonometric functions are also called as **Inverse Circular Functions**.

➢ **Elementary Properties of Inverse Trigonometric Functions :**

Property I

(a) $\sin^{-1}(x) = \text{cosec}^{-1}\left(\dfrac{1}{x}\right), x \in [-1, 1]$

(b) $\text{cosec}^{-1}(x) = \sin^{-1}\left(\dfrac{1}{x}\right), x \in (-\infty, -1] \cup [1, \infty)$

(c) $\cos^{-1}(x) = \sec^{-1}\left(\dfrac{1}{x}\right), x \in [-1, 1]$

(d) $\sec^{-1}(x) = \cos^{-1}\left(\dfrac{1}{x}\right), x \in (-\infty, -1] \cup [1, \infty)$

(e) $\tan^{-1}(x) = \begin{cases} \cot^{-1}\left(\dfrac{1}{x}\right), x > 0 \\ -\pi + \cot^{-1}\left(\dfrac{1}{x}\right), x < 0 \end{cases}$

(f) $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\dfrac{1}{x}\right), x > 0 \\ \pi + \tan^{-1}\left(\dfrac{1}{x}\right), x < 0 \end{cases}$

Scan to know more about this topic

Inverse Trigonometric Function

Property II

(a) $\sin^{-1}(\sin x) = x, -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$

(b) $\cos^{-1}(\cos x) = x, 0 \leq x \leq \pi$

(c) $\tan^{-1}(\tan x) = x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

(d) $\text{cosec}^{-1}(\text{cosec } x) = x,\ -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2},\ x \neq 0$

(e) $\sec^{-1}(\sec x) = x,\ 0 \leq x \leq \pi, x \neq \dfrac{\pi}{2}$

(f) $\cot^{-1}(\cot x) = x, 0 < x < \pi$

Property III

(a) $\sin^{-1}(-x) = -\sin^{-1}x,\ x \in [-1, 1]$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x,\ x \in [-1, 1]$

(c) $\tan^{-1}(-x) = -\tan^{-1}x,\ x \in R$

(d) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x,\ |x| \geq 1$

(e) $\sec^{-1}(-x) = \pi - \sec^{-1}x,\ |x| \geq 1$

(f) $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ x \in \text{R}$

Property IV

(a) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}, x \in [-1, 1]$

(b) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}, x \in R$

(c) $\sec^{-1}x + \text{cosec}^{-1}x = \dfrac{\pi}{2},\ x \in (-\infty, -1] \cup [1, \infty)$

Property V

(a) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(b) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$

Scan to know more about this topic

Properties of Inverse Trigonometric Function

(c) $$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), xy<1\\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), x>0,\ y>0,\ xy>1\\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), x<0,\ y<0,\ xy>1\end{cases}$$

(d) $$\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), xy>-1\\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), x>0,\ y<0,\ xy<-1\\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), x<0,\ y>0,\ xy<-1\end{cases}$$

(e) $$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

Property VI

(a) $$2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}),\text{if } -\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}),\text{if } \dfrac{1}{\sqrt{2}}\le x\le 1\\ -\pi-\sin^{-1}(2x\sqrt{1-x^2}),\text{if } -1\le x\le-\dfrac{1}{\sqrt{2}}\end{cases}$$

(b) $$3\sin^{-1}x=\begin{cases}\sin^{-1}(3x-4x^3),\text{if } -\dfrac{1}{2}\le x\le\dfrac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3),\text{if } \dfrac{1}{2}\le x\le 1\\ -\pi-\sin^{-1}(3x-4x^3),\text{if } -1\le x\le-\dfrac{1}{2}\end{cases}$$

Property VII

(a) $$2\cos^{-1}x=\begin{cases}\cos^{-1}(2x^2-1),\text{ if } 0\le x\le 1\\ 2\pi-\cos^{-1}(2x^2-1),\text{ if } -1\le x\le 0\end{cases}$$

(b) $$3\cos^{-1}x=\begin{cases}\cos^{-1}(4x^3-3x),\text{if } \dfrac{1}{2}\le x\le 1\\ 2\pi-\cos^{-1}(4x^3-3x),\text{if } -\dfrac{1}{2}\le x\le\dfrac{1}{2}\\ 2\pi+\cos^{-1}(4x^3-3x),\text{if } -1\le x\le-\dfrac{1}{2}\end{cases}$$

Property VIII

(a) $$2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\dfrac{2x}{1-x^2}\right),\text{if } -1<x<1\\ \pi-\tan^{-1}\left(\dfrac{2x}{1-x^2}\right),\text{if } x>1\\ -\pi+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right),\text{if } x<-1\end{cases}$$

(b) $$3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right),\text{if } \dfrac{-1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right),\text{if } x>\dfrac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right),\text{if } x<-\dfrac{1}{\sqrt{3}}\end{cases}$$

Know the Formulae

- **Trigonometric Formulae :**
  - **Relation between trigonometric ratios :**

    (a) $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ **(b)** $\tan\theta=\dfrac{1}{\cot\theta}$ **(c)** $\tan\theta.\cot\theta=1$

    (d) $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$ **(e)** $\operatorname{cosec}\theta=\dfrac{1}{\sin\theta}$ **(f)** $\sec\theta=\dfrac{1}{\cos\theta}$
  - **Trigonometric Identities :**

    (a) $\sin^2\theta+\cos^2\theta=1$ **(b)** $\sec^2\theta=1+\tan^2\theta$ **(c)** $\operatorname{cosec}^2\theta=1+\cot^2\theta$
  - **Addition/subtraction/formulae and some related results :**

    (a) $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$ **(b)** $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

OSWAAL BOOKS LEARNING MADE SIMPLE

(c) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

(d) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

(e) $\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$ (f) $\cot(A\pm B)=\dfrac{\cot B\cot A\mp 1}{\cot B\pm\cot A}$

- **Multiple angle formulae involving 2*A* & 3*A* :**

(a) $\sin 2A=2\sin A\cos A$ (b) $\sin A=2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

(c) $\cos 2A=\cos^2 A-\sin^2 A$ (d) $\cos A=\cos^2\dfrac{A}{2}-\sin^2\dfrac{A}{2}$

(e) $\cos 2A=2\cos^2 A-1$ (f) $\cos 2A=1-2\sin^2 A$

(g) $\sin 2A=\dfrac{2\tan A}{1+\tan^2 A}$ (h) $\cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$

(i) $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$ (j) $\sin 3A=3\sin A-4\sin^3 A$

(k) $\cos 3A=4\cos^3 A-3\cos A$ (l) $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

- **Transformation of sums/differences into products & vice-versa :**

(a) $\sin C+\sin D=2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (b) $\sin C-\sin D=2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(c) $\cos C+\cos D=2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (d) $\cos C-\cos D=-2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(e) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$ (f) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(g) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$ (h) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

➢ **Domain and Range of trigonometric functions :**

S. No.	**Function**	**Domain**	**Range**
(i)	sine	R	[–1, 1]
(ii)	cosine	R	[–1, 1]
(iii)	tangent	$R-\left\{x: x=(2n+1)\frac{\pi}{2}; n\in Z\right\}$	R
(iv)	cosecant	$R-\{x: x=n\pi,\ n\in Z\}$	R – (–1, 1)
(v)	secant	$R-\left\{x: x=(2n+1)\frac{\pi}{2}; n\in Z\right\}$	R – (–1, 1)
(vi)	cotangent	$R-\{x: x=n\pi, n\in Z\}$	R

➢ **Relations in different measures of Angle**

(a) Angle in Radian Measure = (Angle in degree measure) × $\dfrac{\pi}{180^\circ}$ radians

(b) Angle in Degree Measure = (Angle in radian measure) × $\dfrac{180^\circ}{\pi}$

(c) θ (in radian measure) = $\dfrac{l}{r}=\dfrac{\text{arc}}{\text{radius}}$

Also following are of importance as well :

(a) 1 right angle = 90° (b) 1° = 60′, 1′ = 60″

(c) $1^\circ=\dfrac{\pi}{180^\circ}$ = 0·01745 radians (Approx.) (d) 1 radian = 57°17′45″ or 206265 seconds.

- **General Solutions :**
  - **(a)** $\sin x = \sin y$ or $x = n\pi + (-1)^n y$, where $n \in Z$.
  - **(b)** $\cos x = \cos y$ or $x = 2n\pi \pm y$, where $n \in Z$.
  - **(c)** $\tan x = \tan y$ or $x = n\pi + y$, where $n \in Z$.
- **Relation in Degree & Radian Measures :**

Angles in Degree	0°	30°	45°	60°	90°	180°	270°	360°
Angles in Radian	0 rad	$\left(\frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}\right)$	(π)	$\left(\frac{3\pi}{2}\right)$	(2π)

- **Trigonometric Ratio of Standard Angles :**

Degree	**0°**	**30°**	**45°**	**60°**	**90°**
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\cot x$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\operatorname{cosec} x$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

OSWAAL BOOKS LEARNING MADE SIMPLE

- **Trigonometric Ratios of Allied Angles :**

Angles (→) **T – Ratios (↓)**	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	**$2\pi-\theta$ or $-\theta$**	**$2\pi+\theta$**
sin	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cot	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$
sec	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\sec\theta$
cosec	$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$

1 Mark Questions

Q. 1 Simplified value of $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ **is:**

(a) $\frac{1}{2}$ **(b)** $\frac{1}{\sqrt{2}}$

(c) $\frac{\sqrt{3}}{2}$ **(s)** $-\frac{\sqrt{3}}{2}$

[ISC Specimen Paper, 2021]

Sol. Correct option : (a)

$$\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]=\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$=\sin\left[\frac{\pi}{2}+\sin^{-1}\left(\sin\frac{\pi}{3}\right)\right]=\sin\left[\frac{\pi}{2}+\frac{\pi}{3}\right]$$

$$=\cos\frac{\pi}{3}=\frac{1}{2}$$

AI Q. 2. Which of the following is the principal value branch of $\cos^{-1}x$?

(a) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ **(b)** $[0,\pi]$

(c) $[0,\pi]$ **(d)** $(0,\pi)-\left\{\frac{\pi}{2}\right\}$

Sol. Correct option : (c)

Explanation : As we know that the principal value of $\cos^{-1}x$ is $[0,\pi]$.

Q. 3. Which of the following is the principal value branch of $\operatorname{cosec}^{-1}x$?

(a) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ **(b)** $[0,\pi]-\left\{\frac{\pi}{2}\right\}$

(c) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ **(d)** $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Sol. Correct option : (d)

Explanation : As we know that the principal value of $\operatorname{cosec}^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$.

AI Q. 4. The domain of function $\cos^{-1}(2x-1)$ is

(a) $[0, 1]$ **(b)** $[-1, 1]$

(c) $(-1, 1)$ **(d)** $[0, \pi]$

Sol. Correct option : (a)

Explanation :

We have $\cos^{-1}(2x-1)$

$\Rightarrow -1 \le 2x-1 \le 1$ $[\because x \in [-1, 1]]$

$\Rightarrow 0 \le 2x \le 2$

$\Rightarrow 0 \le x \le 1$

$\Rightarrow x \in [0, 1]$

Q. 5. The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is

(a) $\frac{\pi}{2}$ **(b)** $\frac{3\pi}{2}$

(c) $\frac{5\pi}{2}$ **(d)** $\frac{7\pi}{2}$

Sol. Correct option : (a)

Explanation : We have, $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$

$$=\cos^{-1}\left[\cos\left(2\pi-\frac{\pi}{2}\right)\right]$$

$$\left[\because \cos\left(2\pi-\frac{\pi}{2}\right)=\cos\frac{\pi}{2}\right]$$

$$=\cos^{-1}\cos\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$$

$$[\because \cos^{-1}(\cos x)=x \text{ if } x\in[0,\pi]]$$

$$\Rightarrow \cos^{-1}\left(\cos\frac{3\pi}{2}\right)=\frac{\pi}{2}$$

AI Q. 6. The value of expression $2\sec^{-1}2+\sin^{-1}\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{6}$ **(b)** $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$ **(d)** 1

Sol. Correct option : (b)

Explanation : We have, $2\sec^{-1}2+\sin^{-1}\left(\frac{1}{2}\right)$

$$=2\sec^{-1}\sec\frac{\pi}{3}+\sin^{-1}\sin\frac{\pi}{6}$$

$$=2\times\frac{\pi}{3}+\frac{\pi}{6} \quad [\because \sec^{-1}(\sec x)=x \text{ and } \sin^{-1}(\sin x)=x]$$

$$=\frac{4\pi+\pi}{6}$$

$$=\frac{5\pi}{6}$$

Q. 7. If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$, then the value of $\cos^{-1}x+\cos^{-1}y$ is ....

(a) $\frac{2\pi}{3}$ **(b)** $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ **(d)** π

Sol. Correct option : (b) $\frac{\pi}{3}$

Explanation :

Given, $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$

i.e.; $\frac{\pi}{2}-\cos^{-1}x+\frac{\pi}{2}-\cos^{-1}y=\frac{2\pi}{3}$

$\cos^{-1}x+\cos^{-1}x=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$

AI Q. 8. Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$.

Sol. $0 \le \cos^{-1}x \le \pi$

$$\cos^{-1}\left(-\frac{1}{2}\right) \le \cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}$$

Examiner's Comment

- Some students take $\cos^{-1}(-x)$ as $-\cos^{-1}x$ instead of $\pi-\cos^{-1}x$ and go wrong.

Answering Tip

- Get thorough with the properties of inverse trigonometric functions.

Q. 9. Find the value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$.

Sol. $\because$ $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

$\therefore$ $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)=\sin^{-1}\sin\left(\pi-\frac{2\pi}{5}\right)$

$$=\sin^{-1}\left(\sin\frac{2\pi}{5}\right)=\frac{2\pi}{5}$$

Examiner's Comment

- Some students write the answer as $\frac{3\pi}{5}$ using the formula $\sin^{-1}(\sin x)=x$.

Answering Tip

- $\sin^{-1}(\sin x) = x$ iff $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$.

AI **Q. 10. If $\sin^{-1}\left(\frac{1}{3}\right)+\cos^{-1}x=\frac{\pi}{2}$, then find the value of x.**

Sol.

$$\sin^{-1}\left(\frac{1}{3}\right)+\cos^{-1}x=\frac{\pi}{2}$$

or $$\sin^{-1}\left(\frac{1}{3}\right)+\frac{\pi}{2}-\sin^{-1}x=\frac{\pi}{2}$$

or $$\sin^{-1}\left(\frac{1}{3}\right)=\sin^{-1}x$$

$$\therefore \qquad x = \frac{1}{3}$$

Examiner's Comment

- Most candidates attempt this question using long calculations instead of using simple fundamental concepts.

Answering Tip

- Make sufficient practice for conversion through diagram and by using formulae.

Q. 11. Write the principal value of $\cos^{-1}[\cos(680°)]$.

Sol. $\cos^{-1}[\cos(680°)] = \cos^{-1}[\cos(720° - 40°)]$

$= \cos^{-1}(\cos 40°)$

$= 40°$

[since $\cos^{-1}(\cos\theta) = \theta$, where $\theta \in [0, \pi]$]

Q. 12. Write the principal value of $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$.

Sol. $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] = \tan^{-1}[-1]$

$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right) = -\frac{\pi}{4}$

$\left[\because \tan^{-1}(\tan\theta) = \theta\ \forall\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).\right]$

Q. 13. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x.

Sol.

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right) = 1$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x = \sin^{-1}(1)$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x = \sin^{-1}x+\cos^{-1}x$$

$$\left[\because \cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}\right]$$

$$\Rightarrow \quad x = \frac{1}{5}$$

Q. 14. Find the value of the following :

$$\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right)$$

Sol. $\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right) = \cot\left[\frac{\pi}{2}-2\cot^{-1}\left(\cot\frac{\pi}{6}\right)\right]$

$= \cot\left[\frac{\pi}{2}-2\left(\frac{\pi}{6}\right)\right]$

$[\because \cot^{-1}(\cot\theta) = \theta\ \forall\ \theta \in (0, \pi)]$

$= \cot\left[\frac{\pi}{2}-\frac{\pi}{3}\right]$

$= \cot\left(\frac{\pi}{6}\right) = \sqrt{3}$

Q. 15. Write the principal value of the following :

$$\left[\tan^{-1}(-\sqrt{3})+\tan^{-1}(1)\right]$$

Sol. $\tan^{-1}(-\sqrt{3})+\tan^{-1}(1)$

$= \tan^{-1}\left[-\tan\left(\frac{\pi}{3}\right)\right]+\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

$= -\frac{\pi}{3}+\frac{\pi}{4} = -\frac{\pi}{12}$ $\left[\because \tan^{-1}(\tan\theta)=\theta\ \forall\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

Q. 16. Write the value of $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right]$.

Sol. $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right] = \tan\left[\tan^{-1}\left(\frac{2\left(\frac{1}{5}\right)}{1-\frac{1}{25}}\right)\right]$

$= \tan\left[\tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right)\right]$

$\left[\because \tan^{-1}(\tan\theta)=\theta\forall\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

$= \left(\frac{5}{12}\right)$

Q. 17. Write the principal value of $\tan^{-1}\left(\tan\frac{9\pi}{8}\right)$.

Sol. $\tan^{-1}\left(\tan\frac{9\pi}{8}\right) = \tan^{-1}\left[\tan\left(\pi+\frac{\pi}{8}\right)\right]$

$= \tan^{-1}\left[\tan\frac{\pi}{8}\right]$

$\left[\because \tan^{-1}(\tan\theta)=\theta\forall\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

$= \frac{\pi}{8}$

Q. 18. Write $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$, $|x| > 1$ in the simplest form.

Sol. Putting $x = \sec\theta \Rightarrow \theta = \sec^{-1}x$

$\therefore \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \cot^{-1}\left(\frac{1}{\sqrt{\sec^2\theta-1}}\right)$

OSWAAL BOOKS
LEARNING MADE SIMPLE

$$= \cot^{-1}\left(\frac{1}{\tan\theta}\right)$$

$$= \cot^{-1}(\cot\theta)$$

$$= \theta$$

$$[\therefore \cot^{-1}(\cot\theta) = \theta \ \forall\ \theta \in (0, \pi)]$$

$$= \sec^{-1} x$$

Q. 19. Write the value of cot $(\tan^{-1}a + \cot^{-1}a)$.

Sol. $\cot(\tan^{-1}a + \cot^{-1}a) = \cot\left(\frac{\pi}{2}\right)$

$$\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$= 0$$

2 Marks Questions

Q 1. Prove that : $\tan^2(\sec^{-1} 2) + \cot^2(\mathrm{cosec}^{-1} 3) = 11$. **[ISC, 2020]**

Sol. L.H.S.

$$= \tan^2(\sec^{-1} 2) + \cot^2(\mathrm{cosec}^{-1} 3)$$

$$= \tan^2(\tan^{-1}\sqrt{3}) + \cot^2(\cot^{-1}2\sqrt{2})$$

$$\begin{bmatrix} \sec^{-1}2 = \tan^{-1}\sqrt{3} \\ \mathrm{cosec}^{-1}3 = \cot^{-1}2\sqrt{2} \end{bmatrix}$$

$$= [\tan(\tan^{-1}\sqrt{3})]^2 + [\cot(\cot^{-1}2\sqrt{2})]^2$$

$$= (\sqrt{3})^2 + (2\sqrt{2})^2$$

$$= 3 + 8 = 11$$

= L.H.S. **Hence proved**

[ISC Marking Scheme, 2020]

Examiner's Comment

- Many candidates didn't able to convert $\sec^{-1}$ and cosec^{-1} in terms of $\tan^{-1}$ and $\cot^{-1}$, respectively and also do algebraic calculation wrong.

Answering Tip

- Learn all the conversions of inverse trigonometric functions.

Q 2. Solve $\sin(2\tan^{-1}x) = 1$. **[ISC, 2019]**

Sol. $\sin(2\tan^{-1}x) = 1$

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\sin\left(\sin^{-1}\frac{2x}{1+x^2}\right) = 1$$

$$\therefore \left(\frac{2x}{1+x^2}\right) = 1 \Rightarrow x^2 - 2x + 1 = 0$$

$$\Rightarrow (x-1)^2 = 0$$

$$\Rightarrow x - 1 = 0$$

$$x = 1$$

2nd Solution:

$$\sin(2\tan^{-1}x) = 1,$$

$$\sin(2\tan^{-1}x) = \sin\frac{\pi}{2}$$

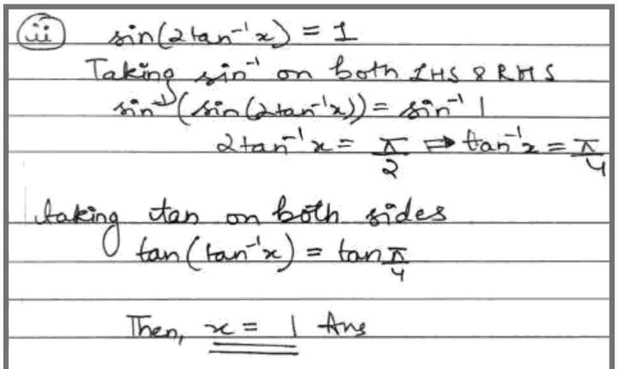
$$\Rightarrow \quad 2\tan^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{4}$$

$$\Rightarrow \quad x = 1$$

[ISC Marking Scheme, 2019]

OR



(ii) $\sin(2\tan^{-1}x) = 1$

Taking $\sin^{-1}$ on both LHS & RHS

$\sin^{-1}(\sin(2\tan^{-1}x)) = \sin^{-1} 1$

$2\tan^{-1}x = \frac{\pi}{2} \Rightarrow \tan^{-1}x = \frac{\pi}{4}$

taking tan on both sides

$\tan(\tan^{-1}x) = \tan\frac{\pi}{4}$

Then, $\underline{\underline{x = 1}}$ Ans

[Topper's Answer, 2019]

AI Q. 3. Solve : $\sin^{-1}\cos(\sin^{-1}x) = \frac{\pi}{3}$ **[SQP, 2018]**

Sol. Given, $\sin^{-1}\cos(\sin^{-1}x) = \frac{\pi}{3}$

$$\Rightarrow \quad \cos(\sin^{-1}x) = \sin\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad \cos\left(\cos^{-1}\sqrt{1-x^2}\right) = \frac{\sqrt{3}}{2}$$

$$[\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}\,]$$

$$\Rightarrow \quad \sqrt{1-x^2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad 1 - x^2 = \frac{3}{4}$$

$$\Rightarrow \quad 1 - \frac{3}{4} = x^2 \Rightarrow x = \pm\frac{1}{2}.$$

Q. 4. Solve : $3\tan^{-1}x + \cot^{-1}x = \pi$ **[ISC, 2018]**

Sol. Given, $3\tan^{-1}x + \cot^{-1}x = \pi$

$$\Rightarrow 2\tan^{-1}x + \tan^{-1}x + \cot^{-1}x = \pi$$

$$\Rightarrow \quad 2\tan^{-1}x + \frac{\pi}{2} = \pi$$

$$\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \qquad 2\tan^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \qquad \tan^{-1}x = \frac{\pi}{4}$$

$$\Rightarrow \qquad \tan^{-1}x = \tan^{-1}(1)$$

$$\Rightarrow \qquad x = 1$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Many candidates tried to solve by using formula of 3 $\tan^{-1} x$ but could not got succeed solving further. Most candidates attempted this question using long calculations instead of using simple fundamental concepts.

Answering Tip

- Understand the concept of conversion of inverse circular functions (one to another form) and do sufficient practice for conversion through diagram and by using formulae.

AI **Q. 5. Prove that :** $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}\sqrt{x}$ **[ISC, 2017]**

Sol. LHS $= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

Let $x = \tan^2\theta$

$$\Rightarrow \text{LHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$= \frac{1}{2} \times 2\theta \qquad \left[\because \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}\right]$$

$$= \theta$$

$$= \tan^{-1}\sqrt{x} \qquad [\because \tan\theta = \sqrt{x} \Rightarrow \theta = \tan^{-1}\sqrt{x}\,]$$

$$= \text{RHS.}$$

Examiner's Comment

- Some candidate used $x = \cos 2\theta$ but left it half way through. Some candidates attempted it by writing in forms of $\tan^{-1}$ function but failed in the conversion.

Answering Tip

- Do adequate practice of questions based on conversions of inverse trigonometric functions.

Q. 6. Solve : $\cos^{-1}\left[\sin(\cos^{-1}x)\right] = \frac{\pi}{6}$ **[ISC, 2015]**

Sol. $\cos^{-1}\left[\sin(\cos^{-1}x)\right] = \frac{\pi}{6}$

$$\Rightarrow \qquad \sin(\cos^{-1}x) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \qquad \sqrt{1-x^2} = \frac{\sqrt{3}}{2}$$

$$(\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2})$$

$$\Rightarrow \qquad 1-x^2 = \frac{3}{4} \Rightarrow x^2 = \frac{1}{4}$$

$$\Rightarrow \qquad x = \pm\frac{1}{2}$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- Many candidates did this question well but a few used wrong value of $\frac{\pi}{6}$, while some wrote incorrect conversion of $\cos^{-1} x$ in terms of $\sin^{-1} x$. A few candidates wrote $\cos^{-1} x$ in terms of $\tan^{-1} x$ which made the expression complicated.

Answering Tip

- Practice the derivations of inverse trigonometric function thoroughly.

Q. 7. Evaluate : $\tan\left[2\tan^{-1}\frac{1}{2} - \cot^{-1}3\right]$ **[ISC, 2014]**

Sol. $\tan\left[2\tan^{-1}\frac{1}{2} - \cot^{-1}3\right]$

$$= \tan\left[\tan^{-1}\frac{1}{1-\frac{1}{4}} - \tan^{-1}\frac{1}{3}\right]$$

$$= \tan\left[\tan^{-1}\frac{4}{3} - \tan^{-1}\frac{1}{3}\right]$$

$$= \tan\left[\tan^{-1}\frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{3}\times\frac{1}{3}}\right]$$

$$= \frac{1}{\frac{13}{9}} = \frac{9}{13}$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- Some candidates could not convert $2\tan^{-1}\frac{1}{2}$ into $\tan^{-1}\frac{4}{3}$.

  Some got the formula for $\tan^{-1} x - \tan^{-1} y$ wrong. Several candidates converted into $\sin^{-1}$ or $\cos^{-1}$ forms and got into more difficulties.

Answering Tip

- A revision of basic trigonometry laws is must. All concept of inverse trigonometry functions must be understood including conversions, combinations, double and triple angle laws through sufficient practice.

OSWAAL BOOKS LEARNING MADE SIMPLE

AI Q. 8. Prove that $\sec^2(\tan^{-1}2)+\text{cosec}^2(\cot^{-1}3)=15$ [ISC, 2012]

Sol. In LHS, let $\tan^{-1}2=\theta$ and $\cot^{-1}3=\alpha$

$\Rightarrow \quad 2=\tan\theta$ and $3=\cot\alpha$

$\therefore \quad \text{LHS}=\sec^2\theta+\text{cosec}^2\alpha$

$=(1+\tan^2\theta)+(1+\cot^2\alpha)$

$=(1+4)+(1+9)$

$=15=\text{RHS}$ **Hence Proved.**

Q. 9. Solve : $\cos^{-1}\left[\sin(\cos^{-1}x)\right]=\dfrac{\pi}{3}$ [ISC, 2010]

Sol. Given, $\cos^{-1}\left[\sin(\cos^{-1}x)\right]=\dfrac{\pi}{3}$

$\Rightarrow \quad \sin(\sin^{-1}\sqrt{1-x^2})=\cos\dfrac{\pi}{3}$

$(\because \cos^{-1}x=\sin^{-1}\sqrt{1-x^2})$

$\Rightarrow \quad \sin(\sin^{-1}\sqrt{1-x^2})=\dfrac{1}{2}$

$\Rightarrow \quad \sqrt{1-x^2}=\dfrac{1}{2}$

$\Rightarrow \quad 1-x^2=\dfrac{1}{4}$

$\Rightarrow \quad 1-\dfrac{1}{4}=x^2$

$\Rightarrow \quad x^2=\dfrac{3}{4}$

$\Rightarrow \quad x=\pm\dfrac{\sqrt{3}}{2}$

Q. 10. Write the value of $\cos^{-1}\left(-\dfrac{1}{2}\right)+2\sin^{-1}\left(\dfrac{1}{2}\right)$.

Sol. $\cos^{-1}\left(-\dfrac{1}{2}\right)+2\sin^{-1}\left(\dfrac{1}{2}\right)$

$=\cos^{-1}\left[-\cos\left(\dfrac{\pi}{3}\right)\right]+2\sin^{-1}\left[\sin\left(\dfrac{\pi}{6}\right)\right]$

$=\cos^{-1}\left[\cos\left(\pi-\dfrac{\pi}{3}\right)\right]+2\left(\dfrac{\pi}{6}\right)$

$\left[\because \sin^{-1}(\sin\theta)=\theta\,\forall\,\theta\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\right]$

$=\pi-\dfrac{\pi}{3}+\dfrac{\pi}{3}$ $[\because \cos^{-1}(\cos\theta)=\theta\ \forall\ \theta\ (0,\pi)]$

$=\pi$

Q. 11. Write the principal value of

$\tan^{-1}(1)+\cos^{-1}\left(-\dfrac{1}{2}\right)$

Sol. $\tan^{-1}(1)+\cos^{-1}\left(-\dfrac{1}{2}\right)$

$=\tan^{-1}\left(\tan\dfrac{\pi}{4}\right)+\cos^{-1}\left(-\cos\dfrac{\pi}{3}\right)$

$=\tan^{-1}\left(\tan\dfrac{\pi}{4}\right)+\cos^{-1}\left[\cos\left(\pi-\dfrac{\pi}{3}\right)\right]$

$\left[\because \tan^{-1}(\tan\theta)=\theta\,\forall\,\theta\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\right.$

$\left.\text{and } \cos^{-1}(\cos\theta)=\theta\,\forall\,\theta\in(0,\pi)\right]$

$=\dfrac{\pi}{4}+\dfrac{2\pi}{3}=\dfrac{11\pi}{12}$

AI Q. 12. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)\right)\right]$.

Sol. $\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)\right)\right]$

$=\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\cos\dfrac{\pi}{6}\right)\right)\right]$

$=\tan^{-1}\left[2\sin\left(2\left(\dfrac{\pi}{6}\right)\right)\right]$

$=\tan^{-1}\left[2\times\dfrac{\sqrt{3}}{2}\right]$

$=\tan^{-1}(\sqrt{3})=\dfrac{\pi}{3}$

Q. 13. Evaluate : $\sin\left(2\cos^{-1}\left(-\dfrac{3}{5}\right)\right)$.

Sol. $\sin\left(2\cos^{-1}\left(-\dfrac{3}{5}\right)\right)$

$[\because \sin 2\theta=2\sin\theta.\cos\theta]$

$=2\sin\left(\cos^{-1}\left(-\dfrac{3}{5}\right)\right)\cos\left(\cos^{-1}\left(-\dfrac{3}{5}\right)\right)$

$=2\sin\left(\cos^{-1}\left(-\dfrac{3}{5}\right)\right)\left(-\dfrac{3}{5}\right)$

$=-2\sin\left(\sin^{-1}\left(\dfrac{4}{5}\right)\right)\left(-\dfrac{3}{5}\right)$

$=-2\cdot\left(\dfrac{4}{5}\right)\left(-\dfrac{3}{5}\right)$

$=\dfrac{24}{25}$

Q. 14. Write the value of the following :

$\tan^{-1}\left(\dfrac{a}{b}\right)-\tan^{-1}\left(\dfrac{a-b}{a+b}\right)$

Sol. $\tan^{-1}\left(\dfrac{a}{b}\right)-\tan^{-1}\left(\dfrac{a-b}{a+b}\right)=\tan^{-1}\left(\dfrac{a}{b}\right)-\tan^{-1}\left(\dfrac{\dfrac{a}{b}-1}{1+\dfrac{a}{b}}\right)$

Let $\dfrac{a}{b}=\tan\theta$

$=\tan^{-1}(\tan\theta)-\tan^{-1}\left(\dfrac{\tan\theta-\tan\left(\dfrac{\pi}{4}\right)}{1+\tan\theta\tan\left(\dfrac{\pi}{4}\right)}\right)$

$=\tan^{-1}(\tan\theta)-\tan^{-1}\left[\tan\left(\theta-\dfrac{\pi}{4}\right)\right]$

$$= \theta - \theta + \frac{\pi}{4} \qquad \left[\because \tan^{-1}(\tan\theta) = \theta \,\forall\, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{4}$$

AI Q. 15. If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$, $xy < 1$, then write the value of $x + y + xy$.

Sol. We have, $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{x+y}{1-xy} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad x + y = 1 - xy$$

$$\Rightarrow \quad x + y + xy = 1$$

Q. 16. Solve for x : $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x), x \neq \frac{\pi}{2}$.

Sol. $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right) = \tan^{-1}(2\sec x)$$

$$\Rightarrow \quad \frac{2\sin x}{\cos^2 x} = 2\sec x$$

$$\Rightarrow \quad \frac{2\sin x}{\cos^2 x} - \frac{2}{\cos x} = 0$$

$$\Rightarrow \quad \frac{2\sin x}{\cos^2 x} - \frac{2\cos x}{\cos^2 x} = 0$$

$$\Rightarrow \quad \sin x - \cos x = 0$$

$$\Rightarrow \quad \tan x = 1$$

$$\Rightarrow \quad x = \frac{\pi}{4}$$

Q. 17. Prove that $\cos^{-1}(x) + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3}$.

Sol. Let, $\cos^{-1}x = \alpha \Rightarrow x = \cos\alpha$

$$\text{LHS} = \alpha + \cos^{-1}\left[\cos\alpha\cos\left(\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\sqrt{1-\cos^2\alpha}\right]$$

$$= \alpha + \cos^{-1}\left[\cos\left(\frac{\pi}{3}\right)\cos\alpha + \sin\frac{\pi}{3}\sin\alpha\right]$$

$$= \alpha + \cos^{-1}\left[\cos\left(\frac{\pi}{3} - \alpha\right)\right]$$

$$= \alpha + \frac{\pi}{3} - \alpha$$

$$= \frac{\pi}{3} = \text{RHS} \qquad [\because \cos^{-1}(\cos\theta) = \theta \,\forall\, \theta \in (0,\pi)]$$

4 Marks Questions

AI Q. 1. Prove that : $\tan^{-1}\frac{1}{2} = \frac{\pi}{4} - \tan^{-1}\frac{1}{3}$

Sol. Now, $\text{RHS} = \frac{\pi}{4} - \tan^{-1}\frac{1}{3}$

$$= \tan^{-1}1 - \tan^{-1}\frac{1}{3}$$

$$= \tan^{-1}\frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \tan^{-1}\frac{2}{4}$$

$$= \tan^{-1}\frac{1}{2} = \text{LHS}$$ **Hence roved.**

Q. 2 If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{2} = \theta$, than proove that $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$ [ISC ,2020]

Sol. $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{2} = \theta$

$$\cos^{-1}\left\{\frac{x}{2}\cdot\frac{y}{3} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right\}$$

$$\left[\cos^{-1}a + \cos^{-1}b = \cos^{-1}\left(ab - \sqrt{1-a^2}\sqrt{1-b^2}\right)\right]$$

$$\frac{xy}{6} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}} = \cos\theta$$

$$\frac{xy}{6} - \cos\theta = \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}$$

Squaring both sides

$$\frac{x^2y^2}{36} + \cos^2\theta - \frac{xy\cos\theta}{3} = \left(1-\frac{x^2}{4}\right)\left(1-\frac{y^2}{4}\right)$$

$$\frac{x^2y^2}{36} + \cos^2\theta - \frac{xy\cos\theta}{3} = 1 - \frac{x^2}{4} - \frac{y^2}{9} + \frac{x^2y^2}{36}$$

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{xy\cos\theta}{3} = 1 - \cos^2\theta$$

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{xy\cos\theta}{3} = \sin^2\theta$$

Multiply by 36

$$9x^2 + 4y^2 - 12xy\cos\theta = 36\sin^2\theta$$

$$9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$$ **Hence Proved.**

[ISC Marking Scheme, 2020]

Examiner's Comment

- Many candidates fail to prove the required equation. Some of them made errors in applying formula of $\cos^{-1}x + \cos^{-1}y$.

OSWAAL BOOKS LEARNING MADE SIMPLE

Answering Tip

- Learn all the formula, laws and inter conversions of inverse circular functions.

Q. 3 Evaluate : $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = \frac{1}{5}$.

[ISC, 2020]

Sol. $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = \frac{1}{5}$

$$= \cos(\cos^{-1}x + \sin^{-1}x + \cos^{-1}x)$$

$$= \cos\left(\frac{\pi}{2} + \cos^{-1}x\right)$$

$$= -\sin(\cos^{-1}x)$$

$$= -\sin\sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$= -\left(\sqrt{1-x^2}\right)$$

$$= -\left(\sqrt{1-\frac{1}{25}}\right)$$

$$= -\sqrt{\frac{24}{25}} = -\frac{2\sqrt{6}}{5}$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Many candidates applied the correct formula, but made errors while solving further hence, couldn't get the required result.

Answering Tip

- Do adequate practice in solving problems on different types of inverse trigonometric function.

Q. 4 Solve for x : $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

Sol. $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2}\cdot\frac{x+1}{x+2}}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1} = 1$$

$$\Rightarrow \quad 2x^2 = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{2}$$

$$\Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$$

[ISC Marking Scheme, 2019]

Q. 5 If $\sec^{-1}x = \text{cosec}^{-1}y$, show that $\frac{1}{x^2} + \frac{1}{y^2} = 1$

Sol. $\sec^{-1}x = \text{cosec}^{-1}y \Rightarrow \cos^{-1}\frac{1}{x} = \sin^{-1}\frac{1}{y} = \theta$

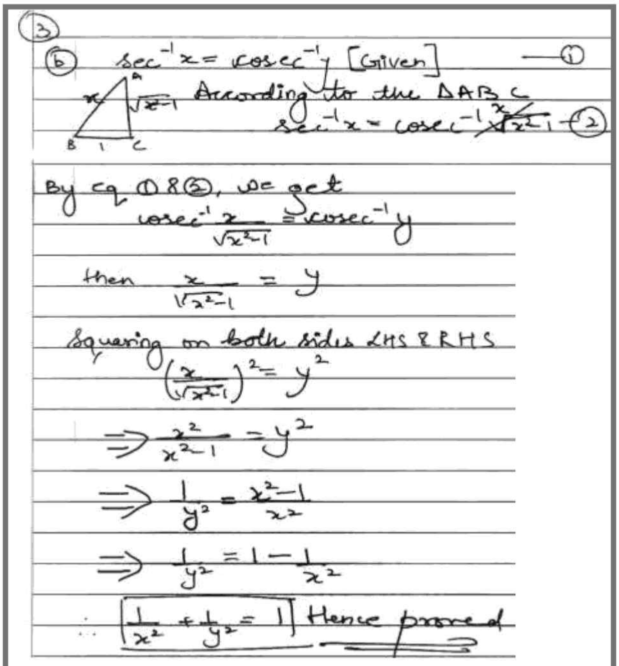
$$\cos\theta = \frac{1}{x}, \sin\theta = \frac{1}{y}$$

$$\Rightarrow \cos^2 + \sin^2\theta = \frac{1}{x^2} + \frac{1}{y^2}$$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} = 1$$

[ISC Marking Scheme 2020]

OR



③

ⓑ $\sec^{-1}x = \text{cosec}^{-1}y$ [Given] —①

According to the ΔABC

$\sec^{-1}x = \text{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$ —②

By eq ① & ②, we get

$\text{cosec}^{-1}\frac{x}{\sqrt{x^2-1}} = \text{cosec}^{-1}y$

then $\frac{x}{\sqrt{x^2-1}} = y$

Squaring on both sides LHS & RHS

$\left(\frac{x}{\sqrt{x^2-1}}\right)^2 = y^2$

$\Rightarrow \frac{x^2}{x^2-1} = y^2$

$\Rightarrow \frac{1}{y^2} = \frac{x^2-1}{x^2}$

$\Rightarrow \frac{1}{y^2} = 1 - \frac{1}{x^2}$

$\therefore \boxed{\frac{1}{x^2} + \frac{1}{y^2} = 1}$ Hence proved

[Topper's Answer, 2019]

AI Q. 6. Prove that : $\tan^{-1}\frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)$

[ISC Specimen Paper, 2018]

Sol. LHS $= \tan^{-1}\frac{1}{2}$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right)$$

$$\left[\because 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1-x^2}\right)\right]$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{\frac{3}{4}}{\frac{5}{4}}\right)$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{4}{5}\right)\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right) = \text{RHS.}$$ **Hence Proved.**

Q. 7. If $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \pi$, Prove that :
$a + b + c = abc$. **[ISC, 2018]**

Sol.

$$\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \pi$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{a+b}{1-ab}\right) = \pi - \tan^{-1}c$$

$$\Rightarrow \quad \frac{a+b}{1-ab} = \tan(\pi - \tan^{-1}c)$$

$$\Rightarrow \quad \frac{a+b}{1-ab} = -c$$

$$\Rightarrow \quad a + b + c = abc \quad \textbf{Hence Proved.}$$

[ISC Marking Scheme, 2017]

OR

Given, $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \pi$

$$\Rightarrow \tan^{-1}\left(\frac{a+b}{1-ab}\right) + \tan^{-1}c = \pi$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{a+b}{1-ab}+c}{1-\left(\frac{a+b}{1-ab}\right)c}\right] = \pi$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{a+b+c-abc}{1-ab-ac-bc}\right) = \pi$$

$$\Rightarrow \quad \frac{a+b+c-abc}{1-ab-ac-bc} = \tan\pi = 0$$

$$\Rightarrow \quad a + b + c = abc \quad \textbf{Hence Proved.}$$

AI Q. 8. Solve the equation for x :
$\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x, x \neq 0$
[ISC 2017, 2013]

Sol. Given, $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$

$$\Rightarrow \quad \sin^{-1}x + \sin^{-1}(1-x) = \frac{\pi}{2} - \sin^{-1}x$$

$$\Rightarrow \quad \sin^{-1}(1-x) = \frac{\pi}{2} - 2\sin^{-1}x$$

$$\Rightarrow \quad 1 - x = \sin\left[\frac{\pi}{2} - 2\sin^{-1}x\right]$$

$$\Rightarrow \quad 1 - x = \cos[2\sin^{-1}x]$$

$$\Rightarrow \quad 1 - x = \cos[\cos^{-1}(1-2x^2)]$$

$$\Rightarrow \quad 1 - x = 1 - 2x^2$$

$$\Rightarrow \quad 2x^2 - x = 0$$

$$\Rightarrow \quad x(2x-1) = 0$$

$$\Rightarrow \quad x = \frac{1}{2} \qquad [\because x \neq 0]$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates made errors in applying the formula of $\sin^{-1}$ A + $\sin^{-1}$ B. In addition, errors took place in simplifying and solving higher degree algebraic equations. Some candidates converted all terms into inverse trigonometric function form and could not handle the resulting equations.

Answering Tip

- Practice thoroughly laws of inverse circular functions and inter conversion of functions. More practice is required for solving different types of higher order equation.

Q. 9. Solve for x, if $\tan(\cos^{-1}x) = \frac{2}{\sqrt{5}}$ **[ISC, 2016]**

Sol. $\therefore \cos^{-1}x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$$\therefore \tan\left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right] = \left(\frac{2}{\sqrt{5}}\right)$$

$$\frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}}$$

Squaring on both sides, we get

$$\frac{1-x^2}{x^2} = \frac{4}{5}$$

$$\Rightarrow \quad 5 - 5x^2 = 4x^2$$

$$\Rightarrow \quad 9x^2 = 5$$

$$\Rightarrow \quad x = \pm\frac{\sqrt{5}}{3}$$

OR

$$\cos^{-1}x = \tan^{-1}\frac{2}{\sqrt{5}}$$

$$= \cos^{-1}\frac{\sqrt{5}}{3}$$

$$\Rightarrow \quad x = \frac{\sqrt{5}}{3}$$

[ISC Marking Scheme, 2016]

Q. 10. Prove that : $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\sin^{-1}\frac{4}{5}$ **[ISC, 2013]**

Sol. LHS $= \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}$

$$= \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}\right)\left(\frac{2}{9}\right)}\right), \qquad \left(\begin{array}{c}\because xy < 1 \\ i.e., \frac{1}{4}\times\frac{2}{9} < 1\end{array}\right)$$

$$= \tan^{-1}\left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right)$$

OSWAAL BOOKS
LEARNING MADE SIMPLE

$$= \tan^{-1}\left(\frac{17}{34}\right)$$

$$= \frac{2}{2}\tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\left[2\tan^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\sin^{-1}\frac{2\left(\frac{1}{2}\right)}{1+\left(\frac{1}{2}\right)^2}\right]$$

$$= \frac{1}{2}\left[\sin^{-1}\frac{1}{1+\frac{1}{4}}\right]$$

$$= \frac{1}{2}\sin^{-1}\frac{4}{5} = \text{RHS}$$

Hence Proved.

AI Q. 11. Prove that: $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$.

Sol. Let, LHS = $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$

$$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}\right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{3}{11}\right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right)$$

$$= \tan^{-1}\left(\frac{1}{3}\right)$$

$$= \cot^{-1}(3) = \text{RHS}$$

AI Q. 12. Find the greatest and least values of: $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$.

Sol. $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$

$$= (\sin^{-1}x + \cos^{-1}x)^2 - 2\sin^{-1}x\cos^{-1}x$$

$$= \left(\frac{\pi}{2}\right)^2 - 2\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$$

$$= \frac{\pi^2}{4} - \pi\sin^{-1}x + 2(\sin^{-1}x)^2$$

$$= 2\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x + \frac{\pi^2}{8}\right]$$

$$= 2\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right]$$

∴ Least value

$$= 2\left(\frac{\pi^2}{16}\right) = \frac{\pi^2}{8}$$

and Greatest value

$$= 2\left[\left(-\frac{\pi}{2} - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right] = \frac{5\pi^2}{4}$$

Q. 13. Show that: $\sin[\cot^{-1}(\cos(\tan^{-1}x))] = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$

Sol. Here, $\cos(\tan^{-1}x) = \cos\left[\cos^{-1}\dfrac{1}{\sqrt{x^2+1}}\right]$

$$= \frac{1}{\sqrt{x^2+1}}$$

$[\because \cos(\cos^{-1}x) = x\ \forall\, x \in [-1, 1]]$

$$\because\quad \cot^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) = \sin^{-1}\left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}\right)$$

$$\therefore\quad \sin\left[\cot^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right] = \sin\left[\sin^{-1}\left(\sqrt{\frac{x^2+1}{x^2+2}}\right)\right]$$

$$= \frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$$

$[\because \sin(\sin^{-1}x) = x\ \forall\ x \in [-1,1]]$

AI Q. 14. Solve for x:

$$\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2\tan^{-1}x.$$

then show that $x = \dfrac{a+b}{1-ab}$.

Sol. We have, $\sin^{-1}\dfrac{2a}{1+a^2} = 2\tan^{-1}a$

and $\sin^{-1}\dfrac{2b}{1+b^2} = 2\tan^{-1}b$

The equation becomes

$$2\tan^{-1}a + 2\tan^{-1}b = 2\tan^{-1}x$$

$$\left(\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad \tan^{-1}a + \tan^{-1}b = \tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{a+b}{1-ab}\right) = \tan^{-1}x$$

$$\Rightarrow \quad x = \frac{a+b}{1-ab}$$

Hence Proved.

Q. 15. Write in the simplest form:

$$\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right],\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Sol. Given, $\tan^{-1}\left[\dfrac{\cos x}{1+\sin x}\right]$

$$\left\{\begin{array}{l} \because\ \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \\ \text{and } 1+\sin x = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2 \end{array}\right\}$$

$$= \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right],$$

Divide by $\cos\frac{x}{2}$, we get

$$= \tan^{-1}\left[\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right] = \frac{\pi}{4}-\frac{x}{2}$$

Q. 16. Show that :

$$\mathbf{\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19} = \frac{\pi}{4}}$$

Sol. LHS $= \tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}$

$$= \tan^{-1}\left(\frac{\frac{15+12}{20}}{\frac{20-9}{20}}\right)-\tan^{-1}\frac{8}{19}$$

$$\left[\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}\frac{27}{11}-\tan^{-1}\frac{8}{19}$$

$$\left[\because \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27\times 8}{11\times 19}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{513-88}{209}}{\frac{209+216}{209}}\right)$$

$$= \tan^{-1}\left(\frac{425}{425}\right)$$

$$= \tan^{-1}1 = \frac{\pi}{4}$$

Q. 17. Show that : $\mathbf{\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{x-y}{x+y} = \frac{\pi}{4}}$

Sol. LHS $= \tan^{-1}\frac{x}{y}-\tan^{-1}\frac{x-y}{x+y}$

$$= \tan^{-1}\left[\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right]$$

$$\left[\because \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right]$$

$$= \tan^{-1}\left[\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right]$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right) = \tan^{-1}1 = \frac{\pi}{4}$$ **Hence Proved.**

Q. 18. Show that : $\mathbf{\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5} = \tan^{-1}\frac{63}{16}}$

Sol. LHS $= \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

$$\because \sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\sin^{-1}\frac{5}{13} = \tan^{-1}\left(\frac{\frac{5}{13}}{\sqrt{1-\frac{25}{169}}}\right)$$

$$= \tan^{-1}\frac{\frac{5}{13}}{\sqrt{\frac{144}{169}}} = \tan^{-1}\frac{5}{12}$$

and $\cos^{-1}x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$$\therefore \quad \cos^{-1}\frac{3}{5} = \tan^{-1}\left(\frac{\sqrt{1-\frac{9}{25}}}{\frac{3}{5}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\frac{16}{25}}}{\frac{3}{5}}\right) = \tan^{-1}\left(\frac{4}{3}\right)$$

Now, LHS $= \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

$$= \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\left[\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5\times 4}{12\times 3}}\right]$$

$$\left[\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

OSWAAL BOOKS LEARNING MADE SIMPLE

$$= \tan^{-1}\left|\frac{\frac{15+48}{36}}{\frac{36-20}{36}}\right|$$

$$= \tan^{-1}\left(\frac{63}{16}\right) = \text{RHS}$$

AI Q. 19. Show that : $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)=3\tan^{-1}\left(\frac{x}{a}\right)$

Sol. LHS= $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

on dividing by a^3,

$$= \tan^{-1}\left(\frac{3\frac{x}{a}-\frac{x^3}{a^3}}{1-3\left(\frac{x^2}{a^2}\right)}\right)$$

$$= \tan^{-1}\left[\frac{3\left(\frac{x}{a}\right)-\left(\frac{x}{a}\right)^3}{1-3\left(\frac{x}{a}\right)^2}\right]$$

Put $\frac{x}{a}=\tan\theta$ or $\theta=\tan^{-1}\frac{x}{a}$

$$= \tan^{-1}\left[\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right) = \text{RHS}$$

Q. 20. Prove that : $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$**; *if* ∈ x[–1, 1]**

Sol. We have $x \in [-1, 1]$

Let $\quad x = \sin\theta \qquad \left\{\because \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right\}$

$\Rightarrow \quad \theta = \sin^{-1}x$

$\Rightarrow \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} \ge -\theta \ge -\frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2}+\frac{\pi}{2} \ge \frac{\pi}{2}-\theta \ge -\frac{\pi}{2}+\frac{\pi}{2}$

$\Rightarrow \quad \pi \ge \frac{\pi}{2}-\theta \ge 0$

Now, $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta = x$

$\Rightarrow \quad \frac{\pi}{2}-\theta = \cos^{-1}x$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2}-\sin^{-1}x$

$\Rightarrow \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, if $x\in[-1, 1]$

Q. 21. Prove that : $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$**; if** $x^2-y^2\le 1$**.**

Sol. Let $\quad \sin^{-1}x = A$ and $\sin^{-1}y = B$

$\Rightarrow \quad x = \sin A$ and $y = \sin B$

$\therefore \quad \cos A = \sqrt{1-x^2}$, $\cos B = \sqrt{1-y^2}$

We have $\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\Rightarrow \quad \sin(A+B) = x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad A + B = \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\therefore \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Q. 22. Write in the simplest form :

$$\sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$$

Sol. Let $\quad x = \cos 2\theta$

$\therefore \quad \sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$

$$= \sin\left[2\tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right]$$

$$= \sin\left[2\tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right]$$

$$\left[\begin{array}{l}\because \cos 2\theta = 1-2\sin^2\theta \\ \text{and } \cos 2\theta = 2\cos^2\theta - 1\end{array}\right]$$

$$= \sin[2\tan^{-1}(\tan\theta)]$$

$$= \sin(2\theta) = \sqrt{1-\cos^2 2\theta}$$

$$= \sqrt{1-x^2}$$

AI Q. 23. Write in the simplest form :

$$\sin^{-1}\left[\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right],\ -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}.$$

Sol. $\sin^{-1}\left[\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right]$

Let $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$= \sin^{-1}\left(\frac{\sin\theta+\sqrt{1-\sin^2\theta}}{\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\frac{\sin\theta+\cos\theta}{\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\sin\theta\times\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos\theta\right)$$

$$= \sin^{-1}\left(\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right)$$

$$= \sin^{-1}\left[\sin\left(\theta+\frac{\pi}{4}\right)\right]$$

$$\Rightarrow \theta+\frac{\pi}{4} = \frac{\pi}{4}+\sin^{-1}x$$

Q. 24. Solve the equation :

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$$

Sol. $$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x \quad \text{[Given]}$$

Put $x = \tan\theta \Rightarrow \tan^{-1}x = \theta$

$$\Rightarrow \quad \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right] = \frac{1}{2}\theta$$

$$\Rightarrow \quad \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] = \frac{\theta}{2}$$

$$\Rightarrow \quad \frac{\pi}{4}-\theta = \frac{\theta}{2}$$

$$\Rightarrow \quad \frac{\pi}{4} = \frac{\theta}{2}+\theta$$

$$\Rightarrow \quad \frac{\pi}{4} = \frac{3\theta}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow \quad x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

6 Marks Questions

Q. 1. If $\sin^{-1}x+\tan^{-1}x=\frac{\pi}{2}$, prove that : $2x^2+1=\sqrt{5}$. **[ISC, 2016]**

Sol. Given, $$\sin^{-1}x+\tan^{-1}x=\frac{\pi}{2}$$

$$\tan^{-1}x=\frac{\pi}{2}-\sin^{-1}x=\cos^{-1}x$$

$$\tan^{-1}x=\tan^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$\Rightarrow \quad x=\frac{\sqrt{1-x^2}}{x}$$

$$\Rightarrow \quad x^2=\sqrt{1-x^2}$$

$$\Rightarrow \quad x^4=1-x^2$$

$$\Rightarrow \quad x^4+x^2-1=0$$

$$\therefore \quad x^2=\frac{-1\pm\sqrt{5}}{2}$$

But x^2 cannot be negative.

$$\therefore \quad x^2=\frac{-1+\sqrt{5}}{2}$$

$$\Rightarrow \quad 2x^2+1=\sqrt{5}$$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Many candidates attempted this question but errors were made while squaring, simplifying and solving higher degree algebraic equations. Most of the candidates failed to prove the result.

Answering Tip

- All algebraic and trigonometric laws need to be revised thoroughly before learning inverse trigonometric functions and their operations. Understood the application of formula for inverse trigonometric functions. Also, learn the domain and range of inverse trigone metric functions.

Q. 2. Solve the equation for x :

$$\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2},\ x\neq 0$$ **[ISC, 2015]**

Sol. $$\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2},\ x\neq 0$$

$$\Rightarrow \quad \sin^{-1}\frac{5}{x}=\frac{\pi}{2}-\sin^{-1}\frac{12}{x}$$

$$\Rightarrow \quad \frac{5}{x}=\sin\left(\frac{\pi}{2}-\sin^{-1}\frac{12}{x}\right)$$

$$\frac{5}{x}=\cos\left(\sin^{-1}\frac{12}{x}\right)$$

$$\Rightarrow \quad \frac{5}{x}=\cos\left(\cos^{-1}\sqrt{1-\left(\frac{12}{x}\right)^2}\right)$$

$$\Rightarrow \quad \frac{5}{x}=\sqrt{1-\left(\frac{12}{x}\right)^2}$$

Squaring on both sides

$$\frac{25}{x^2}=1-\frac{144}{x^2}$$

$$\Rightarrow \quad x^2=169$$

$$\Rightarrow \quad x=\pm 13$$

$$\Rightarrow \quad x=13$$

[as $x = -13$ does not satisfy the given equation.]

[ISC Marking Scheme, 2015]

AI Q. 3. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, prove that : $x^2 + y^2 + z^2 + 2xyz = 1$ **[ISC, 2014]**

Sol. $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$$

$$\Rightarrow \quad \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)=\pi-\cos^{-1}z$$

$$\Rightarrow \quad xy-\sqrt{1-x^2}\sqrt{1-y^2}=\cos(\pi-\cos^{-1}z)$$

$$\Rightarrow \quad xy-\sqrt{1-x^2}\sqrt{1-y^2}=-\cos(\cos^{-1}z)=-z$$

$$\Rightarrow \quad xy+z=\sqrt{1-x^2}\sqrt{1-y^2}$$

Squaring on both sides,

$$x^2y^2+z^2+2xyz=1-x^2-y^2+x^2y^2$$

$$\Rightarrow \quad x^2+y^2+z^2+2xyz=1$$

[ISC Marking Scheme, 2014]

OSWAAL BOOKS LEARNING MADE SIMPLE

Q. 4. Prove that : $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$

[ISC, 2012]

Sol. LHS $=\sin^{-1}\sqrt{1-\left(\frac{63}{65}\right)^2}+\sin^{-1}\frac{2.\frac{1}{5}}{1+\left(\frac{1}{5}\right)^2}$

$$\left[\begin{array}{l}\text{Using}\\ \cos^{-1}x=\sin^{-1}\sqrt{1-x^2}\\ 2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}\end{array}\right]$$

$$=\sin^{-1}\sqrt{1-\frac{3969}{4225}}+\sin^{-1}\frac{\frac{2}{5}}{\frac{26}{25}}$$

$$=\sin^{-1}\sqrt{\frac{4225-3969}{4225}}+\sin^{-1}\frac{5}{13}$$

$$=\sin^{-1}\sqrt{\frac{256}{4225}}+\sin^{-1}\frac{5}{13}$$

$$=\sin^{-1}\frac{16}{65}+\sin^{-1}\frac{5}{13}$$

[Using

$\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\{\sqrt{1-y^2}\}+y\{\sqrt{1-x^2}\})$]

$$\therefore\ \sin^{-1}\frac{16}{65}+\sin^{-1}\frac{15}{13}=\sin^{-1}\left\{\frac{16}{65}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{16}{65}\right)^2}\right\}$$

$$=\sin^{-1}\left(\frac{16}{65}\times\frac{12}{13}+\frac{5}{13}\times\frac{63}{65}\right)$$

$$=\sin^{-1}\left(\frac{16\times32+315}{13\times65}\right)$$

$$=\sin^{-1}\left(\frac{507}{65\times13}\right)=\sin^{-1}\frac{3}{5}=\text{RHS}$$

Q. 5. Prove that : $2\tan^{-1}\frac{1}{5}+\cos^{-1}\frac{7}{5\sqrt{2}}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

[ISC, 2011]

Sol. LHS $=2\tan^{-1}\frac{1}{5}+\cos^{-1}\frac{7}{5\sqrt{2}}+2\tan^{-1}\frac{1}{8}$

$$=2\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right)+\cos^{-1}\frac{7}{5\sqrt{2}}$$

$$=2\tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right]+\cos^{-1}\frac{7}{5\sqrt{2}}$$

$$\left[\because\ \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=2\tan^{-1}\left[\frac{\frac{13}{40}}{\frac{39}{40}}\right]+\tan^{-1}\left[\frac{\sqrt{1-\frac{49}{50}}}{\frac{7}{5\sqrt{2}}}\right]$$

$$\left[\because\ \cos^{-1}x=\tan^{-1}\left(\sqrt{\frac{1-x^2}{x}}\right)\right]$$

$$=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{1}{3}+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}\right)$$

$$=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{\left(\frac{1}{3}+\frac{1}{7}\right)}{\left(1-\frac{1}{3}.\frac{1}{7}\right)}$$

$$=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}$$

$$=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}.\frac{1}{2}}\right)=\tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)=\tan^{-1}1=\frac{\pi}{4}$$

= RHS

Q. 6. Prove that : $\sin\left[2\tan^{-1}\frac{3}{5}-\sin^{-1}\frac{7}{25}\right]=\frac{304}{425}$

[ISC, 2010]

Sol. LHS $=\sin\left[2\tan^{-1}\frac{3}{5}-\sin^{-1}\frac{7}{25}\right]$

$$=\sin\left[\sin^{-1}\frac{2\times\frac{3}{5}}{1+\left(\frac{3}{5}\right)^2}-\sin^{-1}\frac{7}{25}\right]$$

$$\left(\because\ 2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}\right)$$

$$=\sin\left[\sin^{-1}\frac{\frac{6}{5}}{\frac{34}{25}}-\sin^{-1}\frac{7}{25}\right]$$

$$=\sin\left[\sin^{-1}\frac{15}{17}-\sin^{-1}\frac{7}{25}\right]$$

$$=\sin\left[\sin^{-1}\left(\frac{15}{17}\sqrt{1-\frac{49}{625}}-\frac{7}{25}\sqrt{1-\frac{225}{289}}\right)\right]$$

$$\left[\because\ \sin^{-1}x-\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2})-y\sqrt{1-x^2}\right]$$

$$=\frac{15}{17}\times\frac{24}{25}-\frac{7}{25}\times\frac{8}{17}=\frac{360-56}{425}=\frac{304}{425}=\text{RHS}$$

Q. 7. Solve for x :

$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$

[ISC, 2008]

Sol. Given,

$$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$$

$$\Rightarrow\ \tan^{-1}(x-1)+\tan^{-1}(x+1)=\tan^{-1}3x-\tan^{-1}x$$

$$\Rightarrow\ \tan^{-1}\frac{x-1+x+1}{1-x^2+1}=\tan^{-1}\frac{3x-x}{1+3x^2}$$

$$\left(\begin{array}{l}\therefore \quad \tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}\\ \text{and } \tan^{-1}x-\tan^{-1}y=\tan^{-1}\dfrac{x-y}{1+xy}\end{array}\right)$$

$$\Rightarrow \qquad \frac{2x}{2-x^2}=\frac{2x}{1+3x^2}$$

$$\Rightarrow x[1+3x^2-2+x^2]=0$$

$$\Rightarrow \quad x=0 \text{ or } 4x^2-1=0$$

$$\Rightarrow \qquad x^2=\frac{1}{4}$$

$$\Rightarrow \qquad x=\pm\frac{1}{2}, 0$$

Q. 8. Prove that : $2\tan^{-1}\left(\frac{1}{3}\right)+\cot^{-1}(4)=\tan^{-1}\left(\frac{16}{13}\right)$

[ISC, 2007]

Sol. LHS $=2\tan^{-1}\left(\frac{1}{3}\right)+\cot^{-1}(4)$

$$=2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{4}\right)$$

$$=\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\left(\frac{1}{4}\right)$$

$$\left(\begin{array}{l}\because \quad \tan^{-1}\dfrac{1}{x}=\cot^{-1}x\\ \text{and } 2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}\end{array}\right)$$

$$=\tan^{-1}\frac{\frac{2}{3}}{\frac{8}{9}}+\tan^{-1}\left(\frac{1}{4}\right)$$

$$=\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{4}\right)$$

$$=\tan^{-1}\frac{\frac{3}{4}+\frac{1}{4}}{1-\frac{3}{16}}$$

$$\left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right)$$

$$=\tan^{-1}\frac{1}{\frac{13}{16}}=\tan^{-1}\left(\frac{16}{13}\right)=\text{RHS}$$

Q. 9. Prove that : $\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)=\frac{\pi}{4}-\frac{x}{2}$,

where $\pi<x<\frac{3\pi}{2}$.

Sol. LHS $=\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)$

$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right)$$

$$=\tan^{-1}\left(\frac{-\sqrt{2}\cos\frac{x}{2}+\sqrt{2}\sin\frac{x}{2}}{-\sqrt{2}\cos\frac{x}{2}-\sqrt{2}\sin\frac{x}{2}}\right)$$

$$\left(\therefore \pi<x<\frac{3\pi}{2}\Rightarrow\frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}\right.$$

$$\left.\Rightarrow\cos\frac{x}{2}<0,\ \sin\frac{x}{2}>0\right)$$

$$=\tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$$

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)$$

$$=\frac{\pi}{4}-\frac{x}{2} \qquad \left(\therefore -\frac{\pi}{4}>\frac{\pi}{4}-\frac{x}{2}>-\frac{x}{2}\right)$$

Q. 10. If $\sin[\cot^{-1}(x+1)]=\cos(\tan^{-1}x)$**, then find** x**.**

Sol. Given that $\sin[\cot^{-1}(x+1)]=\cos(\tan^{-1}x)$ ...(i)

Since,

$$\cot^{-1}(A)=\sin^{-1}\frac{1}{\sqrt{1+A^2}}$$

Here, $A=x+1$

Applying this identity is equation (i), we have

$$\sin\left[\sin^{-1}\frac{1}{\sqrt{1+(1+x)^2}}\right]=\cos(\tan^{-1}x) \qquad \text{...(ii)}$$

Also,

$$\tan^{-1}A=\cos^{-1}\frac{1}{\sqrt{1+A^2}}$$

Here, $A=x$

Applying this identify in equation (ii), we have

$$\sin\left[\sin^{-1}\frac{1}{\sqrt{1+(1+x)^2}}\right]=\cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \qquad \frac{1}{\sqrt{1+(1+x)^2}}=\frac{1}{\sqrt{1+x^2}}$$

$$\left[\because \sin^{-1}(\sin\theta)=\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right.$$

$$\left.\text{and } \cos^{-1}(\cos\theta)=\theta\ \forall\ \theta[0,\pi]\right]$$

Squaring both sides, we have

$$1+(1+x)^2=1+x^2$$

$$\Rightarrow \qquad 1+1+x^2+2x=1+x^2$$

$$\Rightarrow \qquad 1+2x=0$$

$$\Rightarrow \qquad x=-\frac{1}{2}$$

Q. 11. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$**, find** x**.**

Sol. Given, $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$

OSWAAL BOOKS LEARNING MADE SIMPLE

$$\Rightarrow \quad (\tan^{-1}x)^2+\left(\frac{\pi}{2}-\tan^{-1}x\right)^2=\frac{5\pi^2}{8}$$

$$\Rightarrow 2(\tan^{-1}x)^2-\pi\tan^{-1}x+\frac{\pi^2}{4}-\frac{5\pi^2}{8}=0$$

$$\Rightarrow \quad 2(\tan^{-1}x)^2-\pi\tan^{-1}x-\frac{3\pi^2}{8}=0$$

$$\Rightarrow \quad \tan^{-1}x=\frac{\pi\pm\sqrt{\pi^2+3\pi^2}}{4}$$

$$\Rightarrow \quad \tan^{-1}x=\frac{3\pi}{4},\frac{-\pi}{4}$$

$$\Rightarrow \quad x=-1$$

Q. 12. Show that : $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2}$, $x\in\left(0,\frac{\pi}{4}\right)$.

Sol. Since,

$$1\pm\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\pm2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=\left(\cos\frac{x}{2}\pm\sin\frac{x}{2}\right)^2$$

∴ LHS

$$=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$=\cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right]$$

$$=\cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right]$$

$$=\cot^{-1}\left[\cot\frac{x}{2}\right]$$

$$=\frac{x}{2} \qquad [\because \cot^{-1}(\cot\theta)=\forall\,\theta\in(0,N)]$$

Q. 13. Find the value of the following :

$$\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right],|x|<1,y>0$$

and $xy<1$.

Sol. Let, $Z=\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$

By putting $x=\tan A\Rightarrow A=\tan^{-1}x$
and $y=\tan B\Rightarrow B=\tan^{-1}y$

We get,

$$Z=\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2\tan A}{1+\tan^2A}\right)+\cos^{-1}\left(\frac{1-\tan^2B}{1+\tan^2B}\right)\right]$$

$$Z=\tan\frac{1}{2}[\sin^{-1}(\sin 2A)+\cos^{-1}(\cos 2B)]$$

$$=\tan\frac{1}{2}[2A+2B]$$

$$=\tan[A+B]$$

$$=\tan[\tan^{-1}x+\tan^{-1}y]$$

$$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\frac{x+y}{1-xy} \qquad \left[\because \tan^{-1}(\tan\theta)=\theta\,\forall\,\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

Q. 14. If $y=\cot^{-1}(\sqrt{\cos x})-\tan^{-1}(\sqrt{\cos x})$, then prove that $\sin y=\tan^2\left(\frac{x}{2}\right)$.

Sol. Given,

$$y=\cot^{-1}(\sqrt{\cos x})-\tan^{-1}(\sqrt{\cos x})$$

$$\Rightarrow y=\frac{\pi}{2}-\tan^{-1}(\sqrt{\cos x})-\tan^{-1}(\sqrt{\cos x})$$

$$\Rightarrow y=\frac{\pi}{2}-2\tan^{-1}(\sqrt{\cos x})$$

$$\Rightarrow y=\frac{\pi}{2}-\left[\tan^{-1}\left(\frac{(2\sqrt{\cos x})}{1-(\sqrt{\cos x})^2}\right)\right]$$

$$\Rightarrow \left[\tan^{-1}\left(\frac{(2\sqrt{\cos x})}{1-(\sqrt{\cos x})^2}\right)\right]=\frac{\pi}{2}-y$$

$$\Rightarrow \frac{(2\sqrt{\cos x})}{1-\cos x}=\tan\left(\frac{\pi}{2}-y\right)$$

$$\Rightarrow \quad \cot y=\frac{(2\sqrt{\cos x})}{1-\cos x}$$

$$\Rightarrow \quad \sin y=\frac{1-\cos x}{\sqrt{(2\sqrt{\cos x})^2+(1-\cos x)^2}}$$

$$\Rightarrow \quad \sin y=\frac{1-\cos x}{\sqrt{4\cos x+1+\cos^2x-2\cos x}}$$

$$\Rightarrow \quad \sin y=\frac{1-\cos x}{\sqrt{(1+\cos x)^2}}$$

$$\Rightarrow \quad \sin y=\frac{1-\cos x}{1+\cos x}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\Rightarrow \quad \sin y=\tan^2\frac{x}{2}$$

Hence Proved.

Q. 15. Prove that :

$$\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$$

Sol. Let $\cos^{-1}\frac{a}{b}=\theta$, then $\cos\theta=\frac{a}{b}$

$$\therefore \quad \text{LHS}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}} + \frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}}$$

$$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$= \left(\frac{\left(1+\tan\frac{\theta}{2}\right)^2+\left(1-\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}\right)$$

$$= 2\left(\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right)$$

$$= 2\left(\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}\right)$$

$$= 2 \times \frac{1}{\cos\,(2\,\theta/2)} = \frac{2}{\cos\theta}$$

$$= \frac{2b}{a} = \text{ RHS}$$

Q. 16. Show that :

$$\mathbf{2\tan^{-1}\left(\frac{\sqrt{a-b}}{\sqrt{a+b}}.\tan\frac{\theta}{2}\right) = \cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)}$$

Sol. Let $2\tan^{-1}\left(\frac{\sqrt{a-b}}{\sqrt{a+b}}.\tan\frac{\theta}{2}\right) = \beta$

$$\Rightarrow \quad \frac{\beta}{2} = \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}.\tan\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \tan\frac{\beta}{2} = \sqrt{\frac{a-b}{a+b}}.\tan\frac{\theta}{2}$$

Now,

$$\cos\beta = \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}{1+\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}$$

$$= \frac{(a+b)-(a-b)\tan^2\frac{\theta}{2}}{(a+b)+(a-b)\tan^2\frac{\theta}{2}}$$

$$= \frac{a\left(1-\tan^2\frac{\theta}{2}\right)+b\left(1+\tan^2\frac{\theta}{2}\right)}{a\left(1+\tan^2\frac{\theta}{2}\right)+b\left(1-\tan^2\frac{\theta}{2}\right)}$$

$$= \frac{a\left(\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)+b}{a+b\left(\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)}$$

$\left(\text{Dividing numerator and denominator by } 1+\tan^2\frac{\theta}{2}\right)$

$$\Rightarrow \quad \cos\beta = \frac{a\cos\theta+b}{a+b\cos\theta}$$

$$\Rightarrow \quad \beta = \cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right) = \text{RHS}$$

OSWAAL BOOKS LEARNING MADE SIMPLE

SELF ASSESSMENT TEST

UNIT–I

Maximum Marks : 22 **Timing : 1 Hour**

Q. 1. **The $\cos\left(\tan^{-1}x+\cot^{-1}\sqrt{3}\right)=0$, then find value of x.**

Q. 2. **Find the value of $\cos\left(\sin^{-1}x+\cos^{-1}x\right), |x|\le 1$.**

Q. 3. **A binary operation $*$ defined on Q–{1} is given by $a * b = a + b - ab$. Find the identity element.**

Q. 4. **How many equivalence relations on the set {1, 2, 3} containing (1, 2) and (2, 1) are there in all? Justify your answer.**

Q. 5. **Show that $\sin^{-1}\frac{\sqrt{3}}{2}+2\tan^{-1}\frac{1}{\sqrt{3}}=\frac{2\pi}{3}$**

Q. 6. **Solve : $\tan^{-1}(2+x)+\tan^{-1}(2-x)=\tan^{-1}\frac{2}{3}$**

Q. 7. **Lef $f : N \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N ; y = 4x + 3$, for some $x \in N\}$. Show that f is invertible. Find its inverse.**

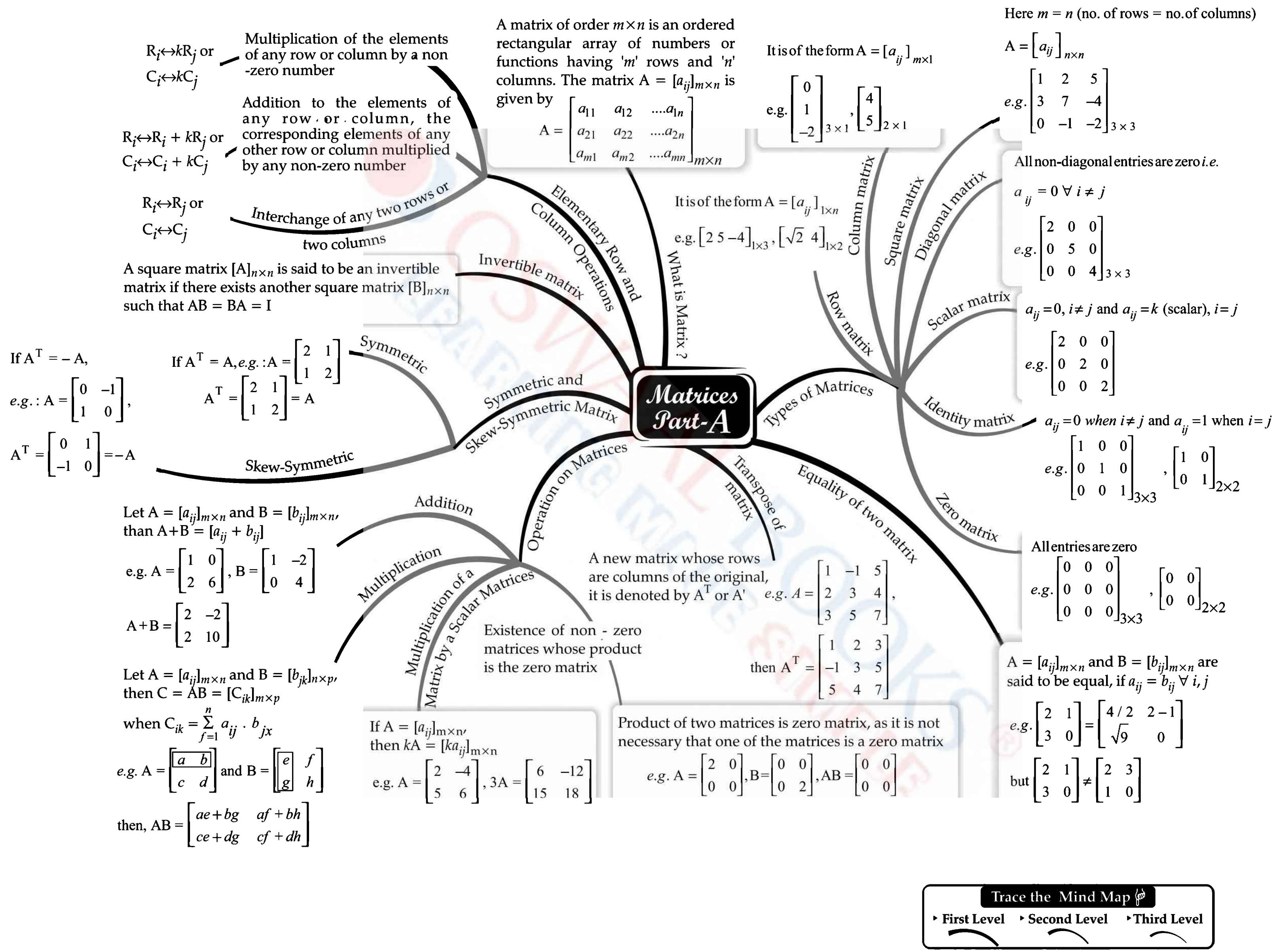
Q. 8. **Prove that : $2\left(\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\right)=\pi$**

Q. 9. **Show that : $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{2}{\sqrt{5}}=\cot^{-1}\frac{2}{11}$**

OSWAAL PREPARATION TOOLS

Solutions for Self-assessment Test

Visit : *http://qr.w69b.com/g/pi8gdC0qk* **OR Scan the Code**



Matrices Part-A

What is Matrix ?

A matrix of order $m \times n$ is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A = [a_{ij}]_{m\times n}$ is given by $A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ a_{m1} & a_{m2} & \dots a_{mn} \end{bmatrix}_{m\times n}$

Types of Matrices

Row matrix

It is of the form $A = [a_{ij}]_{1\times n}$

e.g. $[2\ 5\ -4]_{1\times 3}, [\sqrt{2}\ \ 4]_{1\times 2}$

Column matrix

It is of the form $A = [a_{ij}]_{m\times 1}$

e.g. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$

Square matrix

Here $m = n$ (no. of rows = no.of columns)

$A = \left[a_{ij}\right]_{n\times n}$

e.g. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times 3}$

Diagonal matrix

All non-diagonal entries are zero *i.e.*

$a_{ij} = 0 \ \forall\ i \neq j$

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$

Scalar matrix

$a_{ij} = 0, i \neq j$ and $a_{ij} = k$ (scalar), $i = j$

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Identity matrix

$a_{ij} = 0$ *when* $i \neq j$ and $a_{ij} = 1$ when $i = j$

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

Zero matrix

All entries are zero

e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

Equality of two matrix

$A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are said to be equal, if $a_{ij} = b_{ij} \ \forall\ i, j$

e.g. $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4/2 & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$

but $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

Transpose of matrix

A new matrix whose rows are columns of the original, it is denoted by A^T or A'

e.g. $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$,

then $A^T = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \\ 5 & 4 & 7 \end{bmatrix}$

Operation on Matrices

Addition

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, than $A+B = [a_{ij} + b_{ij}]$

e.g. $A = \begin{bmatrix} 1 & 0 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$

$A+B = \begin{bmatrix} 2 & -2 \\ 2 & 10 \end{bmatrix}$

Multiplication

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, then $C = AB = [C_{ik}]_{m\times p}$

when $C_{ik} = \sum_{f=1}^{n} a_{ij} \cdot b_{jx}$

e.g. $A = \begin{bmatrix} \boxed{\begin{matrix} a & b \end{matrix}} \\ \begin{matrix} c & d \end{matrix} \end{bmatrix}$ and $B = \begin{bmatrix} \boxed{\begin{matrix} e \\ g \end{matrix}} & \begin{matrix} f \\ h \end{matrix} \end{bmatrix}$

then, $AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

Multiplication of a Matrix by a Scalar Matrices

If $A = [a_{ij}]_{m\times n}$, then $kA = [ka_{ij}]_{m\times n}$

e.g. $A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix}, 3A = \begin{bmatrix} 6 & -12 \\ 15 & 18 \end{bmatrix}$

Existence of non - zero matrices whose product is the zero matrix

Product of two matrices is zero matrix, as it is not necessary that one of the matrices is a zero matrix

e.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Symmetric and Skew-Symmetric Matrix

Symmetric

If $A^T = A$, *e.g.* : $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A$

Skew-Symmetric

If $A^T = -A$,

e.g. : $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A$

Invertible matrix

A square matrix $[A]_{n\times n}$ is said to be an invertible matrix if there exists another square matrix $[B]_{n\times n}$ such that AB = BA = I

Elementary Row and Column Operations

Multiplication of the elements of any row or column by a non -zero number

$R_i \leftrightarrow kR_j$ or $C_i \leftrightarrow kC_j$

Addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number

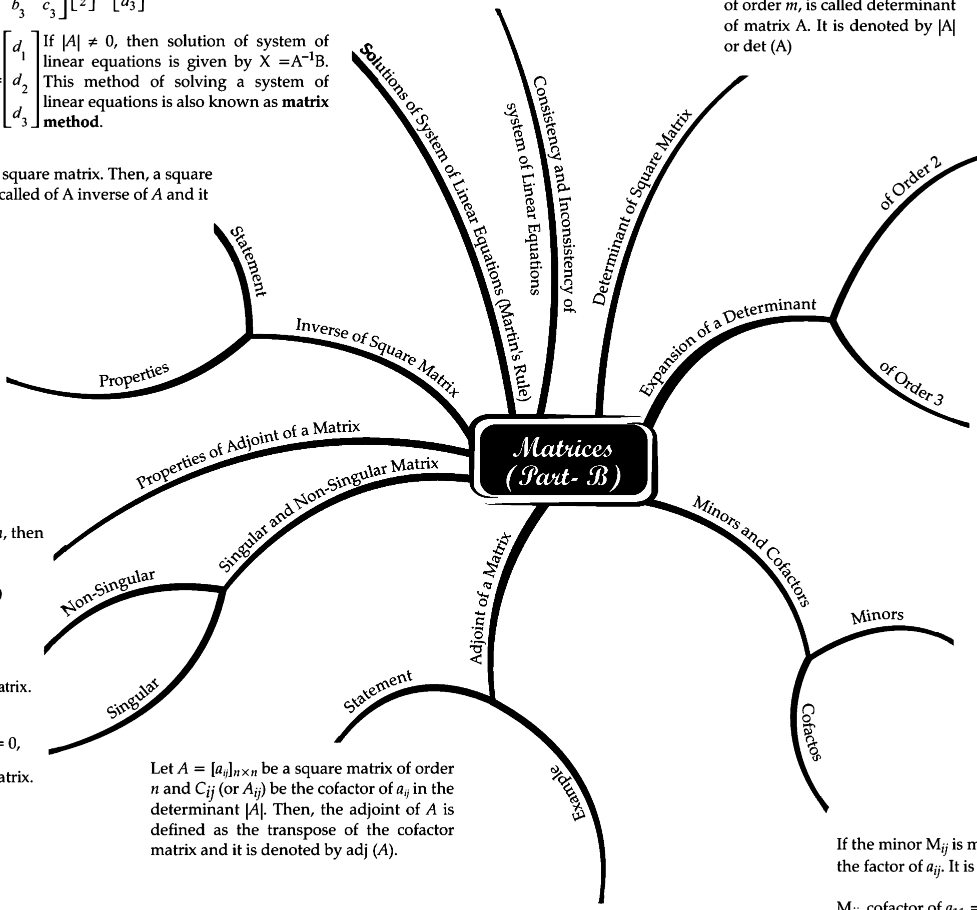
$R_i \leftrightarrow R_i + kR_j$ or $C_i \leftrightarrow C_i + kC_j$

Interchange of any two rows or two columns

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

Trace the Mind Map

- First Level
- Second Level
- Third Level



Matrices (Part- B)

Solutions of System of Linear Equations (Martin's Rule)

Let system of linear equation in three variables be

$a_1x+b_1y+c_1z = d_1$, $a_2x+b_2y+c_2z = d_2$ and $a_3x+b_3y+c_3z = d_3$

It can be written in the matrix form as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ or AX = B

Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ If $|A| \neq 0$, then solution of system of linear equations is given by $X = A^{-1}B$. This method of solving a system of linear equations is also known as **matrix method**.

Consistency and Inconsistency of system of Linear Equations

A system of equations is consistent if it has one or more solutions while it is inconsistence if it has no solutions

Determinant of Square Matrix

A unique number (real or complex) can be associated, to every square matrix A = [aij] of order *m*, is called determinant of matrix A. It is denoted by |A| or det (A)

Expansion of a Determinant

of Order 2

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{22}$$

of Order 3

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\,(a_{22}.a_{33} - a_{32}.a_{23}) - a_{12}\,(a_{21}.a_{33} - a_{31}.a_{23}) + a_{13}\,(a_{21}.a_{32} - a_{31}.a_{22})$$

Minors and Cofactors

Minors

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minor of $a_{11} = M_{11}$

$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, Minor of $a_{12} = M_{12} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ and so on.

Cofactos

If the minor M_{ij} is multiplied by $(-1)^{i+j}$, then we obtain the factor of a_{ij}. It is denoted by A_{ij} or C_{ij}, i.e., $A_{ij} = (-1)^{i+j}$

M_{ij}, cofactor of $a_{11} = (-1)^{1+1} M_{11}\, A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

Adjoint of a Matrix

Statement

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order *n* and C_{ij} (or A_{ij}) be the cofactor of a_{ij} in the determinant |*A*|. Then, the adjoint of *A* is defined as the transpose of the cofactor matrix and it is denoted by adj (*A*).

Example

If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then adj $A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$,

where A_{ij}' s are coafactors of *A*.

Singular and Non-Singular Matrix

Non-Singular

If A is a square matrix and $|A| \neq 0$,

then A is known as non-singular matrix.

Singular

If A is a square matrix, and $|A| = 0$,

then A is known as singular matrix.

Properties of Adjoint of a Matrix

If A is a square matrix of order $n \times n$, then

- (i) $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$
- (ii) adj $(A^T) = (\text{adj } A)^T$
- (iii) $|\text{adj } A| = |A|^{n-1}$, provided $|A| \neq 0$

Inverse of Square Matrix

Statement

Let *A* be a non-singular (*i.e.* $|A| \neq 0$) square matrix. Then, a square matrix *B*, such that $AB = BA = I$ is called of A inverse of *A* and it is denoted by A^{-1}

i.e. $A^{-1} = \dfrac{A}{|A|}$ adj (*A*).

Properties

Let A and B be square matrices of same order, then

- (i) A square matrix is invertible if and only if it is non-singular
- (ii) $(A^{-1})^{-1} = A$
- (iii) $(A')^{-1} = (A^{-1})'$
- (iv) $(AB)^{-1} = B^{-1}A^{-1}$
- (v) $|A^{-1}| = |A|^{-1}$
- (vi) $AA^{-1} = A^{-1}A = I$
- (vii) $(kA)^{-1} = \frac{1}{k}A^{-1}$, if $k \neq 0$

Trace the Mind Map

‣ First Level ‣ Second Level ‣ Third Level

UNIT-2

Algebra

CHAPTER

3

SYLLABUS

- ***Concept, notation, order, equality, types of matrices: zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices. Operations on matrices : Addition, multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. Non-commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 3). Concept of elementary row and column operations. Invertible matrices and proof of the uniqueness of inverse, if it exists; (here, all matrices will have real entries).***
- ***Types of matrices ($m \times n$; $m, n \leq 3$), order ; Identity matrix, Diagonal matrix.***
- ***Symmetric, Skew symmetric.***
- ***Operation - Addition, Subtraction, Multiplication of matrix with scalar, Multiplication of two matrices (the compatibility).***

  e.g. $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ ***= AB (say) but BA is not possible.***
- ***Singular and non-singular matrices.***
- ***Existence of two non-zero matrices whose product is a zero matrix.***
- ***Inverse of*** $(2 \times 2,\ 3 \times 3)$ ***,matrices*** $A^{-1} = \dfrac{Adj\, A}{|A|}$.
- ***Martin's Rule (i.e., using matrices)***
- $a_1x + b_1y + c_1z = d_1$

  $a_2x + b_2y + c_2z = d_2$

  $a_3x + b_3y + c_3z = d_3$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1}B$$

Problems based on above.

- ***Note 1 : The conditions for consistency of equations in two and three variables, using matrices, are to be covered.***
- ***Note 2 : Inverse of a matrix by elementary operations to be covered.***

OSWAAL BOOKS LEARNING MADE SIMPLE

Trend Analysis

List of Concepts	2016	2017	2018	2019	2020
Matrices and Operations	1 Q (2 marks)	1 Q (2 marks)	1 Q (2 marks)	1 Q (2 marks)	1 Q (2 marks)
Invertible Matrices and Martin's Rule	1 Q (6 marks) 1 Q (2 marks)	1 Q (6 marks)	1 Q (6 marks)	1 Q (6 marks)	1 Q (6 marks)

TOPIC-1
Matrices and Operations

Revision Notes

TOPIC - 1
Matrices and Operations .... **P. 54**
TOPIC - 2
Invertible Matrices and Martin's Rule .... **P. 63**
TOPIC - 3
Elementary Operation or Transformation of a Matrix .... **P. 78**

➢ **Matrix :**

A matrix is an ordered rectangular array of numbers (real or complex) or functions which are known as elements or the entries of the matrix. It is denoted by the uppercase letters *i.e.*, *A*, *B*, *C*, etc.

Consider a matrix *A* given as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Here in matrix *A* depicted above, the horizontal lines of elements are said to constitute rows of the matrix *A* and vertical lines of elements are said to constitute columns of the matrix. Thus, matrix *A* has *m* **rows** and *n* **columns**.

Scan to know more about this topic

What is a matrix?

The array is enclosed by square brackets [ ], the parentheses ( ) or the double vertical bars || ||.

- A matrix having *m* rows and *n* columns is called a matrix of order *m* × *n* (read as '*m* by *n*' matrix). A matrix 'A' of order *m* × *n* is depicted as $A = [a_{ij}]_{m\times n}$; $i, j \in N$.
- Also, in general, a_{ij} means an element lying in the *i*th row and *j*th column.
- No. of elements in the matrix $A = [a_{ij}]_{m\times m}$ is given as $(m)(n)$.

➢ **Types of Matrices :**

(i) Column matrix : A matrix having only one column is called a **column matrix** or **column vector**.

e.g., $A = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}$, $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$

General notation : $A = [a_{ij}]_{m\times 1}$

(ii) Row matrix : A matrix having only one row is called a **row matrix** or **row vector**.

e.g., $A = [2\ \ 5\ -4]_{1\times 3}, B = [\sqrt{2}\ \ 4]_{1\times 2}$

General notation : $A = [a_{ij}]_{1\times n}$

(iii) Square matrix : It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $n\times n$ matrix is said to constitute a square matrix of order $n \times n$ and is known as a **square matrix of order** '*n*'.

e.g., $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times 3}$ is a square matrix of order 3.

General notation : $A = [a_{ij}]_{n\times n}$

(iv) Diagonal matrix : A square matrix $A = [a_{ij}]_{m\times m}$ is said to be diagonal matrix if all the elements, except those in the leading diagonal are zero *i.e.*, $a_{ij} = 0$ for all $i \neq j$.

e.g., $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$ is a diagonal matrix of order 3.

- Also, there are more notation specifically used for the diagonal matrices. For instance, consider the matrix depicted above, it can also be written as diag (2 5 4) or diag [2, 5, 4]
- Note that the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$ of a square matrix $A = [a_{ij}]_{m\times m}$ of order m × m are said to constitute the principal diagonal or simply **the diagonal of the square matrix** *A*. These elements are known as **diagonal elements of matrix** *A*.

(v) Scalar matrix : A diagonal matrix $A = [a_{ij}]_{m\times m}$ is said to be a scalar matrix if its diagonal elements are equal.

i.e., $a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \text{ for some constant } k \end{cases}$

e.g., $A = \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix}_{3\times 3}$ is a scalar matrix of order 3.

(vi) Unit or Identity matrix : A square matrix $A = [a_{ij}]_{m\times m}$ is said to be an identity matrix if $a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.

A unit matrix can also be defined as the scalar matrix each of whose diagonal elements is unity. We denote the identity matrix of order m by I_m or I.

e.g., $I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

(vii) Zero matrix or Null matrix: A matrix is said to be a **zero matrix** or **null matrix** if each of its elements is '0'.

e.g., $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$, $C = [0 \;\; 0]_{1\times 2}$

(viii) Triangular matrix :

(a) Lower triangular matrix : A square matrix is called a lower triangular matrix if all the entries above the main diagonal are zero.

e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 3 & 5 \end{bmatrix}$

(b) Upper triangular matrix : A square matrix is called a **upper triangular matrix** if all the entries below the main diagonal are zero.

e.g., $A = \begin{bmatrix} 1 & -8 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

➤ **Equality of Matrices :**

Two matrices A and B are said to be equal and written as $A = B$, if they are of the same order and their corresponding elements are identical *i.e.*, $a_{ij} = b_{ij}$ *i.e.*, $a_{11} = b_{11}$, $a_{22} = b_{22}$, $a_{32} = b_{32}$, etc.

➤ **Addition of Matrix :**

If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by '$A + B$'.

Thus, if $A = [a_{ij}]$, $B = [b_{ij}] \Rightarrow A + B = [a_{ij} + b_{ij}]$.

Properties of matrix addition :

- Commutative property : $A + B = B + A$
- Associativity property : $A + (B + C) = (A + B) + C$
- Cancellation law : **(i)** Left cancellation : $A + B = A + C \Rightarrow B = C$
  (ii) Right cancellation : $B + A = C + A \Rightarrow B = C$

➤ **Multiplication of a Matrix by a Scalar :**

If a $m \times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus, if $A = [a_{ij}]$, and it is multiplied by a scalar k, then $kA = [ka_{ij}]$, *i.e..* $A = [a_{ij}]$.

$\Rightarrow kA = [ka_{ij}]$

e.g., $A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix} \Rightarrow 3A = \begin{bmatrix} 6 & -12 \\ 15 & 18 \end{bmatrix}$

Scan to know more about this topic

Matrix multiplication

➤ **Multiplication of Two Matrices :**

Let $A = [a_{ij}]$ be a $m \times n$ matrix and $B = [b_{jk}]$ be a $n \times p$ matrix such that the number of columns in A is equal to the number of rows in B, then the $m \times p$ matrix $C = [c_{ij}]$ such that $C_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$ is said to be the product of the matrices A and B in that order and it is denoted by AB, *i.e.*, $C = AB$.

OSWAAL BOOKS LEARNING MADE SIMPLE

Properties of matrix multiplication :

- Note that the product AB is defined only when the number of columns in matrix A is equal to the number of rows in matrix B.
- If A and B are $m \times n$ and $n \times p$ matrices, respectively, then the matrix AB will be an $m \times p$ matrix, *i.e.*, order of matrix AB will be $m \times p$.
- In the product AB, A is called the **pre-factor** and B is called the **post-factor**.
- If two matrices A and B are such that AB is possible, then it is not necessary that the product BA is also possible.
- If A is an $m \times n$ matrix and both AB as well as BA are defined, then B will be an $n \times m$ matrix.
- If A is an $n \times n$ matrix and I_n be the unit matrix of order n, then $AI_n = I_n A = A$.
- Matrix multiplication is **associative**, *i.e.*, $A(BC) = (AB)C$.
- Matrix multiplication is **distributive** over the **addition**, *i.e.*, $A.(B+C) = AB + AC$.
- Matrix multiplication is not commutative.

➢ **Existence of non-zero matrices whose product is zero.**

The product of two matrices can be zero without either factor being a zero matrix.

e.g. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

Here, $A \neq 0$ and $B \neq 0$.

Also, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

➢ **Transpose of a Matrix :**

If $A = [a_{ij}]_{m\times m}$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a transpose of matrix A. The transpose of A is denoted by A' or A^T, *i.e.*, if $A^T = [a_{ji}]_{n\times m}$.

For example, if $A = \begin{bmatrix} 5 & -4 & 1 \\ 0 & \sqrt{5} & 3 \end{bmatrix}$ then $A^T = \begin{bmatrix} 5 & 0 \\ -4 & \sqrt{5} \\ 1 & 3 \end{bmatrix}$

Properties of Transpose of Matrices :

- $(A+B)^T = A^T+B^T$
- $(A^T)^T = A$
- $(kA)^T = kA^T$, where k is any constant
- $(AB)^T = B^TA^T$
- $(ABC)^T = C^TB^TA^T$

➢ **Symmetric matrix :** A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix if $A^T = A$, *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = [a_{ij}]$.

For example, $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $B = \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$

Note

- For any matrices AA^T and A^TA are symmetric matrices
- If A and B are two symmetric matrices of same order, then
  - (i) AB is symmetric if and only if $AB = BA$.
  - (ii) $A \pm B$, $AB + BA$ are also symmetric matrices.

➢ **Skew Symmetric Matrix :**

A square matrix $A = [a_{ij}]$ is said to be **a skew symmetric matrix** if $A^T = -A$ *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = -[a_{ij}]$.

For example, $A = \begin{bmatrix} 0 & 1 & -5 \\ -1 & 0 & 5 \\ 5 & -5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Note

- All the diagonal elements in a skew-symmetric matrix are zero.
- If A and B are two symmetric matrices, then $AB - BA$ is a skew symmetric matrix.

Know the Formulae

➢ For any square matrix A, the matrix $A + A^T$ is a symmetric and $A - A^T$ is always a skew symmetric matrix.

➢ A square matrix can be expressed as the sum of a symmetric and skew symmetric matrix, *i.e.*, $A = \frac{1}{2}(P) + \frac{1}{2}(Q)$, where $P = A + A^T$ is a symmetric matrix and $Q = A - A^T$ is a skew symmetric matrix.

Mnemonics

Type of Matrices

'Remember Crist Subah Dophar Syam'

R	C	S	D	S	N	I	T	E
↓	↓	↓	↓	↓	↓	↓	↓	↓
Row Matrix	**Column Matrix**	**Square Matrix**	**Digonal Matrix**	**Scalar Matrix**	**Null Matrix**	**Identity Matrix**	**Triangular Matrix**	**Equal Matrix**

1 Mark Questions

Q. 1 If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A^2 is equal to :

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

[ISC Specimen Paper, 2021]

Sol. Correct option : (b)

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Q. 2. $A = \left[a_{ij}\right]_{m \times n}$ *is a square matrix, if*

(a) $m < n$ (b) $m > n$

(c) $m = n$ (d) None of these

Sol. Correct option : (c)

Explanation : It is known that a given matrix is said to be a square matrix if the number of rows is equal to the number of columns.

Therefore,

$A = \left[a_{ij}\right]_{m \times n}$ is a square matrix, if $m = n$.

AI Q. 3. Which of the given values of x and y make the following pair of matrices equal

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

(a) $x = \frac{-1}{3}, y = 7$ (b) Not possible to find

(c) $y = 7, x = \frac{-2}{3}$ (d) $x = \frac{-1}{3}, y = \frac{-2}{3}$

Sol. Correct option : (b)

Explanation : It is given that

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix} = \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

Equating the corresponding elements, we get:

$$3x + 7 = 0$$

$$\Rightarrow \quad x = -\frac{7}{3}$$

$$5 = y - 2$$

$$\Rightarrow \quad y = 7$$

$$y + 1 = 8$$

$$\Rightarrow \quad y = 7$$

$$2 - 3x = 4$$

$$\Rightarrow \quad x = -\frac{2}{3}$$

We find that on comparing the corresponding elements of the two matrices, we get two different values of x, which is not possible.

Hence, it is not possible to find the values of x and y for which the given matrices are equal.

Q. 4. The number of all possible matrices of order 3 × 3 with each entry 0 or 1 is :

(a) 27 (b) 18

(c) 81 (d) 512

Sol. Correct option : (d)

Explanation : The given matrix of the order 3 × 3 has 9 elements and each of these elements can be either 0 or 1.

Now, each of the 9 elements can be filled in two possible ways.

Therefore, by the multiplication principle, the required number of possible matrices is $2^9 = 512$.

Q. 5. Suppose P and Q are two different matrices of order $3 \times n$ and $n \times p$, then the order of the matrix $P \times Q$ is ?

(a) $3 \times p$ (b) $p \times 3$

(c) $n \times n$ (d) 3×3

Sol. Correct option : (a)

Explanation :

Order of $P = 3 \times n$

Order of $Q = n \times p$

Since the number of columns of P is equal to number of rows of Q, PQ is defined.

Order of PQ is $3 \times p$

AI Q. 6. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then $A + A' = I$, then the value of α is :

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) π (d) $\frac{3\pi}{2}$

Sol. Correct option : (b)

Explanation:

Given that, $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Also $A + A' = I$

$$\Rightarrow \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries, we have

$$\Rightarrow 2\cos\alpha = 1$$

$$\Rightarrow \cos\alpha = \frac{1}{2}$$

$$\Rightarrow \cos\alpha = \cos\frac{\pi}{3}$$

$$\therefore \quad \alpha = \frac{\pi}{3}$$

Q. 7. If A is a square matrix such that $A^2=A$, then write the value of $7A-(I+A)^3$, where I is an identity matrix.

Sol. Since, $A^2 = A$

$$7A - (I + A)^3 = 7A - I^3 - 3A^2I - 3AI^2 - A^3$$

$$= 7A - I - 3A - 3A - A^2A$$

$$[\because I^3 = I^2 = I \text{ and } A^2I = AI = A]$$

$$= 7A - I - 3A - 3A - A.A$$

$$= 7A - I - 3A - 3A - A$$

$$= 7A - I - 7A = -I$$

Q. 8. If $\begin{bmatrix} 2x & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$, then find x.

Sol. Given, $\begin{bmatrix} 2x & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} 2x-9 & 4x \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$$

$$\Rightarrow [2x^2 - 9x + 12x] = 0$$

$$\Rightarrow 2x^2 + 3x = 0$$

$$\Rightarrow x(2x+3) = 0$$

$$\Rightarrow x = 0 \text{ or } x = -\frac{3}{2}$$

Q. 9. Give an example of a skew symmetric matrix of order 3.

Sol. A skew symmetric matrix of order 3 is

$$A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Q. 10. Write the element a_{23} of a 3 × 3 matrix $A = (a_{ij})$ whose elements a_{ij} are given by $a_{ij} = \frac{|i-j|}{2}$

Sol. Given, $a_{ij} = \frac{|i-j|}{2}$

$$\therefore \quad a_{23} = \frac{|2-3|}{2}$$

$$= \frac{1}{2}$$

AI 11. If $(2x \quad 4)\begin{pmatrix} x \\ -8 \end{pmatrix} = 0$, find the positive value of x.

Sol. $(2x \quad 4)\begin{pmatrix} x \\ -8 \end{pmatrix} = 0$

$$\Rightarrow (2x^2 - 32) = 0$$

By equating,

$$2x^2 - 32 = 0$$

$$\Rightarrow 2x^2 = 32$$

$$\Rightarrow x^2 = 16$$

$$\Rightarrow x = \pm 4 \Rightarrow x = 4$$

Q. 12. Find the matrix A if :

$$\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

Sol. Given, $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & -1-2 & 4+1 \\ -2+0 & 1-4 & 3-9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$$

AI Q. 13. Write the value of $x + y + z$ if

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Sol. Given, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad (\because IA = A)$$

By equating, $x = 1, y = -1, z = 0$

$$\therefore \quad x + y + z = 1 - 1 + 0 = 0$$

Q. 14. For a 2 × 2 matrix $A = [a_{ij}]$, whose elements are given by $a_{ij} = \frac{(i+2j)^2}{4}$, write the value of a_{21}.

Sol. Given, $a_{ij} = \frac{(i+2j)^2}{4}$

$$\therefore \quad a_{21} = \frac{[2+2(1)]^2}{4}$$

$$\Rightarrow a_{21} = \frac{[2+2]^2}{4}$$

$$\Rightarrow a_{21} = \frac{16}{4}$$

$$\Rightarrow a_{21} = 4$$

Q. 15. If $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$**, then find the matrix** A**.**

Sol. Given, $$3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$\Rightarrow \quad 3A - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 3A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$\Rightarrow \quad 3A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Q. 16. If matrix $A = [a_{ij}]_{2\times2}$**, where** $a_{ij} = \begin{cases} 2, & i \neq j \\ 0, & i = j \end{cases}$**, write the matrix** A**.**

Sol. Given, $$a_{ij} = \begin{cases} 2, & i \neq j \\ 0, & i = j \end{cases}$$

$$\therefore \quad a_{11} = a_{22} = 0, a_{12} = a_{21} = 2$$

Hence, $$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

2 Marks Questions

Q. 1. If $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$**, find** x**.**

Sol. Given $$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 2-6 & -6+12 \\ 5-14 & -15+28 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} -4 & 6 \\ -9 & 13 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$$

$$\Rightarrow \quad x = 13$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Mostly candidates commit errors in calculation as they confuses in multiplication of two matrices.

Answering Tip

- Give ample practice on problems based on multiplication of two matrices.

Q. 2. Show that $(A + A')$ **is symmetric matrix, if**

A = $\begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$ **[ISC, 2019]**

Sol.

Given, $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$. $\Rightarrow A' = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

$$\Rightarrow \quad A + A' = \begin{pmatrix} 4 & 7 \\ 7 & 10 \end{pmatrix}$$

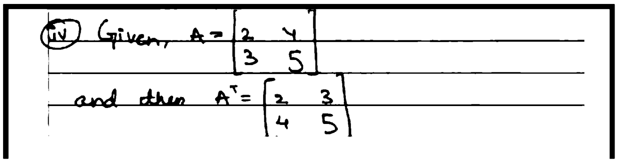
$$\Rightarrow \quad (A + A')' = \begin{pmatrix} 4 & 7 \\ 7 & 10 \end{pmatrix}$$

$$\because \quad (A + A') = (A + A')'$$

Therefore, $(A + A')$ is a symmetric matrix.

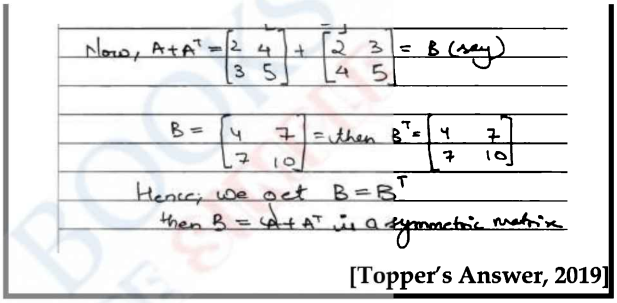
[ISC Marking Scheme, 2019]

OR



ⓘⓥ Given, $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

and then $A^T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$



Now, $A + A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = B$ (say)

$B = \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix}$ = then $B^T = \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix}$

Hence, we get $B = B^T$

then $B = A + A^T$ is a symmetric matrix

[Topper's Answer, 2019]

AI Q. 3. If $A = \begin{bmatrix} 5 & a \\ b & 0 \end{bmatrix}$ **and** A **is symmetric matrix, show that** $a = b$**.** **[ISC, 2018]**

Sol. Given, $$A = \begin{bmatrix} 5 & a \\ b & 0 \end{bmatrix}$$

If A is symmetric, then $A = A'$

Now, $$A' = \begin{bmatrix} 5 & b \\ a & 0 \end{bmatrix}$$

$$\because \quad A = A'$$

$$\therefore \quad \begin{bmatrix} 5 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} 5 & b \\ a & 0 \end{bmatrix}$$

On equating corresponding elements, we get

$a = b$ **Hence Proved.**

[ISC Marking Scheme 2018]

Examiner's Comment

- A few candidates could not solve the problem based on symmetric matrix.

Answering Tip

- Understand the basic concepts and definition of different types of matrices and thoroughly practice the problems based on fundamental concepts.

Q. 4. If the matrix $\begin{bmatrix} 6 & -x^2 \\ 2x-15 & 10 \end{bmatrix}$ **is symmetric, find the value of** x**.** **[ISC 2017]**

Sol. Let $A = \begin{bmatrix} 6 & -x^2 \\ 2x-15 & 10 \end{bmatrix}$

Since, given A is symmetric matrix *i.e.*, $A = A'$

$$\therefore \quad \begin{bmatrix} 6 & -x^2 \\ 2x-15 & 10 \end{bmatrix} = \begin{bmatrix} 6 & 2x-15 \\ -x^2 & 10 \end{bmatrix}$$

Equating corresponding elements, we get

$$-x^2 = 2x - 15$$

$$\Rightarrow \quad x^2 + 2x - 15 = 0$$

$$\Rightarrow \quad (x + 5)(x - 3) = 0$$

$$\Rightarrow \quad x = -5 \text{ or } x = 3$$

[ISC Marking Scheme, 2017]

Q. 5. Find the value of k if $M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $M^2 - kM - I_2 = 0$.

[ISC, 2015; 2018]

Sol. Given, $M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $M^2 - kM - I_2 = 0$

Here, $$M^2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$$

$$\because \quad M^2 = kM + I_2$$

$$\therefore \quad \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} = k\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} = \begin{bmatrix} k+1 & 2k \\ 2k & 3k+1 \end{bmatrix}$$

On equating corresponding elements, we get

$$k + 1 = 5$$

$$\Rightarrow \quad k = 4$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- A number of candidates wrote M^2 as $\begin{bmatrix} 1 & 4 \\ 4 & 9 \end{bmatrix}$ instead of $\begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$. Some candidates took '*k*' as matrix instead of '*k*' as scalar value while some candidates wrote I_2 as $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ which was not correct. I_2 was an identity matrix of the order 2 × 2.

Answering Tips

- Revise all matrix operations and practice matrix multiplication.
- Identity matrix, null matrix and their order should be understand thoroughly.

AI **Q. 6. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find the values of x and y such that $A^2 + xI_2 = yA$.** **[ISC, 2014]**

Sol. Given, $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$

$$\therefore \quad A^2 = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

$$\because \quad A^2 + xI_2 = yA$$

$$\therefore \quad \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 16+x & 8 \\ 56 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

Comparing the corresponding entries of equal matrices, we have $\Rightarrow y = 8$ and $16 + x = 3y$

$x = 3 \times 8 - 16$

$= 24 - 16$

$= 8$

$\Rightarrow y = 8$, $x = 8$ **[ISC Marking Scheme, 2014]**

Examiner's Comment

- Some candidates found A^2 by squaring the elements of A, instead of multiplying A by A. I_2 was taken as $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ by a few instead of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Answering Tip

- Practice fundamental matrix operations thoroughly.

Q. 7. If $(A - 2I)(A - 3I) = 0$, where $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find the value of x. **[ISC, 2013]**

Sol. Here,

$$A - 2I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix}$$

and $$A - 3I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix}$$

Given, $(A - 2I)(A - 3I) = 0$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2x-2 \\ -x+1 & x^2-5x+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equating, $2x - 2 = 0, -x + 1 = 0, x^2 - 5x + 4 = 0$

$$\Rightarrow \quad x = 1,\ x = 1,\ (x-1)(x-4) = 0$$

$$\Rightarrow \quad x = 1 \text{ or } x = 4$$

$$\therefore \quad x = 1$$

Q. 8. Solve for x and y, if $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} + 2\begin{bmatrix} 2x \\ 3y \end{bmatrix} = 3\begin{bmatrix} 7 \\ -3 \end{bmatrix}$. **[ISC, 2012]**

Sol. Given, $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} + 2\begin{bmatrix} 2x \\ 3y \end{bmatrix} = 3\begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} + \begin{bmatrix} 4x \\ 6y \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} x^2 + 4x \\ y^2 + 6y \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

By equating,

$\Rightarrow \quad x^2+4x=21, \qquad y^2+6y=-9$

$\Rightarrow \quad x^2+4x-21=0, \quad y^2+6y+9=0$

$\Rightarrow \quad (x+7)(x-3)=0, \qquad (y+3)^2=0$

$\Rightarrow \quad x=-7 \text{ or } x=3, \qquad y=-3$

AI **Q. 9. If $M(\theta)=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$, show that $M(x).M(y)=M(x+y)$. [ISC, 2009]**

Sol. $\text{L.H.S.}=M(x)\cdot M(y)$

$$=\begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}\cdot\begin{bmatrix}\cos y & \sin y\\ -\sin y & \cos y\end{bmatrix}$$

$$=\begin{bmatrix}\cos x\cos y-\sin x\sin y & \cos x\sin y+\sin x\cos y\\ -(\sin x\cos y+\cos x\sin y) & \cos x\cos y-\sin x\sin y\end{bmatrix}$$

$$=\begin{bmatrix}\cos(x+y) & \sin(x+y)\\ -\sin(x+y) & \cos(x+y)\end{bmatrix}$$

$=M(x+y)=\text{R.H.S.}$

Hence Proved.

Q. 10. Find x and y if $x+y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ and $x-y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$. [ISC, 2008]

Sol. Given, $x+y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ ...(i)

and $x-y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$ ...(ii)

Adding (i) and (ii),

$$2x=\begin{bmatrix}10 & 0\\ 2 & 8\end{bmatrix} \quad \therefore \quad x=\begin{bmatrix}5 & 0\\ 1 & 4\end{bmatrix}$$

Subtracting (ii) from (i),

$$2y=\begin{bmatrix}4 & 0\\ 2 & 2\end{bmatrix} \quad \therefore \quad y=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$$

Q. 11. If $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$, show that $A^2-3I=2A$. [ISC, 2007]

Sol. Given, $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$

$$\therefore \quad A^2=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}=\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}$$

$$3I=3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

Now, $A^2-3I=\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}-\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$

$$=\begin{bmatrix}2 & 4\\ 4 & 2\end{bmatrix}=2\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$$

$=2A$ **Hence Proved.**

AI **Q. 12. If $\begin{bmatrix}x-y & 2y\\ 2y+z & x+y\end{bmatrix}=\begin{bmatrix}1 & 4\\ 9 & 5\end{bmatrix}$, write the value of $x+y+z$.**

Sol. $\begin{bmatrix}x-y & 2y\\ 2y+z & x+y\end{bmatrix}=\begin{bmatrix}1 & 4\\ 9 & 5\end{bmatrix}$

By equating,

$x-y=1$ ...(i)

and $2y=4\Rightarrow y=2$ ...(ii)

Put $y=2$ in (i) $\Rightarrow x-2=1\Rightarrow x=3$

$2y+z=9$

$\Rightarrow \quad 2(2)+z=9$

$\Rightarrow \quad z=9-4\Rightarrow z=5$

$\therefore \quad x+y+z=3+2+5=10$

Q. 13. For what value of k, the matrix $\begin{bmatrix}2k+3 & 4 & 5\\ -4 & 0 & -6\\ -5 & 6 & -2k-3\end{bmatrix}$ is a skew symmetric matrix?

Sol. Since given matrix $\begin{bmatrix}2k+3 & 4 & 5\\ -4 & 0 & -6\\ -5 & 6 & -2k-3\end{bmatrix}$ is a skew symmetric matrix,

then $A=-A^T$

$$\Rightarrow \begin{bmatrix}2k+3 & 4 & 5\\ -4 & 0 & -6\\ -5 & 6 & -2k-3\end{bmatrix}=-\begin{bmatrix}2k+3 & 4 & 5\\ -4 & 0 & -6\\ -5 & 6 & 2k+3\end{bmatrix}^T$$

$$\Rightarrow \begin{bmatrix}2k+3 & 4 & 5\\ -4 & 0 & -6\\ -5 & 6 & -2k-3\end{bmatrix}=-\begin{bmatrix}2k+3 & -4 & -5\\ 4 & 0 & 6\\ 5 & -6 & -2k-3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2k+3 & 4 & 5\\ -4 & 0 & -6\\ -5 & 6 & -2k-3\end{bmatrix}=\begin{bmatrix}-2k-3 & 4 & 5\\ -4 & 0 & -6\\ -5 & 6 & 2k+3\end{bmatrix}$$

By equating, we get $2k+3=-2k-3$

i.e., $2(2k+3)=0\Rightarrow 2k+3=0\Rightarrow k=-\frac{3}{2}$

Q. 14. If $A^T=\begin{bmatrix}3 & 4\\ -1 & 2\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$, then find A^T-B^T.

Sol. Given $A^T=\begin{bmatrix}3 & 4\\ -1 & 2\\ 0 & 1\end{bmatrix}$

and $B=\begin{bmatrix}-1 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$

$\Rightarrow \quad B^T=\begin{bmatrix}-1 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}$

$$\Rightarrow \quad A^T-B^T=\begin{bmatrix}3 & 4\\ -1 & 2\\ 0 & 1\end{bmatrix}-\begin{bmatrix}-1 & 1\\ 2 & 2\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}4 & 3\\ -3 & 0\\ -1 & -2\end{bmatrix}$$

Q. 15. Simplify :

$$\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

Sol. $\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$

$$=\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ -\cos\theta\sin\theta & \cos^2\theta\end{bmatrix}+\begin{bmatrix}\sin^2\theta & -\sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\sin\theta\cos\theta\\ -\cos\theta\sin\theta+\sin\theta\cos\theta & \cos^2\theta+\sin^2\theta\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

OSWAAL BOOKS LEARNING MADE SIMPLE

Q. 16. If matrix $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$, then write the value of λ.

Sol. Given, $A^2 = \lambda A$, where $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+9 & -9-9 \\ -9-9 & 9+9 \end{bmatrix} = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix}$$

Since, $A^2 = \lambda A \Rightarrow \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \begin{bmatrix} 3\lambda & -3\lambda \\ -3\lambda & 3\lambda \end{bmatrix}$

By equating corresponding elements, we get

$\Rightarrow \quad 18 = 3\lambda \Rightarrow \lambda = 6$

4 Marks Questions

Q. 1. Express the matrix $A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrix.

Sol.
$$A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$$

then
$$A' = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}$$

Let
$$P = \frac{1}{2}(A + A')$$
$$= \frac{1}{2}\left(\begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}\right)$$
$$= \frac{1}{2}\begin{bmatrix} 4 & 11 & -5 \\ 11 & 6 & 3 \\ -5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix}$$

Since, $P' = P$

$\therefore$ P is a symmetric matrix.

Let
$$Q = \frac{1}{2}(A - A')$$
$$= \frac{1}{2}\left(\begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}\right)$$
$$= \frac{1}{2}\begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 7 \\ 7 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}$$

Since, $Q' = -Q$

$\therefore$ Q is a skew symmetric matrix.

Also, $\quad P + Q$

$$= \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$$

AI Q. 2. Find the values of x, y, and z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A'A = I$.

Sol. Here,
$$A' = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

Given, $A'A = I$

$$\Rightarrow \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^2+x^2 & xy-xy & -xz+xz \\ xy-xy & 4y^2+y^2+y^2 & 2yz-yz-yz \\ -zx+zx & 2yz-zy-zy & z^2+z^2+z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On equating corresponding elements, we get

$\Rightarrow \quad 2x^2 = 1, 6y^2 = 1$ and $3z^2 = 1$

$\Rightarrow \quad x^2 = \frac{1}{2}, y^2 = \frac{1}{6}$ and $z^2 = \frac{1}{3}$

$\Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{6}}$ and $z = \pm\frac{1}{\sqrt{3}}$

Q. 3. Solve the matrix equation $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ -9 \end{bmatrix}$.

Sol. We have, $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ -9 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} x^2-3x \\ y^2-6y \end{bmatrix} = \begin{bmatrix} -2 \\ -9 \end{bmatrix}$$

By equating corresponding elements, we get

$$\begin{aligned} x^2-3x &= -2 \text{ and } y^2-6y = -9 \\ \Rightarrow \quad x^2-3x+2 &= 0 \text{ and } y^2-6y+9 = 0 \\ \Rightarrow \quad x^2-2x-x+2 &= 0 \text{ and } y^2-3y-3y+9 = 0 \\ \Rightarrow \quad x(x-2)-1(x-2) &= 0 \text{ and } y(y-3)-3(y-3) = 0 \\ \Rightarrow \quad (x-2)(x-1) &= 0 \text{ and } (y-3)(y-3) = 0 \\ \therefore \quad x &= 1, 2 \text{ and } y = 3 \end{aligned}$$

Q. 4. Find the value of X and Y if

$$X+Y = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}, X-Y = \begin{bmatrix} 6 & 5 \\ 7 & 3 \end{bmatrix}.$$

Sol. We have, $X+Y = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}, X-Y = \begin{bmatrix} 6 & 5 \\ 7 & 3 \end{bmatrix}$

$$(X+Y)+(X-Y) = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 5 \\ 7 & 3 \end{bmatrix}$$

$$2X = \begin{bmatrix} 8 & 8 \\ 12 & 4 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} 4 & 4 \\ 6 & 2 \end{bmatrix}$$

and $$(X+Y)-(X-Y) = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 7 & 3 \end{bmatrix}$$

$$2Y = \begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$$

$$Y = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}$$

Q. 5. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix}$, show that $AB \neq BA$.

Sol. We have, $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0 & 8+2 \\ -3+0 & 12+8 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 10 \\ -3 & 20 \end{bmatrix}$$

and $$BA = \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2+12 & -1+16 \\ 0+6 & 0+8 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 15 \\ 6 & 8 \end{bmatrix} \neq AB$$

$\therefore \quad AB \neq BA$ **Hence Proved.**

AI Q. 6. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = 0$.

[ISC, 2015]

Sol. Getting $A^2 = A \cdot A$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\therefore A^2 - 5A + 4I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix}$$

Since, $A^2 - 5A + 4I + X = 0$

$$\therefore \quad X = -(A^2 - 5A + 4I)$$

or $$X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- Most of the students attempt this question incorrectly. They make errors while calculating A^2. Sometimes they add two matrices to solve A^2.

Answering Tip

- Stress upon developing logical and reasoning skills to apply the connect property of matrix.

TOPIC-2
Invertible Matrices and Martin's Rule

Revision Notes

- **Determinant :** A unique number (real or complex) can be associated to every square matrix is known as its determinant. The determinant of matrix A is denoted by det A or $|A|$.

OSWAAL BOOKS LEARNING MADE SIMPLE

eg. : **(i)** If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2,

then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

(ii) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3,

then $|A| = (-1)^{1+1} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

- **Singular Matrix & Non-Singular Matrix :**

  (a) Singular matrix: A square matrix A is said to be singular if $|A| = 0$, *i.e.*, its determinant is zero.

  e.g.,
  $$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{vmatrix}$$
  $$= 1(15-12) - 2(12-12) + 3(4-5)$$
  $$= 3 - 0 - 3 = 0$$

  $\therefore$ A is singular matrix.

  Scan to know more about this topic

  Invertible matrices

  (b) Non-singular Matrix : A square matrix A is said to be non-singular if $|A| \neq 0$.

  e.g.,
  $$A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$
  $$= 0(0-1) - 1(0-1) + 1(1-0)$$
  $$= 0 + 1 + 1 = 2 \neq 0$$

  $\therefore$ A is a non-singular matrix.

  Scan to know more about this topic

  Martin's rule

- ● A square matrix A is **invertible** if and only if A is **non-singular.**
- **Minors :** Minors of an element a_{ij} of a determinant (*or a determinant corresponding to matrix A*) is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence, we can get 9 minors corresponding to the 9 elements of a third order (*i.e.*, 3 × 3) determinant.
- **Co-factors :** Cofactor of an element a_{ij} denoted by A_{ij} is defined by $A=(-1)^{(i+j)}M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the cofactor of element a_{ij}.
- **Adjoint of a Square Matrix :**

  Let $A = [a_{ij}]$ be a square matrix. Also, assume $B = [A_{ij}]$, where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the **adjoint of matrix** A and it is denoted by ***adj*** **(*A*)**.

  To find adjoint of a 2 × 2 matrix: If the adjoint of a square matrix of order 2 can be obtained by interchanging the diagonal elements and changing the signs of off-diagonal elements, *i.e.*, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

  For example, consider a square matrix of order 3 as $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then in order to find the adjoint matrix A, we find a matrix B (formed by the co-factors of elements of matrix A as mentioned above in the definition)

  i.e., $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adj A = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$

- **Algorithm to find A^{-1} by Determinant Method:**

  Step 1 : Find $|A|$.

  Step 2 : If $|A| = 0$, then, write "A is a singular matrix and hence not invertible". Else write "A is non-singular matrix and hence invertible".

  Step 3 : Calculate the co-factors of elements of matrix A.

  Step 4 : Write the matrix of co-factors of elements of A and then obtain its transpose to get *adj A* (*i.e.*, adjoint A).

Step 5 : Find the inverse of A by using the relation : $A^{-1}=\frac{1}{|A|}(adj\,A)$

- **Properties associated with various Operations of Matrices & Determinants:**

  (a) $AB = I = BA$
  (b) $AA^{-1} = I$ or $A^{-1}I = A^{-1}$
  (c) $(AB)^{-1} = B^{-1}A^{-1}$
  (d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
  (e) $(A^{-1})^{-1} = A$
  (f) $(A^T)^{-1} = (A^{-1})^T$
  (g) $A(adj\,A) = (adj\,A)\,A = |A|\,I$
  (h) $adj(AB)=adj\,(B)\,adj\,(A)$
  (i) $adj\,(A^T)= (adj\,A)^T$
  (j) $(adj\,A)^{-1} = (adj\,A^{-1})$
  (k) $|adj\,A| = |A|^{n-1},\ if\ |A| \neq 0$, where n is of the order of A
  (l) $|AB| = |A|\,|B|$
  (m) $|A\ adj\,A| = |A|^n$, where n is of the order of A
  (n) $|A^{-1}|=\frac{1}{|A|}$, if matrix A is invertible
  (o) $|A|= |A^T|$

- $|kA| = k^n|A|$, where n is of the order of square matrix A and k is any scalar.
- If A is non-singular matrix of order n, then $adj\ (adj$ A) $= |A|^{n-2} A$.

- **Uniqueness Theorem :** Let A be an invertible square matrix of order n. Suppose B and C are the two inverse of A.

Then $\quad AB = BA = I_n \quad$ (by definition of inverse matrix)

$$AC = CA = I_n$$

Now, $\quad B = BI_n = B(AC) \quad [\because$ Matrix multiplication is associative]

$$\begin{aligned} &= (BA)C \\ &= I_nC \\ &= C \end{aligned}$$

$\therefore B = C$, *i.e.*, any two inverse of A are equal matrices.

Hence, the inverse of A is unique.

- **Solving System of Equations by Matrix Method [Martin's Rule]**

  (a) Consistent and Inconsistent system : A system of equations is consistent if it has one or more solutions otherwise it is said to be an inconsistent system. In other words, an inconsistent system of equations has no solution.

  (b) Homogeneous and Non-homogeneous system : A system of equations $AX= B$ is said to be a homogeneous system if $B = 0$. Otherwise it is called a non-homogeneous system of equations.

  Let given system of equations is :

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Step 1: Assume $\quad A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Step 2 : Find $|A|$. Now there may be following situations :

(i) If $|A|\neq 0 \Rightarrow A^{-1}$ exists, then the given system of equations is consistent and therefore, the system has **unique solution.** In that case, write

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B \quad \left[\text{where } A^{-1}=\frac{1}{|A|}(adj\,A)\right]$$

(i) If $|A| = 0 \Rightarrow A^{-1}$ does not exist, then the given system of equations is either inconsistent or it has infinitely many solutions. In order to check proceed as follow :

- Compute $(adj\,A)\,B$.
- If $(adj\,A)B \neq 0$, then the given system of equations is inconsistent, *i.e.*, it has no solution.
- If $(adj\,A)B = 0$, then the given system of equations is consistent with infinitely many solutions.
  [In order to find these infinitely many solutions, replace one of the variables by some real number and proceed in the same manner in the new two variables system of equations]

OSWAAL BOOKS LEARNING MADE SIMPLE

1 Mark Questions

Q.1. For what value of k inverse does not exist for the matrix $\begin{bmatrix} 1 & 2 \\ k & 6 \end{bmatrix}$:

(a) 0 (b) 3 (c) 6 (d) 2

[ISC Specimen Paper, 2021]

Sol. Correct Option: (b)

Explanation:

Given inverse of $\begin{bmatrix} 1 & 2 \\ k & 6 \end{bmatrix}$ does not exist

$\Rightarrow$ matrix is singular

$$\begin{bmatrix} 1 & 2 \\ k & 6 \end{bmatrix} = 0$$

$$\Rightarrow \quad 6 - 2k = 0$$

$$\Rightarrow \quad k = 3$$

AI **Q. 2. Matrix which does not have an inverse by solving it, is classified as which of the following?**

(a) unidentified matrix (b) linear matrix

(c) non-singular matrix (d) singular matrix

Sol. Correct Option: (d)

Explanation: This is because singular matrices do not have inverse, because the value of their determinant is zero.

Q. 3. Multiplicative inverse matrix generalized form to solve is which of the following?

(a) $AA^{-1} = A^{-1}A = I$ (b) $AA^{-2} = A^{-3}A = I$

(c) $3AA^4 = A^{-1}A = I$ (d) $AA^{-1} = 2A^{-5}A = Q$

Sol. Correct Option: (a)

Explanation: This is because product of a matrix and its inverse is always an identity matrix.

AI **Q. 4. Which of the following relation describes the solution of system of equation in form of $AX = B$?**

(a) $A^{-1}AX = A^{-1}B$ (b) $B^{-1}AX = A^{-1}B$

(c) $AB^{-1}AX = AB^{-1}X$ (d) $X^{-1}AB = A^{-1}BX$

Sol. Correct Option: (a)

Explanation: We have, $A^{-1}AX = A^{-1}B$ which is same as $X = A^{-1}B$, and this is the primary relation employed for finding inverse.

Q. 5. If A is an invertible square matrix of order 3 and $|A| = 5$, then find the value of $|adj\,A|$.

Sol. $|adj\,A| = |A|^{n-1}$, where n is the order of the matrix

$$\therefore \quad |adj\,A| = 5^2 \Rightarrow |adj\,A| = 25$$

Q. 6. If A is a square matrix and $|A| = 2$, then write the value of $|AA'|$, where A' is the transpose of matrix A.

Sol. $\because \quad |AA'| = |A||A'|$

$$= 2 \times 2 \qquad [\because |A| = |A'|]$$

$$= 4$$

Q. 7. If A is a square matrix of order 3 such that $|adj\,A| = 225$, find $|A'|$.

Sol. $|adj\,A| = |A|^{n-1}$, where n is the order of the matrix.

$$|A|^2 = 15^2 \Rightarrow |A| = \pm 15 \Rightarrow |A'| = \pm 15$$

$$[\because |A| = |A'|]$$

Q. 8. The value of the determinant of a matrix A of order 3×3 is 4. Find the value of $|5A|$.

Sol. Since, $\quad |kA| = k^n|A|$

$$\therefore \quad |5A| = 5^3(4) = 125(4) = 500$$

Q. 9. A is a square matrix of order 3 and $|2A| = k|A|$, then find the value of k.

Sol. Since $|kA| = k^n|A|$

$$\therefore \quad |2A| = 2^3|A| \qquad ...(i)$$

$$\therefore \text{ Given,} \quad |2A| = k|A| \qquad ...(ii)$$

From eq. (i) and (ii), we get

$$k = 2^3 = 8$$

Q. 10. For what value of x, the matrix $\begin{bmatrix} 1+x & 7 \\ 3-x & 8 \end{bmatrix}$ is a singular matrix ?

Sol. Since, A is a singular matrix

$$\begin{vmatrix} 1+x & 7 \\ 3-x & 8 \end{vmatrix} = 0$$

$$\Rightarrow \quad 8(1+x) - 7(3-x) = 0$$

$$\Rightarrow \quad 8 + 8x - 21 + 7x = 0$$

$$\Rightarrow \quad 15x - 13 = 0$$

$$\Rightarrow \quad x = \frac{13}{15}$$

2 Marks Questions

Q. 1. Find the matrix X for which :

$$\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

[ISC, 2016]

Sol. Given, $\quad \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

It is of the form $AX = B$, *i.e.*, $A^{-1}AX = A^{-1}B$ $\Rightarrow X = A^{-1}B$

Now, $\quad \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$

$$\left[\because \text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$$

$$\therefore \quad X = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\text{or} \quad X = \begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Errors were made by candidates while finding inverse and correct $X = A^{-1}B$ form.

Answering Tip

- Understand basic operations of matrices. More practice is required in inverse and multiplication of matrix by matrix.

Q. 2. If the matrix $A = \begin{bmatrix} 6 & x & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{bmatrix}$ **is a singular matrix, find the value of** x. **[ISC, 2010]**

Sol. A is a singular matrix $\Rightarrow |A| = 0$

$$\therefore \quad \begin{vmatrix} 6 & x & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad 6(-2-10) - x(4+20) + 2(10-10) = 0$$

$$\Rightarrow \quad -72 - 24x = 0$$

$$\Rightarrow \quad x = -3$$

Q. 3. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, **show that** $A^{-1} = \dfrac{1}{19}A$.

Sol. Given, $$A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -4 - 15$$

$$= -19 \neq 0$$

$\therefore A^{-1}$ exists.

Now, co-factors of elements of $|A|$ are :

$A_{11} = -2, A_{12} = -5, A_{21} = -3, A_{22} = 2$

$$\therefore \quad adj\,A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

Now, $$A^{-1} = \frac{1}{|A|} adj\,A$$

$$\Rightarrow \quad A^{-1} = \frac{1}{-19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{19}A$$ **Hence Proved.**

Q. 4. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, **find the numbers** a **and** b **such that** $A^2 + aA + bI = 0$. **Hence, find** A^{-1}.

Sol. Given, $$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Now, $A^2 + aA + bI = 0$ ...(i)

$$\Rightarrow \quad \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 11+3a+b & 8+2a+0 \\ 4+a+0 & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now, equating corresponding elements, we get

$11 + 3a + b = 0, 8 + 2a = 0, 4 + a = 0, 3 + a + b = 0$

$$\Rightarrow \quad a = -4 \text{ and } b = 1$$

Putting values of 'a' and 'b' in eq (i), we get

$$A^2 - 4A + I = 0$$

$$\Rightarrow \quad 4A - A^2 = I$$

$$\Rightarrow \quad A(4I - A) = I$$

$$\Rightarrow \quad A^{-1} = 4I - A$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Q. 5. If $A = \begin{bmatrix} 4 & 6 \\ 7 & 5 \end{bmatrix}$, **then what is the value of** $A.(adj\,A)$ **?**

Sol. Given, $$A = \begin{bmatrix} 4 & 6 \\ 7 & 5 \end{bmatrix}$$

$$adj\,A = \begin{bmatrix} 5 & -6 \\ -7 & 4 \end{bmatrix}$$

$$A \cdot (adj\,A) = \begin{bmatrix} 4 & 6 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 5 & -6 \\ -7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 20-42 & -24+24 \\ 35-35 & -42+20 \end{bmatrix}$$

$$= \begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix}$$

Q. 6. If $A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$, **then write** A^{-1}.

Sol. $$A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\,A)$$

Here, $$|A| = \begin{vmatrix} 3 & 10 \\ 2 & 7 \end{vmatrix} = 21 - 20 = 1$$

and $$adj\,A = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$$

Hence, $$A^{-1} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$$

OSWAAL BOOKS LEARNING MADE SIMPLE

4 Marks Questions

Q. 1. Test the consistency for the system of equations :

$$4x - 5y - 2z = 2$$
$$5x - 4y + 2z = -2$$
$$2x + 2y + 8z = -1$$

[ISC, 2007]

Sol. The given system of equations can be written in matrix form as :

$$\begin{bmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

i.e., $AX = B$

where, $A = \begin{bmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{vmatrix}$

$$= 4(-32 - 4) + 5(40 - 4) - 2(10 + 8)$$
$$= -144 + 180 - 36 = -180 + 180 = 0$$

Since $|A| = 0$, A^{-1} does not exist and system has either no solution or infinite number of solutions.

To check, we find $(adj\ A)\ B$

Co-factors of elements $|A|$ are :

$A_{11} = -36, A_{12} = -36, A_{13} = 18, A_{21} = 36, A_{22} = 36, A_{23} = -18, A_{31} = -18, A_{32} = -18, A_{33} = 9$

$$\therefore \quad adj\ A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -36 & 36 & -18 \\ -36 & 36 & -18 \\ 18 & -18 & 9 \end{bmatrix}$$

Now, $(adj\ A).B = \begin{bmatrix} -36 & 36 & -18 \\ -36 & 36 & -18 \\ 18 & -18 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} -72 - 72 + 18 \\ -72 - 72 + 18 \\ 36 + 36 - 9 \end{bmatrix} = \begin{bmatrix} -126 \\ -126 \\ 63 \end{bmatrix} \neq 0$$

Since $|A| = 0$ and $(adj\ A).B \neq 0$, therefore given system has no solution and hence has inconsistent solutions.

Q 2. Use matrix method to examine the following system of equations for consistency or inconsistency.

$$2x + 5y = 7$$
$$6x + 15y = 13$$

Sol. Given system of equation is

$$2x + 5y = 7$$
$$6x + 15y = 13$$

It can be written as

$$\begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$$

i.e., $AX = B$

where, $A = \begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & 5 \\ 6 & 15 \end{vmatrix} = 30 - 30 = 0$

$\because$ A is singular, then the given system has no solution or has infinite number of solutions.

To check, we find $(adj\ A)\ B$.

Co-factors of elements of $|A|$ are :

$A_{11} = 15, A_{12} = -6, A_{21} = -5, A_{22} = 2$

$$\therefore \quad adj\ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}$$

$$\therefore \quad (adj\ A)\ B = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} 105 - 65 \\ -42 + 26 \end{bmatrix} = \begin{bmatrix} 40 \\ -16 \end{bmatrix} \neq 0$$

Hence, the given system has no solution and is therefore inconsistent.

AI Q. 3. If $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ and $A.(adj\mathrm{A}) = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then find the value of k.

Sol. Given, $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

To find $adj\ A$, let us find co-factors of $|A|$.

$\therefore$ Co-factors of $|A|$ are :

$$A_{11} = (-1)^{1+1}\cos x = \cos x$$
$$A_{12} = (-1)^{1+2}(-\sin x) = \sin x$$
$$A_{21} = (-1)^{2+1}(\sin x) = -\sin x$$
$$A_{22} = (-1)^{2+2}\cos x = \cos x$$

$$\therefore \quad adj\ A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}^T = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

Now given, $A.adj(A) = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos^2 x + \sin^2 x & -\sin x\cos x + \sin x\cos x \\ -\sin x\cos x + \sin x\cos x & \sin^2 x + \cos^2 x \end{bmatrix} = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\Rightarrow \quad k = 1$$

Q. 4. Three schools *A, B* and *C* organised a *mela* for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled material at a cost of ₹ 25, ₹ 100 and ₹ 50 each. The number of articles sold are given below :

School / Articles	*A*	*B*	*C*
Fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by selling the above articles. Also, find the total funds collected for the purpose.

Sol.
$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} HF & M & P \end{array} \\ \begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix} \end{array}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 1000+5000+1000 \\ 625+4000+1500 \\ 875+5000+2000 \end{bmatrix}$$

$$= \begin{bmatrix} 7000 \\ 6125 \\ 7875 \end{bmatrix}$$

Funds collected by school *A* : ₹ 7,000

Funds collected by school *B* : ₹ 6,125

Funds collected by school *C* : ₹ 7,875

Total collection: ₹ 7,000 + ₹ 6,125 + ₹ 7,875 = ₹ 21,000

Q. 5. A trust fund has ₹ 35,000 is to be invested in two different types of bonds. The first bond pays 8% interest per annum which will be given to orphanage and second bond pays 10% interest per annum which will be given to an N.G.O. (Cancer Aid Society). Use matrix multiplication, determine how to divide ₹ 35,000 among two types of bonds if the trust fund obtains an annual total interest of ₹ 3,200.

Sol. Let investment in first type of bond be ₹ x.

$\therefore$ The investment in second type of bond

$$= ₹\,(35{,}000 - x)$$

$$\therefore \quad \begin{bmatrix} x & 35{,}000 - x \end{bmatrix}\begin{bmatrix} \frac{8}{100} \\ \frac{10}{100} \end{bmatrix} = [3{,}200]$$

or $$\frac{8}{100}x + (35{,}000 - x)\frac{10}{100} = 3{,}200$$

or $$x = ₹\,15{,}000$$

$\therefore$ Investment in first bond = ₹ 15,000

and investment in second bond

$$= ₹(35{,}000 - 15{,}000)$$

$$= ₹\,20{,}000$$

Q. 6. To promote the making of toilets for women, an organisation tried to generate awareness through (i) house calls (ii) letters and (iii) announcements. The cost for each mode per attempt is given below :

(i) ₹ 50 (ii) ₹ 20 (iii) ₹ 40

The number of attempts made in three villages *X*, *Y* and *Z* are given below :

	(i)	(ii)	(iii)
X	400	300	100
Y	300	250	75
Z	500	400	150

Find the total cost incurred by the organisation for the three villages separately, using matrices.

Sol. Let *A* be the matrix that represent the number of attempts made in three villages *X*, *Y* and *Z* and let *B* be the matrix that represent the cost for each mode per attempt.

Then, the matrices *A* and *B* can be represented as

$$\begin{array}{c} \\ A = \end{array}\begin{array}{c} \begin{array}{ccc} \text{House} & \text{Calls Letters} & \text{Announcements} \end{array} \\ \begin{bmatrix} 400 & 300 & 100 \\ 300 & 250 & 75 \\ 500 & 400 & 150 \end{bmatrix} \end{array}$$

and $$B = \begin{bmatrix} 50 \\ 20 \\ 40 \end{bmatrix}$$

Now, the cost incurred by the organisation for three villages can be represented as.

$$AB = \begin{bmatrix} 400 & 300 & 100 \\ 300 & 250 & 75 \\ 500 & 400 & 150 \end{bmatrix}\begin{bmatrix} 50 \\ 20 \\ 40 \end{bmatrix}$$

$$= \begin{bmatrix} 400 \times 50 + 300 \times 20 + 100 \times 40 \\ 300 \times 50 + 250 \times 20 + 75 \times 40 \\ 500 \times 50 + 400 \times 20 + 150 \times 40 \end{bmatrix}$$

$$= \begin{bmatrix} 20000 + 6000 + 4000 \\ 15000 + 5000 + 3000 \\ 25000 + 8000 + 6000 \end{bmatrix} = \begin{bmatrix} 30000 \\ 23000 \\ 39000 \end{bmatrix}$$

Thus, the total cost incurred by the organisation for the three villages *X*, *Y* and *Z* separately are ₹ 30000, ₹ 23000 and ₹ 39000, respectively.

6 Marks Questions

Q. 1. Using matrices, solve the following system of equations

$2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$ and $x + y - 2z = -3$

Sol. The given system of linear equation

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Now, $$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

We know,

$$AX = B$$

$$\therefore \quad X = A^{-1}B$$

Now, $|A| = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} = -1 \neq 0$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -4+4 = 0$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -(-6+4) = 2$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = (3-2) = 1$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -(6-5) = -1$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = (-4-5) = -9$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2+3) = -5$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = (12-10) = 2$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = -(-8-15) = 23$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = (4+9) = 13$$

$$\therefore \quad \text{adj}\, A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj}\, A}{|A|}$$

$$= \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore \quad X = A^{-1}B$$

$$X = -\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = -\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = 1, y = 2, z = 3$

Q. 2. Solve the following system of linear equations using matrices :
$x - 2y = 10, 2x - y - z = 8, -2y + z = 7$ [ISC, 2020]

Sol. $x - 2y = 10, 2x - y - z = 8$ and $-2y + z = 7$

Equations can be rearrange in matrix form

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$AX = B$$

$$\left[\text{where } A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ x \end{bmatrix} \text{ and } B = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}\right]$$

Now, $A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= 1\begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix}$$

$= 1(-1-2) + 2(2+0) + 0$

$= -3 + 4 = 1$

$|A| = 1$ A^{-1} exist.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} = -1-2 = -3$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = -(2+0) = -2$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} = -4+0 = -4$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} = -(-2+0) = 2$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1(1-0) = 1$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 0 & -2 \end{vmatrix} = -(-2+0) = 2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 0 \\ -1 & -1 \end{vmatrix} = 2+0 = 2$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -(-1-0) = 1$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = (-1+4) = 3$$

$$A_{adj} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}^t = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{A_{adj}}{|A|} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}$$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -30+16+14 \\ -20+8+7 \\ -40+16+21 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

$\therefore$ $x = 0, y = -5$ and $z = -3$

[ISC Marking Scheme, 2020]

Q. 3. Solve the following system of linear equations using matrix method:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9;\ \frac{2}{x}+\frac{5}{y}+\frac{7}{z}=52;\ \frac{2}{x}+\frac{1}{y}-\frac{1}{z}=0$$

[ISC, 2019]

Sol. The equations are

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 9$$

$$\frac{2}{x}+\frac{5}{y}+\frac{7}{z} = 52$$

$$\frac{2}{x}+\frac{1}{y}-\frac{1}{z} = 0$$

$$\frac{1}{x} = a,\ \frac{1}{y} = b,\ \frac{1}{z} = c$$

Equation are $a + b + c = 9$

$2a + 5b + 7c = 52$

$2a + b - c = 0$

Matrix equation

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix},\ X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$AX = B$

$|A| = = -4 \neq 0$

$X = A^{-1} B$

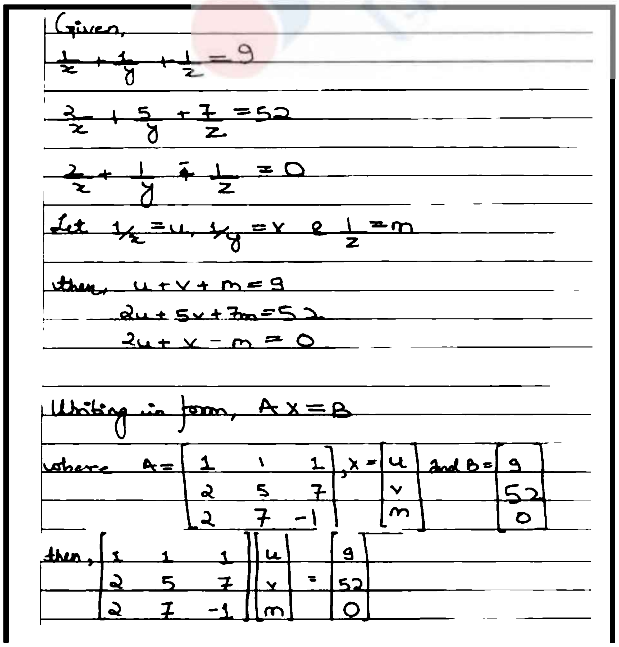
$$A^{-1} = -\frac{1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -4 \\ -12 \\ -20 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$a = 1, b = 3, c = 5 \Rightarrow x = 1, y = \frac{1}{3}, z = \frac{1}{5}$$

[ISC Marking Scheme, 2019]

OR



Given,

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9$

$\frac{2}{x} + \frac{5}{y} + \frac{7}{z} = 52$

$\frac{2}{x} + \frac{1}{y} - \frac{1}{z} = 0$

Let $1/x = u$, $1/y = v$ & $\frac{1}{z} = m$

then, $u + v + m = 9$

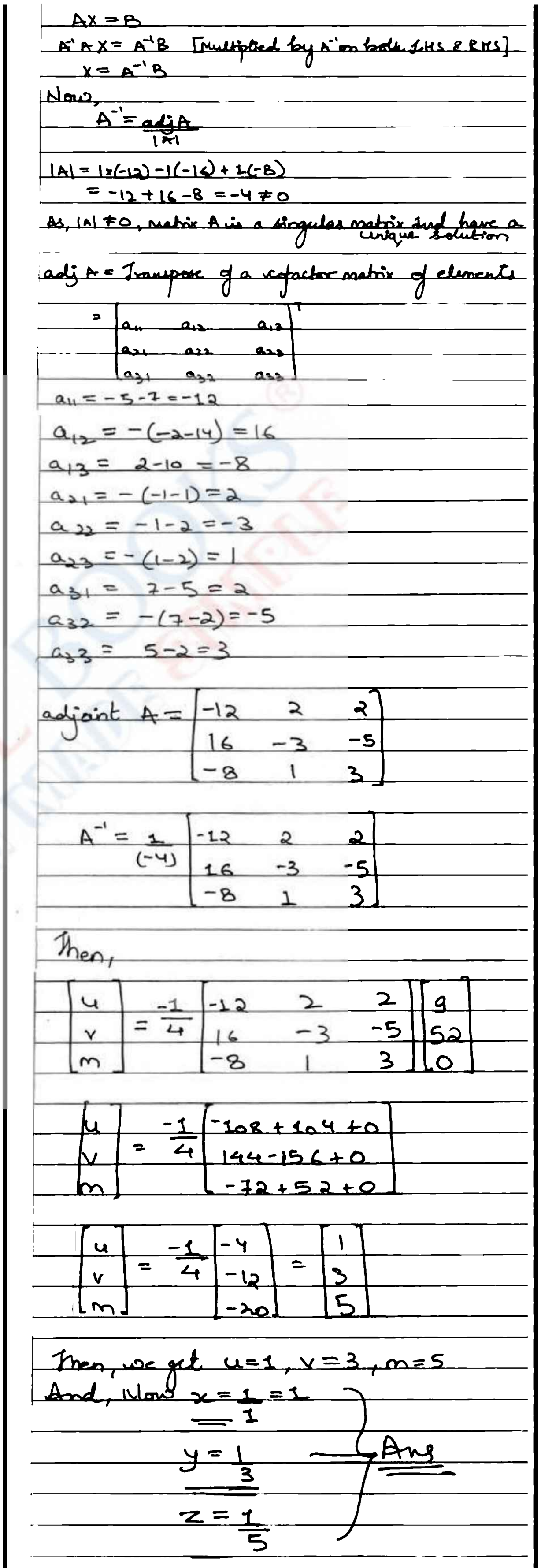
$2u + 5v + 7m = 52$

$2u + v - m = 0$

Writing in form, $AX = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 7 & -1 \end{bmatrix}$, $X = \begin{bmatrix} u \\ v \\ m \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$

then, $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 7 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ m \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$



$AX = B$

$A^{-1}AX = A^{-1}B$ [Multiplied by A^{-1} on both LHS & RHS]

$X = A^{-1}B$

Now,

$A^{-1} = \frac{adj A}{|A|}$

$|A| = 1\times(-12) - 1(-16) + 1(-8)$

$= -12 + 16 - 8 = -4 \neq 0$

As, $|A| \neq 0$, matrix A is a singular matrix and have a unique solution

adj A = Transpose of a cofactor matrix of elements

$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T$

$a_{11} = -5 - 7 = -12$

$a_{12} = -(-2-14) = 16$

$a_{13} = 2 - 10 = -8$

$a_{21} = -(-1-1) = 2$

$a_{22} = -1 - 2 = -3$

$a_{23} = -(1-2) = 1$

$a_{31} = 7 - 5 = 2$

$a_{32} = -(7-2) = -5$

$a_{33} = 5 - 2 = 3$

adjoint $A = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{(-4)}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$

Then,

$\begin{bmatrix} u \\ v \\ m \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$

$\begin{bmatrix} u \\ v \\ m \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -108 + 104 + 0 \\ 144 - 156 + 0 \\ -72 + 52 + 0 \end{bmatrix}$

$\begin{bmatrix} u \\ v \\ m \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -4 \\ -12 \\ -20 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Then, we get $u = 1$, $v = 3$, $m = 5$

And, Now $x = \frac{1}{1} = 1$

$y = \frac{1}{3}$

$z = \frac{1}{5}$

Ans

[Topper's Answer 2019]

Examiner's Comment

- Some candidates made simplification errors in the process of finding the values of variables x, y and z. A few candidates made errors while calculating co-factors and inverse of a matrix.

Answering Tip

- Do adequate practice in computing co-factors, inverse of a given matrix and in finding solution of a system of equations in the matrix form.

Q. 4. Using matrices, solve the following system of equations :

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

[ISC, 2018]

Sol.

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$AX = B$$

and $\quad X = A^{-1} B$

Here, $\quad |A| = -1$

$$adj\,A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} adj\,A.B$$

$$= \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \; x = 1, y = 2, z = 3$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Most of the candidates attempted the question correctly. A few candidates made error while calculating co-factors which resulted in incorrect adjoint matrix.

Answering Tip

- The basics with regards to co-factors of elements, obtaining adjoint and inverse and meaning of pre and post multiplication of matrices must be learn thoroughly; *e.g.* if $AX = B$ then $X = A^{-1} B$ and not BA^{-1} as such multiplication of matrices is not commutative.

Detailed Answer :

Given system of equations is

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

It can be written as : $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$, which

is of the form, $\quad AX = B$

Here, $\quad A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Now, $\quad |A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$$= 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$$
$$= 0 - 6 + 5 = -1 \neq 0$$

$\therefore A^{-1}$ exists and $A^{-1} = \dfrac{adjA}{|A|}$

Co-factors of $|A|$ are :

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -4 + 4 = 0$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -(-6 + 4) = 2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -(6 - 5) = -1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -4 - 5 = -9$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2 + 3) = -5$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 12 - 10 = 2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = -(-8 - 15) = 23$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 4 + 9 = 13$$

$$\therefore \quad adj\,A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now, $\quad X = A^{-1} B$

$$\Rightarrow \quad X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore\ x = 1, y = 2, z = 3$

Q. 5. Evaluate $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

Hence, Solve the system of equations,
$3x - 2y + 3z = 8, 2x + y - z = 1, 4x - 3y + 2z = 4.$
[ISC Specimen Paper, 2018]

Sol. Given,

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} = \begin{bmatrix} -17 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & -17 \end{bmatrix}$$

$$= -17\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AC = -17\,I \qquad \text{...(i)}$$

The given system of equations is

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

i.e., $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$\Rightarrow \qquad AX = B$

Where $\qquad A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix},$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$\Rightarrow \qquad X = A^{-1}B \qquad$ ...(ii)

From (i), $\qquad AC = -17I,$

Here, $\qquad C = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

$\Rightarrow \qquad \frac{-1}{17}AC = I$

$\Rightarrow \qquad \left(\frac{-1}{17}A\right)C = I$

$\Rightarrow \qquad A\left(\frac{-1}{17}C\right) = I$

$\Rightarrow \qquad A^{-1} = \frac{-1}{17}C$

From eqn. (ii), $\qquad X = A^{-1}B$

$$= \left(\frac{-1}{17}C\right)B$$

$$= \frac{-1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -8-5-4 \\ -64-6+36 \\ -80+1+28 \end{bmatrix}$$

$$= \frac{-1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix}$$

$\Rightarrow \qquad x = 1, y = 2, z = 3$

Q. 6. Given that : $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix},$

find *AB*. Using this result, solve the following system of equation :

$x - y = 3, 2x + 3y + 4z = 17$ and $y + 2z = 7$
[ISC, 2017]

Sol. $\qquad AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I$$

Now, $\quad x - y = 3, 2x + 3y + 4z = 17, y + 2z = 7$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$\therefore \qquad AB = 6I$

$\Rightarrow \qquad A\left(\frac{1}{6}B\right) = I$

$\Rightarrow \qquad A^{-1} = \frac{1}{6}B = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$

$\therefore \qquad X = A^{-1}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$

$$= \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -2 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$\therefore \qquad x = 2, y = -1, z = 4$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates did not know how to use the product *AB*, to solve the given equations. Made errors in substituting $A^{-1} = \frac{1}{6}B$. Some candidates solve the equations by finding A^{-1} by conventional method, *i.e.,* $\frac{\text{adj } A}{|A|}$. A few candidates solve it using cramer's Rule.

OSWAAL BOOKS LEARNING MADE SIMPLE

Answering Tip

- Understand the basic concepts of inverse of a matrix and its properties. Do adequate practice of solving equations by inverse matrix method.

Q. 7. Solve the following system of linear equations using matrix method :
$3x+y+z=1,\ 2x+2z = 0, 5x+y+2z = 2$ [ISC, 2016]

Sol. Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 2 \\ 5 & 1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 1 & 1 \\ 2 & 0 & 2 \\ 5 & 1 & 2 \end{vmatrix}$$

$$= 3(0-2)-1(4-10)+1(2-0) = 2$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -2 & -1 & 2 \\ 6 & 1 & -4 \\ 2 & 2 & -2 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\Rightarrow \quad X = \frac{1}{2}\begin{bmatrix} -2 & -1 & 2 \\ 6 & 1 & -4 \\ 2 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad X = \frac{1}{2}\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \therefore x = 1, y = -1, z = -1$$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Several candidates made errors in finding the co-factors of elements of matrix. For finding the unknown matrix X, some candidates used post-multiplication with inverse of A. A few candidates solved using Cramer's Rule.

Answering Tip

- Logical and reasoning skills should be used to apply the correct matrix property.

Q. 8. Given two matrices A and B, $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 4 & 1 \\ 1 & -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 11 & -5 & -14 \\ -1 & -1 & 2 \\ -7 & 1 & 6 \end{bmatrix}$. Find AB and use this result to solve the following system of equations:
$x-2y+3z = 6, x+4y+z = 12, x-3y+2z = 1$

[ISC 2015]

Sol. $AB = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 4 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} 11 & -5 & -14 \\ -1 & -1 & 2 \\ -7 & 1 & 6 \end{bmatrix} = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{bmatrix}$

$$\Rightarrow \quad AB = -8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow AB = -8\,I$$

$$\Rightarrow \quad A^{-1} = \frac{1}{-8}B$$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 4 & 1 \\ 1 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } C = \begin{bmatrix} 6 \\ 12 \\ 1 \end{bmatrix}$$

$$X = A^{-1}C$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-8}\begin{bmatrix} 11 & -5 & -14 \\ -1 & -1 & 2 \\ -7 & 1 & 6 \end{bmatrix}\begin{bmatrix} 6 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x = 1,\ y = 2,\ z = 3$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- A few candidates found the product of AB incorrectly. Many did not use the product of AB to solve the equation system. They found A^{-1} by using matrix inverse method. Several candidates found incorrect co-factors hence their values of x, y, z were incorrect. Some candidates could not obtain adjoint and inverse of a matrix correctly.

Answering Tip

- Inverse of a square matrix needs to be learnt step by step. Utilization of the inverse to correctly find the unknown matrix needs to be grasped properly. Product of two matrices need attention. Sufficient practice is a must.

Q. 9. Using matrix method, solve the following system of equation:
$x-2y = 10,\ 2x+y+3z = 8$ and $-2y+z = 7$

[ISC, 2014]

Sol. $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 11$$

co-factors of $|A|$ are:

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = (1+6) = 7$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = -(2-0) = -2$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} = (-4+0) = -4$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} = -(-2+0) = 2$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} = (1-0) = 1$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix}1 & -2\\0 & -2\end{vmatrix} = -(-2+0) = 2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}2 & 0\\1 & 3\end{vmatrix} = (-6-0) = -6$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix}1 & 0\\2 & 3\end{vmatrix} = -(3-0) = -3$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}1 & -2\\2 & 1\end{vmatrix} = (1+4) = 5$$

$$\text{adj A} = \begin{bmatrix}7 & -2 & -4\\2 & 1 & 2\\-6 & -3 & 5\end{bmatrix}^{T} = \begin{bmatrix}7 & 2 & -6\\-2 & 1 & -3\\-4 & 2 & 5\end{bmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix}7 & 2 & -6\\-2 & 1 & -3\\-4 & 2 & 5\end{bmatrix}$$

$$X = A^{-1}B$$

$$\therefore \quad X = \frac{1}{11}\begin{bmatrix}7 & 2 & -6\\-2 & 1 & -3\\-4 & 2 & 5\end{bmatrix}\begin{bmatrix}10\\8\\7\end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix}44\\-33\\11\end{bmatrix}$$

$$= \begin{bmatrix}4\\-3\\1\end{bmatrix}$$

$x = 4, y = -3, z = 1$

[ISC Marking Scheme, 2014]

Examiner's Comment

- A number of candidates could not calculate $|A|$ correctly. Some could not obtain the adjoint and the inverse correctly owing to faulty application of cross multiplication rule. For, $AB = C$ some concluded $B = C.\,A^{-1}$ instead of $B = A^{-1}C$. Some used Cramer's rule which was not asked.

Answering Tip

- Inverse of a square matrix needs to be learnt and done step by step. Utilization of the inverse to correctly find the unknown matrix needs to be understood. Sufficient practice is a must.

Q. 10. Find A^{-1}, where $A = \begin{bmatrix}4 & 2 & 3\\1 & 1 & 1\\3 & 1 & -2\end{bmatrix}$. Hence, solve the following system of linear equations :

$4x + 2y + 3z = 2,\ x + y + z = 1,\ 3x + y - 2z = 5$

[ISC, 2013]

Sol. Try yourself similar to Q. 4 in 6 marks questions.

Q. 11. Find the product of the matrices A and B where:

$A = \begin{bmatrix}-5 & 1 & 3\\7 & 1 & -5\\1 & -1 & 1\end{bmatrix}, B = \begin{bmatrix}1 & 1 & 2\\3 & 2 & 1\\2 & 1 & 3\end{bmatrix}$. Hence,solve the following equations by matrix method :

$$x + y + 2z = 1$$
$$3x + 2y + z = 7$$
$$2x + y + 3z = 2$$

[ISC, 2012]

Sol. Try yourself similar to Q. 5 in 6 marks questions.

AI Q. 12. Solve the following system of equations using matrix method :

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$
$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$
$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

[ISC, 2011]

Sol. Let $\frac{1}{x} = p, \frac{1}{y} = q$ and $\frac{1}{z} = r$ hence, given system of equations can be written as

$\therefore\ 2p + 3q + 10r = 4$ ...(i)

$4p - 6q + 5r = 1$ ...(ii)

$6p + 9q - 20r = 2$ ...(iii)

Writing the given system of equations in matrix form, *i.e.*, $AX = B$

$$\text{or} \quad \begin{bmatrix}2 & 3 & 10\\4 & -6 & 5\\6 & 9 & -20\end{bmatrix}\begin{bmatrix}p\\q\\r\end{bmatrix} = \begin{bmatrix}4\\1\\2\end{bmatrix}$$

where,

$$A = \begin{bmatrix}2 & 3 & 10\\4 & -6 & 5\\6 & 9 & -20\end{bmatrix}, B = \begin{bmatrix}4\\1\\2\end{bmatrix}, X = \begin{bmatrix}p\\q\\r\end{bmatrix}$$

$$\therefore \quad |A| = 2(120-45) - 3(-80-30) + 10(36+36)$$
$$= 150 + 330 + 720 = 1200$$

$\therefore\ |A| \neq 0$

$\Rightarrow A^{-1}$ exists.

Now co-factors of A are :

$$A_{11} = 75,\ A_{12} = 110,\ A_{13} = 72$$
$$A_{21} = 150,\ A_{22} = -100,\ A_{23} = 0$$
$$A_{31} = 75, A_{32} = 30,\ A_{33} = -24$$

$$adj\,A = \begin{bmatrix}75 & 110 & 72\\150 & -100 & 0\\75 & 30 & -24\end{bmatrix}^{T} = \begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}$$

$$\text{Now,} \quad X = A^{-1}B = \frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\1\\2\end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix}600\\400\\240\end{bmatrix} = \begin{bmatrix}1/2\\1/3\\1/5\end{bmatrix}$$

OSWAAL BOOKS LEARNING MADE SIMPLE

$\therefore \quad p=\frac{1}{2}, q=\frac{1}{3}, r=\frac{1}{5}$

$\Rightarrow \quad \frac{1}{x}=\frac{1}{2}, \frac{1}{y}=\frac{1}{3}, \frac{1}{z}=\frac{1}{5}$

$\Rightarrow \quad x=2, y=3, z=5$

Q. 13. Using matrix method, solve the following equations :

$$\begin{aligned} 5x+3y+z&=16\\ 2x+y+3z&=19\\ x+2y+4z&=25 \end{aligned}$$

Sol. Try yourself similar to Q. 4 in 6 marks questions.

Q. 14. Solve the following linear equations using the matrix method :

$$\begin{aligned} x+y+z&=9\\ 2x+5y+7z&=52\\ 2x+y-z&=0 \end{aligned}$$

[ISC, 2009]

Sol. Try yourself similar to Q. 4 in 6 marks questions.

Q. 15. If $A=\begin{bmatrix}1 & 2 & -3\\ 2 & 3 & 2\\ 3 & -3 & -4\end{bmatrix}$, find A^{-1} and hence solve the following system of linear equations :

$$\begin{aligned} x+2y-3z&=-4\\ 2x+3y+2z&=2\\ 3x-3y-4z&=11 \end{aligned}$$

[ISC, 2008]

Sol. Given,

$$\begin{aligned} x+2y-3z&=-4\\ 2x+3y+2z&=2\\ 3x-3y-4z&=11 \end{aligned}$$

Here, $A=\begin{bmatrix}1 & 2 & -3\\ 2 & 3 & 2\\ 3 & -3 & -4\end{bmatrix}$

$|A|=-6+28+45=67\neq 0$, therefore A^{-1} exists.

$$\begin{aligned} &A_{11}=-6, && A_{12}=14, && A_{13}=-15\\ &A_{21}=17, && A_{22}=5, && A_{23}=9\\ &A_{31}=13, && A_{32}=-8, && A_{33}=-1 \end{aligned}$$

$$adj.\,A=\begin{bmatrix}-6 & 14 & -15\\ 17 & 5 & 9\\ 13 & -8 & -1\end{bmatrix}^T=\begin{bmatrix}-6 & 17 & 13\\ 14 & 5 & -8\\ -15 & 9 & -1\end{bmatrix}$$

$$\therefore \quad A^{-1}=\frac{adjA}{|A|}$$

$$\therefore \quad A^{-1}=\frac{1}{67}\begin{bmatrix}-6 & 17 & 13\\ 14 & 5 & -8\\ -15 & 9 & -1\end{bmatrix}$$

Now the given system of equations is :

$$\begin{aligned} x+2y-3z&=-4\\ 2x+3y+2z&=2\\ 3x-3y-4z&=11 \end{aligned}$$

It can be written as,

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 3 & 2\\ 3 & -3 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-4\\ 2\\ 11\end{bmatrix}$$, which is of the form

$AX=B\Rightarrow X=A^{-1}B$

$$\therefore \quad \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{67}\begin{bmatrix}-6 & 17 & 13\\ 14 & 5 & -8\\ -15 & 9 & -1\end{bmatrix}\begin{bmatrix}-4\\ 2\\ 11\end{bmatrix}$$

$$=\frac{1}{67}\begin{bmatrix}24+34+143\\ -56+10-88\\ 60+18-11\end{bmatrix}$$

$$=\frac{1}{67}\begin{bmatrix}201\\ -134\\ 67\end{bmatrix}=\begin{bmatrix}3\\ -2\\ 1\end{bmatrix}$$

$\Rightarrow \quad x=3,\ y=-2,\ z=1$

Q. 16. Solve the following system of equations using matrices :

$$\begin{aligned} x+y+z&=6\\ x-y+z&=2\\ 2x+y-z&=1 \end{aligned}$$

[ISC, 2008]

Sol. Try yourself similar to Q. 4 in 6 marks questions.

Q. 17. A shopkeeper has 3 varieties of pen '*A*', '*B*' and '*C*'. Meenu purchased 1 pen of each variety for a total of ₹ 21. Jeevan purchased 4 pens of '*A*' variety, 3 pen say '*B*' variety and 2 pens of '*C*' variety for ₹ 60. While Shikha purchased 6 pens of '*A*' variety, 2 pens of 'B' variety and 3 pens of '*C*' variety for ₹ 70. Using matrix method, find cost of each variety of pen.

Sol. Let the cost of '*A*' variety pen be x, the cost of '*B*' variety pen be y, the cost of '*C*' variety pen be z.

According to the question,

$$\begin{aligned} x+y+z&=21\\ 4x+3y+2z&=60\\ 6x+2y+3z&=70 \end{aligned}$$

It can be written as,

$$\begin{bmatrix}1 & 1 & 1\\ 4 & 3 & 2\\ 6 & 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}21\\ 60\\ 70\end{bmatrix}$$

i.e., $AX=B$

where, $A=\begin{bmatrix}1 & 1 & 1\\ 4 & 3 & 2\\ 6 & 2 & 3\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}21\\ 60\\ 70\end{bmatrix}$

Now, $|A|=\begin{vmatrix}1 & 1 & 1\\ 4 & 3 & 2\\ 6 & 2 & 3\end{vmatrix}$

$$=1(9-4)-1(12-12)+1(8-18)$$
$$=5-10=-5\neq 0$$

$\Rightarrow A^{-1}$ exists.

$$\begin{aligned} A_{11}&=(9-4)=5\\ A_{12}&=-(12-12)=0\\ A_{13}&=(8-18)=-10\\ A_{21}&=-(3-2)=-1\\ A_{22}&=(3-6)=-3\\ A_{23}&=-(2-6)=4\\ A_{31}&=(2-3)=-1\\ A_{32}&=-(2-4)=2\\ A_{33}&=3-4=-1 \end{aligned}$$

$$adj\ A = \begin{bmatrix} 5 & 0 & -10 \\ -1 & -3 & 4 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\ A)$$

Also, $AX = B$

$\Rightarrow$ $X = A^{-1}B$

$\Rightarrow$ $A^{-1} = \frac{1}{|A|}(adj\ A)B$

$$= \frac{1}{|A|}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}\begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} 105-60-70 \\ 0-180+140 \\ -210+240-70 \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} -25 \\ -40 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore$ $x = 5, y = 8, z = 8$

$\therefore$ Cost of 'A' variety pen = ₹ 5

'B' variety pen = ₹ 8

'C' variety pen = ₹ 8

Q. 18. A trust invested some money in two types of bonds. The first bond pays 10% interest and second bond pays 12% interest. The trust received ₹ 2800 as interest. However, if trust had interchanged money in bonds, they would have got ₹ 100 less as interest. Using matrix method find the amount invested by the trust.

Sol. According to the questions :

$$[x\ y]\begin{bmatrix} \frac{10}{100} \\ \frac{12}{100} \end{bmatrix} = [2800]$$

i.e., $\frac{10x}{100} + \frac{12y}{100} = 2800$

$\Rightarrow$ $10x + 12y = 280000$

$\Rightarrow$ $5x + 6y = 140000$ ...(i)

Also,

$$[y\quad x]\begin{bmatrix} \frac{10}{100} \\ \frac{12}{100} \end{bmatrix} = [2700]$$

i.e., $\frac{10y}{100} + \frac{12x}{100} = 2700$

$\Rightarrow$ $10y + 12x = 270000$

$\Rightarrow$ $5y + 6x = 135000$ ...(ii)

By matrix method,

$$\begin{bmatrix} 5 & 6 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 140000 \\ 135000 \end{bmatrix}$$

where, $A = \begin{bmatrix} 5 & 6 \\ 6 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 140000 \\ 135000 \end{bmatrix}$

It is of the form $AX = B$

$\Rightarrow$ $X = A^{-1}B$

$$|A| = \begin{vmatrix} 5 & 6 \\ 6 & 5 \end{vmatrix}$$

$= 25 - 36 = -11 \neq 0$

$\therefore$ A is invertible

co-factors of $|A|$ are

$A_{11} = 5, A_{12} = -6,$

$A_{21} = -6, A_{22} = 5$

$\therefore$ $adj\ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{vmatrix} 5 & -6 \\ -6 & 5 \end{vmatrix}$

$A^{-1} = \frac{1}{|A|}(adj\ A)$ and

$X = A^{-1}B$

$$= \frac{1}{-11}\begin{bmatrix} 5 & -6 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 140000 \\ 135000 \end{bmatrix}$$

$$= \frac{1}{-11}\begin{bmatrix} 700000-810000 \\ -840000+675000 \end{bmatrix}$$

$$= \frac{-1}{11}\begin{bmatrix} -110000 \\ -165000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10000 \\ 15000 \end{bmatrix}$$

$\therefore$ Amount invested in type one bond ₹ x = ₹ 10000

and in type two bond ₹ y = ₹ 15000

Total amount invested is

₹ x + ₹ y = ₹ 10000 + ₹ 15000

= ₹ 25,000

Q. 19. The monthly incomes of Advit and Ridhaan are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves ₹ 15,000 per month, find their monthly incomes using matrix method.

Sol. Let monthly income of Advit = ₹ $3x$

Let monthly income of Ridhaan = ₹ $4x$

Monthly expenditure of Advit = ₹ $5y$

Monthly expenditure of Ridhaan = ₹ $7y$

Advit's saving = ₹ $(3x - 5y)$

Ridhaan's saving = ₹ $(4x - 7y)$

According to the question,

$3x - 5y = 15000$

$4x - 7y = 15000$

Using matrices,

$$\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

i.e., $AX = B$ or $X = A^{-1}B$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

Now, $|A| = -21 + 20$

$= -1 \neq 0 \Rightarrow \therefore A^{-1}$ exists

OSWAAL BOOKS LEARNING MADE SIMPLE SAMPLE

Here, $adj\,A = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} adj\,A = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105000 - 75000 \\ 60000 - 45000 \end{bmatrix} = \begin{bmatrix} 30000 \\ 15000 \end{bmatrix}$$

$\Rightarrow \quad x = 30000$ and $y = 15000$

Therefore,

Advit's income = $3x = 3 \times 30000$ = ₹ 90000

Ridhaan's income = $4x = 4 \times 30000$ = ₹ 120000

AI Q. 20. Test for consistency the system of equations and solve them.

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

Sol. The given system of equations can be written in matrix form $AX = B$, *i.e.*,

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{vmatrix}$$

$= 5(260 - 4) - 3\,(30 - 14) + 7\,(6 - 182)$

$= 1280 - 48 - 1232 = 0$

$\because |A| = 0$, hence A^{-1} does not exist and the given system has either no solution or has infinite number of solutions.

To check, we final (adj A). B

Co-factors of elements of $|A|$ are :

$A_{11} = 256, A_{12} = -16, A_{13} = -176, A_{21} = -16, A_{22} = 1,$
$A_{23} = 11, A_{31} = -176, A_{32} = 11, A_{33} = 121.$

$$\therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

$$\therefore \quad (\text{adj } A).\,B = \begin{bmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1024 - 144 - 880 \\ -64 + 9 + 55 \\ -704 + 99 + 605 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

Here, $|A| = 0$ and (adj A). $B = 0$, the system is consistent and has infinite number of solutions.

To find the solution, let $z = k$ and solve any two of given equations as follows :

$$5x + 3y + 7z = 4 \qquad \text{...(i)}$$
$$3x + 26y + 2z = 9 \qquad \text{...(ii)}$$

On solving eqs. (i) & (ii),

$$x = \frac{1}{11}(7 - 16k) \text{ and } y = \frac{1}{11}(k + 3)$$

Substituting these values of x and y in the remaining equation, *i.e.*, $7x + 2y + 10z = 5$, we get

$$7.\frac{1}{11}(7 - 16k) + 2\frac{1}{11}(k + 3) + 10k = 5 \qquad (\text{Here, } z = k)$$

$\Rightarrow \quad 49 - 112k + 2k + 6 + 110k = 55$

$\Rightarrow \quad 55 = 55$, which is true

Hence, the given system of equations has infinite number of solution given by

$$x = \frac{1}{11}(7 - 16k),\ y = \frac{1}{11}(k + 3),\ z = k,$$

where k is any real number.

TOPIC-3
Elementary Operations or Transformation of Matrix

Scan to know more about this topic

Elementary operations or transformation of matrix

Revision Notes

- **Elementary Operations or Transformations of a Matrix:** The following three operations applied on the row (or column) of a matrix are called elementary row (or column) transformations.

(a) Interchange of any two rows (or columns) : When i^{th} row (or column) of a matrix is interchanged with the j^{th} row (or column), it is denoted as $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$).

(b) Multiplying all elements of a row (or column) of a matrix by a non-zero scalar : When the i^{th} row (or column) of a matrix is multiplied by a scalar k, it is denoted as $R_i \to kR_i$ (or $C_i \to k\,C_i$).

Important Facts

- In case, after applying one or more elementary row (or column) operations on $A = IA$ (or ($A=AI$), if we obtain all zeros in one or more rows of the matrix A on LHS, then A^{-1} does not exist.

(c) Adding to the elements of a row (or column), the corresponding elements of any other row (column) multiplied by an scalar k : When k times the elements of i^{th} row (or column) is added to the corresponding elements of the j^{th} row (or column), it is denoted as $R_i \to R_i + kR_j$ (or $C_i \to C_i + k\,C_j$).

➢ **Algorithm to find Inverse of a Matrix by Elementary Operations or Transformations**

- **By Row Transformations :**

  Step 1 : Write the given square matrix as $A = I_nA$.

  Step 2 : Perform a sequence of elementary row operations successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result $I_n = BA$.

  Step 3 : Matrix B is the inverse of A. So, write $A^{-1} = B$.

- **By Column Transformations :**

  Step 1 : Write the given square matrix as $A = AI_n$.

  Step 2 : Perform a sequence of elementary row operations successively on A on the LHS and post-factor I_n on the RHS till we obtain the result $I_n = AB$.

  Step 3 : Matrix B is the inverse of A. So, write $A^{-1} = B$.

2 Marks Questions

Q. 1. Find the inverse of $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ using elementary transformations.

Sol. Let $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

We write $A = IA$

$$\therefore \quad \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + R_1$, we get

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Q. 2. Find the inverse of $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$ using elementary transformations.

Sol. Let $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

We write $A = IA$

$$\therefore \quad \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$, we get

$$\begin{bmatrix} 1 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - R_1$, we get

$$\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + 2R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix} A$$

Applying $R_2 \to 1/2\, R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1/2 & 1 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -1 & 3 \\ -1/2 & 1 \end{bmatrix}$$

Q. 3. Using elementary transformations. Find the inverse of $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

We write, $A = IA$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying $C_1 \to C_1 - 2\,C_2$, we get

OSWAAL BOOKS LEARNING MADE SIMPLE

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = A\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

Clearly, all the elements of column C_1 are zeros, so A^{-1} does not exist.

6 Marks Questions

Q. 1. Use elementary transformation, find the inverse of matrix.

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

Ans. Since,

$$A = I_3A$$

Let $$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}A$$

Applying $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix}A$$

Applying $R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix}A$$

Applying $R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix}A$$

Applying $R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 3 & -1 \\ 5 & -3 & 1 \end{bmatrix}A$$

Applying $R_1 \to R_1 + \frac{1}{2}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ 5 & -3 & 1 \end{bmatrix}A$$

Applying $R_3 \to \frac{1}{2}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Q. 2. Using elementary transformation, find the inverse of the matrix $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and use it to solve the following system of lines as equations :

$$8x + 4y + 3z = 19$$
$$2x + y + z = 5$$
$$x + 2y + 2z = 7$$

Sol. We know that

$$A = I_3 A$$

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_3 \to 2R_3$,

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}A$$

Applying $R_3 \to R_3 - R_2$,

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}A$$

Applying $R_2 \to 4R_2$,

$$\begin{bmatrix} 8 & 4 & 3 \\ 8 & 4 & 4 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 2 \end{bmatrix}A$$

Applying $R_2 \to R_2 - R_1$,

$$\begin{bmatrix} 8 & 4 & 3 \\ 0 & 0 & 1 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & -1 & 2 \end{bmatrix}A$$

Applying $R_2 \to R_2 + R_3$,

$$\begin{bmatrix} 8 & 4 & 3 \\ 0 & 3 & 4 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 2 \\ 0 & -1 & 2 \end{bmatrix}A$$

Applying $R_3 \to R_3 - R_2$,

$$\begin{bmatrix} 8 & 4 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 2 \\ 1 & -4 & 0 \end{bmatrix}A$$

Applying $R_1 \to R_1 + 3R_3$,

$$\begin{bmatrix} 8 & 4 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -12 & 0 \\ -1 & 3 & 2 \\ 1 & -4 & 0 \end{bmatrix}A$$

Applying $R_2 \to R_2 + 4R_3$,

$$\begin{bmatrix}8&4&0\\0&3&0\\0&0&-1\end{bmatrix}=\begin{bmatrix}4&-12&0\\3&-13&2\\1&-4&0\end{bmatrix}A$$

Applying $R_1 \to R_1 - \frac{4}{3}R_2$,

$$\begin{bmatrix}8&0&0\\0&3&0\\0&0&-1\end{bmatrix}=\begin{bmatrix}0&\frac{16}{3}&-\frac{8}{3}\\3&-13&2\\1&-4&0\end{bmatrix}$$

Applying $R_1 \to \frac{1}{8}R_1$, $R_2 \to \frac{1}{3}R_2$ and $R_3 = -R_3$,

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&\frac{2}{3}&-\frac{1}{3}\\1&-\frac{13}{3}&\frac{2}{3}\\-1&4&0\end{bmatrix}A$$

$$\text{Hence, } A^{-1}=\begin{bmatrix}0&\frac{2}{3}&-\frac{1}{3}\\1&-\frac{13}{3}&\frac{2}{3}\\-1&4&0\end{bmatrix}$$

Matrix representation of given linear equation is

$$\begin{bmatrix}8&4&3\\2&1&1\\1&2&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}19\\5\\7\end{bmatrix}$$

$\Rightarrow$ $AX = B$

$\Rightarrow$ $X = A^{-1}B$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0&\frac{2}{3}&-\frac{1}{3}\\1&-\frac{13}{3}&\frac{2}{3}\\-1&4&0\end{bmatrix}\begin{bmatrix}19\\5\\7\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0+\frac{10}{3}-\frac{7}{3}\\19-\frac{65}{3}+\frac{14}{3}\\-19+20+0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$\therefore$ $x = 1, y = 2, z = 1$

Q. 3. Using elementary column operations, find the inverse of the following matrix : $\begin{bmatrix}-1&1&2\\1&2&3\\3&1&1\end{bmatrix}$.

Sol. Let $A=\begin{bmatrix}-1&1&2\\1&2&3\\3&1&1\end{bmatrix}$

$A = I_3 A$

$$\therefore \begin{bmatrix}-1&1&2\\1&2&3\\3&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A$$

Applying $C_1 \leftrightarrow C_2$,

$$\begin{bmatrix}1&-1&2\\2&1&3\\1&3&1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}A$$

Applying $C_2 \to C_2 + C_1$ and $C_3 \to C_3 - 2C_1$,

$$\begin{bmatrix}1&0&0\\2&3&-1\\1&4&-1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&1&-2\\0&0&1\end{bmatrix}A$$

Applying $C_1 \to C_1 + 2C_3$,

$$\begin{bmatrix}1&0&0\\0&3&-1\\-1&4&-1\end{bmatrix}=\begin{bmatrix}0&1&0\\-3&1&-2\\2&0&1\end{bmatrix}A$$

Applying $C_2 \to C_2 + 2C_3$

$$\begin{bmatrix}1&0&0\\0&1&-1\\-1&2&-1\end{bmatrix}=\begin{bmatrix}0&1&0\\-3&-3&-2\\2&2&1\end{bmatrix}A$$

Applying $C_3 \to C_3 + C_2$,

$$\begin{bmatrix}1&0&0\\0&1&0\\-1&2&1\end{bmatrix}=\begin{bmatrix}0&1&1\\-3&-3&-5\\2&2&3\end{bmatrix}A$$

Applying $C_1 \to C_1 + C_3$ and $C_2 \to C_2 - 2C_3$,

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&-1&1\\-8&7&-5\\5&-4&3\end{bmatrix}A$$

$$\Rightarrow \quad A^{-1}=\begin{bmatrix}1&-1&1\\-8&7&-5\\5&-4&3\end{bmatrix}$$

Q. 4. Using elementary transformations, find inverse of $\begin{bmatrix}2&-3&3\\2&2&3\\3&-2&2\end{bmatrix}$.

Sol. Try yourself, similar to Q. 1 in 6 marks questions.

Q. 5. Find the inverse of $\begin{bmatrix}1&3&-2\\-3&0&-5\\2&5&0\end{bmatrix}$ using elementary transformations.

Sol. Try yourself, similar to Q. 1 in 6 marks questions.

Q. 6. Using elementary transformations, find the inverse of $\begin{bmatrix}2&0&-1\\5&1&0\\0&1&3\end{bmatrix}$.

Sol. Let $A=\begin{bmatrix}2&0&-1\\5&1&0\\0&1&3\end{bmatrix}$

We write $A = I_3 A$

OSWAAL BOOKS LEARNING MADE SIMPLE

$$\therefore \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $C_1 \to C_1 + C_3$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 0 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

Apply $C_3 \to C_3 + C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 5 \\ 3 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} A$$

Apply $C_1 \to C_1 - C_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ -3 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} A$$

Apply $C_3 \to C_3 - 5C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ -1 & 0 & 2 \end{bmatrix} A$$

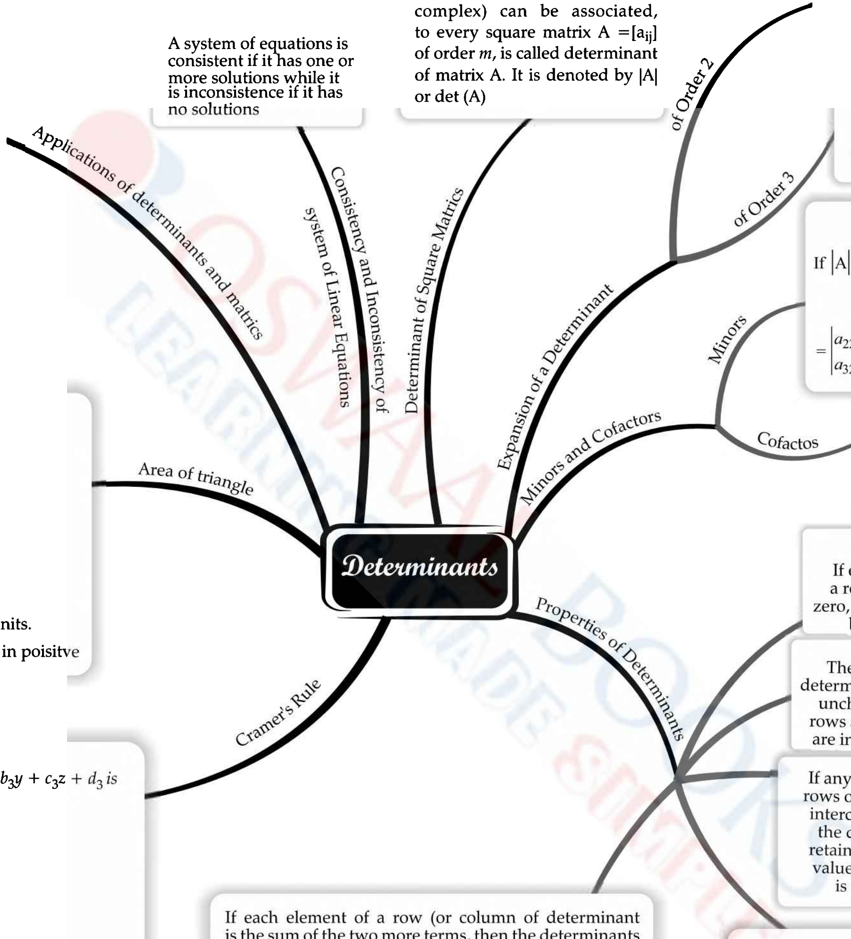
Apply $C_2 \to C_2 - C_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 6 & -5 \\ -1 & -2 & 2 \end{bmatrix} A$$

Apply $C_1 \to C_1 + 3C_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

$$\therefore \qquad A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$


Determinants
Determinant of Square Matrics
Expansion of a Determinant
of Order 2
of Order 3
Minors and Cofactors
Minors
Cofactos
Properties of Determinants
Cramer's Rule
Area of triangle
Applications of determinants and matrics
Consistency and Inconsistency of system of Linear Equations

A unique number (real or complex) can be associated, to every square matrix A =[aij] of order *m*, is called determinant of matrix A. It is denoted by |A| or det (A)

$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$

$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\,(a_{22}.a_{33} - a_{32}.a_{23}) - a_{12}$
$(a_{21}.a_{33} - a_{31}.a_{23}) + a_{13}\,(a_{21}.a_{32} - a_{31}.a_{22})$

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minor of $a_{11} = M_{11}$
$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, Minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

If the minor M_{ij} is multiplied by $(-1)^{i+j}$, then we obtain the factor of a_{ij}. It is denoted by A_{ij} or C_{ij}, i.e., $A_{ij} = (-1)^{i+j}$ M_{ij}, cofactor of $a_{11} = (-1)^{1+1} M_{11}$ or $A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

If each element of a row or column is zero, then determinant becomes zero

e.g. $A = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \therefore |A| = 0$

The value of a determinant remains unchanged if its rows and columns are interchanged.

e.g. $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

If any two adjaceat rows or columns are interchanged then the determinant retains its absolute value, but its sign is changed.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_1 & c_1 \end{vmatrix}$

$\Rightarrow \Delta_1 = -1\Delta$

If each element of a row or a column of a determinant is multiplied by a constant k, then the value of new determinant are k times the value of the original determinant

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$
$= k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \Delta_1 = k\Delta$

If each element of a row (or column of determinant is the sum of the two more terms, then the determinants can be expressed as the sum of the two or more determinants.

e.g. $\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$

The solution of the system
$a_1x + b_1y + c_1z + d$, $a_2x + b_2y + c_2z + d_2$ and $a_3x + b_3y + c_3z + d_3$ *is*
given by $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ and $z = \frac{D_z}{D}$, where

If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

provided that $D \neq 0$

If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of triangle,

Area of $\Delta = \frac{1}{2}\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$

For eg: if (1, 2), (3, 4) amd (–2, 5) are the vertices, then area of the triangle is

Area of $\Delta = \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{bmatrix}$

$= \frac{1}{2}|1(4-5) - 2(3+2) + 1(15+18) = 6$ sq. units.

we take positie value of the determinant because area in poisitve

- *If* $a_1x + b_1y + c_1z = d_1$,
  $a_2x + b_2y + c_2z = d_2$,
  $a_3x + b_3y + c_3z = d_3$
  then we an write $AX + B$,

Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

- Unique solution of $AX = B$ is $X = A^{-1}B$, $|A| \neq 0$
- $AX = B$ is consistent or inconsistent according as the solution existes or not.
- For a square matrix A in $AX = B$, if
  (i) $|A| \neq 0$ then there exists unique solution.
  (ii) $|A| = 0$ and $(adj.\ A)\ B \neq 0$, then no solution.
  (iii) if $|A| = 0$ and $(adj.\ A).B = 0$ then system may or may not be consistent.

A system of equations is consistent if it has one or more solutions while it is inconsistence if it has no solutions

Trace the Mind Map
‣ First Level ‣ Second Level ‣ Third Level

CHAPTER

4 DETERMINANTS

SYLLABUS

- ***Determinant of square matrix (upto 3 × 3 matrices), Properties of determinants, Minors, Co-factors and Applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix.***

  Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

- ***Determinants***
  - ***Order***
  - ***Minors***
  - ***Cofactors***
  - ***Expansion***
  - ***Applications of determinants in finding the area of triangle and collinearity***
  - ***Properties of determinants. Problems based on properties of determinants.***

Trend Analysis

List of Concepts	2016	2017	2018	2019	2020
Determinants and its Properties	1 Q (4 marks)	1 Q (6 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (4 marks)	1 Q (2 marks) 1 Q (4 marks)
Applications of Determinants				1 Q (2 marks)	

TOPIC-1

Determinants and Its Properties

TOPIC - 1
Determinants and Its Properties.... **P. 84**
TOPIC - 2
Applications of Determinants .... **P. 104**

Revision Notes

- **Determinant :** A unique number (real or complex) can be associated to every square matrix $A = [a_{ij}]$ of order m. This number is called the determinant of the square matrix A, where $a_{ij} = (i, j)^{\text{th}}$ element of A.

  The determinant of matrix A is denoted by det A or $|A|$.

  - **Determinant of a square matrix of order 2**

    If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}$

    It follows that the value of a determinant of order 2 is equal to the product of the elements along the principal diagonal *minus* the product of the off diagonal elements.

    e.g., : Let $A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$, then

    $$|A| = 5 \times 3 - (-2) \times 4 = 15 + 8 = 23.$$

Scan to know more about this topic

What is determinant?

- **Determinant of a square matrix of order 3**

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3,

then $|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

or $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{33}a_{21} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

It follows that the value of a determinant of order 3 is the sum of the product of elements a_{ij} in first row with $(-1)^{i+j}$ times the determinant of a 2 × 2 sub matrix obtained by leaving the first row and column passing through elements.

Note

- **Minor :** In a determinant, the minor of an element is a determinant left after deleting the complete row and column in which the element exists and is denoted by corresponding capital letter.
- **Cofactor :** In a determinant, the cofactor of an element is a minor of that element with its respective sign. More briefly co-factors are signed minors.

➢ **Elementary properties of Determinants**

Property I : The value of a determinant is not altered by inter changing its rows into columns and columns into rows, *i.e.*,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & b_3 \end{vmatrix}$$

Property II : If any two adjacent rows or columns of a determinant are interchanged, the sign of the determinant get changed but its numerical value remains unaltered, *i.e.*,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1) \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property III : If any two rows or columns of a determinant are identical, the value of determinant is zero, *i.e.*,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ or, } \begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_1 \end{vmatrix} = 0$$

1 Mark Questions

Q. 1. Which of the following statements is correct :

(a) Determinant is number associated to a matrix

(b) Determinant is a square matrix.

(c) Determinant is a number associated to a square matrix

(d) None of the above

Sol. Correct options : (c)

Explanation: We know that, to every square matrix $A = [a]^{ij}$ of order n, we can associated a number (real or complex) called determinant of the square matrix A

$a^{ij} = (i, j)$th element of A.

Q. 2. Which of the following is not a property of determinant?

(a) The value of determinant changes if all of its rows and columns are interchanged

(b) The value of determinant changes if any two rows or columns are interchanged

(c) The value of determinant is zero if any two rows and columns are identical

(d) The value of determinant gets multiplied by k, if each element of row or column is multiplied by k

Sol. Correct Option : (a)

Explanation: The value of determinant remains unchanged if all of its rows and columns are interchanged *i.e.* $|A| = |A'|$, where A is a square matrix and A' is the transpose of the matrix A.

AI Q. 3. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & x & y \\ 1 & x & -y \\ 1 & -x^2 & y^2 \end{bmatrix}$$

(a) $(x+1)$ **(b)** $-2xy(x+1)$

(c) $xy(x+1)$ **(d)** $2xy(x+1)$

Sol. Correct Option : (b)

OSWAAL BOOKS LEARNING MADE SIMPLE

Explanation: Given that, $A = \begin{bmatrix} 1 & x & y \\ 1 & x & -y \\ 1 & -x^2 & y^2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & x & y \\ 1 & x & -y \\ 1 & -x^2 & y^2 \end{vmatrix}$$

Taking x common from C_2 and y common from C_3, we get

$$|A| = xy\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -x & y \end{vmatrix}$$

Expanding along R_1, we get

$$|A| = xy\{1(y-x) - 1(y+1) + 1(-x-1)\}$$
$$= xy(y - x - y - 1 - x - 1)$$
$$= xy(-2x - 2) = -2xy(x + 1).$$

Q. 4. Evaluate $A = \begin{vmatrix} x^2 & x^3 & x^4 \\ x & y & z \\ x^2 & x^3 & x^4 \end{vmatrix}$

(a) 0 **(b) 1**

(c) xyz **(d)** x^2yz^3

Sol. Correct Option : (a)

Explanation:

$$|A| = \begin{vmatrix} x^2 & x^3 & x^4 \\ x & y & z \\ x^2 & x^3 & x^4 \end{vmatrix}$$

If the elements of any two rows or columns are identical, then the value of determinant is zero. Here, the elements of row 1 and row 3 are identical. Hence, its determinant is 0.

AI **Q. 5. Evaluate A=** $\begin{vmatrix} \cos\theta & -\cos\theta & 1 \\ \sin^2\theta & \cos^2\theta & 1 \\ \sin\theta & -\sin\theta & 1 \end{vmatrix}$

Sol. $|A| = \begin{vmatrix} \cos\theta & -\cos\theta & 1 \\ \sin^2\theta & \cos^2\theta & 1 \\ \sin\theta & -\sin\theta & 1 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2$

$$|A| = \begin{vmatrix} \cos\theta - \cos\theta & -\cos\theta & 1 \\ \sin^2\theta + \cos^2\theta & \cos^2\theta & 1 \\ \sin\theta - \sin\theta & -\sin\theta & 1 \end{vmatrix} = \begin{vmatrix} 0 & -\cos\theta & 1 \\ 1 & \cos^2\theta & 1 \\ 0 & -\sin\theta & 1 \end{vmatrix}$$

Expanding along C_1, we get

$0 - 1(\cos\theta + \sin\theta) = \sin\theta - \cos\theta.$

Q. 6. Evaluate $A = \begin{vmatrix} b-c & b & c \\ a & c-a & c \\ a & b & a-b \end{vmatrix}$

Sol. $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2b & b & c \\ 2c & c-a & c \\ 2a & b & a-b \end{vmatrix} = 2\begin{vmatrix} b & b & c \\ c & c-a & c \\ a & b & a-b \end{vmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= 2\begin{vmatrix} b & 0 & c-b \\ c & -a & 0 \\ a & b & -b \end{vmatrix}$$

$$= 2\,[b(ab - 0) - 0 + (c - b)(bc - ac + a^2)]$$
$$= 2\,[ab^2 + bc^2 - ac^2 + a^2c - b^2c + abc - a^2b]$$
$$= 2\,(ab^2 + abc - a^2b)$$
$$= 2ab\,(b + c - a)$$

Q. 7. Evaluate $A = \begin{vmatrix} -a & b & c \\ -2a+4x & 2b-4y & 2c+4z \\ x & -y & z \end{vmatrix}$

Sol. $|A| = \begin{vmatrix} -a & b & c \\ -2a+4x & 2b-4y & 2c+4z \\ x & -y & z \end{vmatrix}$

Using the properties of determinants, the given determinant can be expressed as a sum of two determinants.

$$|A| = \begin{vmatrix} -a & b & c \\ -2a & 2b & 2c \\ x & -y & z \end{vmatrix} + \begin{vmatrix} -a & b & c \\ 4x & -4y & 4z \\ x & -y & z \end{vmatrix}$$

$$= 2\begin{vmatrix} -a & b & c \\ -a & b & c \\ x & -y & z \end{vmatrix} + 4\begin{vmatrix} -a & b & c \\ x & -y & z \\ x & -y & z \end{vmatrix}$$

Since two rows are similar in each of the determinants, the determinant is 0.

AI **Q. 8. Evaluate** $A = \begin{vmatrix} 4 & 8 & 12 \\ 6 & 12 & 18 \\ 7 & 14 & 21 \end{vmatrix}$

Sol. $|A| = \begin{vmatrix} 4 & 8 & 12 \\ 6 & 12 & 18 \\ 7 & 14 & 21 \end{vmatrix}$

Taking 4, 6 and 7 from R_1, R_2, R_3 respectively

$$|A| = 4 \times 6 \times 7\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

Since the elements of all rows are identical, the determinant is zero.

Q. 9. If A_{ij} **is the cofactor of the element** a_{ij} **of the determinant** $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$**, then write the value of** $a_{32}A_{32}$**.**

Sol. $$a_{32}A_{32} = -5\begin{vmatrix}2 & 5\\ 6 & 4\end{vmatrix}$$

$$= -5(8-30)$$

$$= -5(-22) = 110$$

Q. 10. Write the value of the determinant :

$$\begin{vmatrix}\mathbf{102} & \mathbf{18} & \mathbf{36}\\ \mathbf{1} & \mathbf{3} & \mathbf{4}\\ \mathbf{17} & \mathbf{3} & \mathbf{6}\end{vmatrix}$$

Sol. Let $$\Delta = \begin{vmatrix}102 & 18 & 36\\ 1 & 3 & 4\\ 17 & 3 & 6\end{vmatrix}$$

$$= \begin{vmatrix}17 & 3 & 6\\ 1 & 3 & 4\\ 17 & 3 & 6\end{vmatrix} \qquad \left[R_1 \to \frac{1}{6}R_1\right]$$

$$= 0 \qquad (\because R_1 \text{ and } R_3 \text{ are identical})$$

Q. 11. If $\Delta = \begin{vmatrix}5 & 3 & 8\\ 2 & 0 & 1\\ 1 & 2 & 3\end{vmatrix}$, write the minor of the element a_{23}.

Sol. Given, $$A = \begin{vmatrix}5 & 3 & 8\\ 2 & 0 & 1\\ 1 & 2 & 3\end{vmatrix}$$

$$\therefore \text{ Minor of } a_{23} = \begin{vmatrix}5 & 3\\ 1 & 2\end{vmatrix}$$

$$= 10 - 3 = 7$$

Q. 12. If $\Delta = \begin{vmatrix}1 & 2 & 3\\ 2 & 0 & 1\\ 5 & 3 & 8\end{vmatrix}$, write the minor of the element a_{22}.

Sol. Given, $$A = \begin{vmatrix}1 & 2 & 3\\ 2 & 0 & 1\\ 5 & 3 & 8\end{vmatrix}$$

$$\therefore \quad \text{Minor of } a_{22} = \begin{vmatrix}1 & 3\\ 5 & 8\end{vmatrix}$$

$$\Rightarrow \qquad = 8 - 15$$

$$\Rightarrow \qquad = -7$$

Q. 13. Write the value of $\begin{vmatrix}2 & 7 & 65\\ 3 & 8 & 75\\ 5 & 9 & 86\end{vmatrix}$.

Sol. Let $$\Delta = \begin{vmatrix}2 & 7 & 65\\ 3 & 8 & 75\\ 5 & 9 & 86\end{vmatrix}$$

$$= \begin{vmatrix}2 & 7 & 0\\ 3 & 8 & 0\\ 5 & 9 & 0\end{vmatrix} \quad [C_3 \to C_3 - 9C_2 - C_1]$$

$$= 0$$

[Since all values of C_3 is zero so determinant value is zero]

2 Marks Questions

Q 1. Without expanding at any stage, find the value of the determinant :

$$\Delta = \begin{vmatrix}20 & a & b+c\\ 20 & b & a+c\\ 20 & c & a+b\end{vmatrix}$$ **[ISC, 2020]**

Sol. $$\Delta = \begin{vmatrix}20 & a & b+c\\ 20 & b & c+a\\ 20 & c & a+b\end{vmatrix}$$

$$= 20\begin{vmatrix}1 & a & b+c\\ 1 & b & c+a\\ 1 & c & a+b\end{vmatrix}$$

$$\Delta = 20\begin{vmatrix}1 & a & a+b+c\\ 1 & b & a+b+c\\ 1 & c & a+b+c\end{vmatrix} \qquad [C_3 \to C_3 + C_2]$$

$$\Delta = 20(a+b+c)\begin{vmatrix}1 & a & 1\\ 1 & b & 1\\ 1 & c & 1\end{vmatrix}$$

Here, $C_1 = C_3$ so, the determinant value will be zero

$\therefore \quad \Delta = 20(a+b+c)(0) = 0$

$\Delta = 0$ **[ISC marking Scheme, 2020]**

Examiner's Comment

- Many candidates expanded the determinants to solve it, though it was clearly mentioned in the question find the value of determinant without expansion at any stage.

Answering Tip

- Do extensive practice of different types of questions based on properties of determinants.

AI **Q. 2. Without expanding at any stage, find the value of $\begin{vmatrix}a & b & c\\ a+2x & b+2y & c+2z\\ x & y & z\end{vmatrix}$** **[ISC, 2018]**

Sol. Let $$\Delta = \begin{vmatrix}a & b & c\\ a+2x & b+2y & c+2z\\ x & y & z\end{vmatrix}$$

Applying $R_1 = R_1 + 2R_3$, we get

$$\Delta = \begin{vmatrix}a+2x & b+2y & c+2z\\ a+2x & b+2y & c+2z\\ x & y & z\end{vmatrix} = 0$$

Since, R_1 and R_2 are identical, therefore value of Δ is zero. **[ISC Marking Scheme 2018]**

OSWAAL BOOKS® LEARNING MADE SIMPLE

Q. 3. Without expanding at any stage, find the value of the determinant :

$$\Delta = \begin{vmatrix} 2 & x & y+z \\ 2 & y & z+x \\ 2 & z & x+y \end{vmatrix}$$

[ISC Specimen Paper, 2018]

Sol. Given, $\Delta = \begin{vmatrix} 2 & x & y+z \\ 2 & y & z+x \\ 2 & z & x+y \end{vmatrix}$

By applying, $C_3 = C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2 & x & x+y+z \\ 2 & y & x+y+z \\ 2 & z & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix}$$

$$= 2 \times 0 = 0$$

[Since, $C_1 = C_2 \therefore \Delta = 0$]

Q. 4. Find the maximum value of $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1+\sin\theta} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1+\cos\theta} \end{vmatrix}$

Sol. Let the maximum value be

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$$

By Expanding, we get

$$\Delta = 1\{(1 + \sin\theta)(1 + \cos\theta) - 1\}$$
$$- 1\{(1 + \cos\theta) - 1\} + 1\{1 - (1 + \sin\theta)\}$$
$$= (1 + \sin\theta + \cos\theta + \sin\theta\cos\theta - 1)$$
$$- (\cos\theta) - \sin\theta$$
$$= \sin\theta . \cos\theta$$
$$= \frac{2\sin\theta.\cos\theta}{2}$$
$$= \frac{1}{2}\sin 2\theta$$

Maximum value

$$= \frac{1}{2} \qquad [\text{as} -1 \le \sin 2\theta \le 1]$$

Q. 5. If $x \in N$ and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, then find the value of x.

Sol. Given, $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$

$$\Rightarrow \quad 2x(x+3)-(-2)(-3x) = 8$$
$$\Rightarrow \quad 2x^2+6x-6x = 8$$
$$\Rightarrow \quad x^2 = \frac{8}{2}$$
$$\Rightarrow \quad x^2 = 4$$
$$\Rightarrow \quad x^2 = \pm 2 \qquad [\therefore x \in N]$$
$$\therefore \quad x = 2$$

Q. 6. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x.

Sol. Given, $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$$2x^2 - 40 = 18 + 14$$
$$\Rightarrow \quad 2x^2 = 32 + 40$$
$$\Rightarrow \quad 2x^2 = 72$$
$$\Rightarrow \quad x^2 = 36$$
$$\Rightarrow \quad x = \pm 6$$

AI **Q. 7. If $\begin{vmatrix} \sin\alpha & \cos\beta \\ \cos\alpha & \sin\beta \end{vmatrix} = -\frac{1}{2}$ where α and β are acute angles, then write the value of $\alpha + \beta$.**

Sol. Given, $\begin{vmatrix} \sin\alpha & \cos\beta \\ \cos\alpha & \sin\beta \end{vmatrix} = -\frac{1}{2}$

$$\Rightarrow \quad \sin\alpha\sin\beta - \cos\alpha\cos\beta = -\frac{1}{2}$$
$$\Rightarrow \quad -\cos(\alpha + \beta) = -\frac{1}{2}$$
$$\Rightarrow \quad \cos(\alpha + \beta) = \cos\frac{\pi}{3}$$
$$\Rightarrow \quad \cos(\alpha + \beta) = \cos\frac{\pi}{3}$$
$$\Rightarrow \quad (\alpha + \beta) = \frac{\pi}{3}$$

4 Marks Questions

Q. 1. Using properties of determinants, show that

$$\begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2 + px - 2q^2)$$

[ISC, 2020]

Sol. L.H.S. $= \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix}$

$C_1 : C_1 - C_2$

$$= \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix}$$

$$= (x-p)\begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$

$R_2 : R_1 + R_2$

$$= (x-p)\begin{vmatrix} 1 & p & q \\ 0 & \ddot{u}+ & 2 \\ 0 & q & x \end{vmatrix}$$

Expand towards C_1

$$= (x-p)\,\mathrm{t}\begin{vmatrix} \ddot{u}+ & 2 \\ q & x \end{vmatrix}$$

$$= (x-p)(x^2 + px - 2q^2)$$

= R.H.S. **Hence Proved.**

AI Q. 2. Using properties of determinants prove that:

$$\begin{vmatrix} x & x(x^2+1) & x+1 \\ y & y(y^2+1) & y+1 \\ z & z(z^2+1) & z+1 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

[ISC, 2019]

$$\begin{vmatrix} x & x(x^{\text{ü}}+1) & x+1 \\ y & y(y^{\text{ü}}+1) & y+1 \\ z & z(z^{\text{ü}}+1) & z+1 \end{vmatrix} = \begin{vmatrix} x & x(x\ \ +1) & x \\ y & y(y\ \ +1) & y \\ z & z(z\ \ +1) & z \end{vmatrix} + \begin{vmatrix} x & x(x\ \ +1) & 1 \\ y & y(y\ \ +1) & 1 \\ z & z(z\ \ +1) & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix} + \begin{vmatrix} x & x & 1 \\ y & y & 1 \\ z & z & 1 \end{vmatrix} \Rightarrow \begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix}$$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\begin{vmatrix} x-y & x^3-y^3 & \text{ü} \\ y-z & y^3-z^3 & 0 \\ z & z^3 & 1 \end{vmatrix} = (x-y)(y-z)\begin{vmatrix} & x^2+y^2+xy & \\ 1 & y^2+z^2+yz & 0 \\ z & z^3 & 1 \end{vmatrix}$$

Again $R_2 \to R_2 - R_1$

$$(x-y)(y-z)\begin{vmatrix} 1 & x^2+y^2+xy & 0 \\ 0 & z^2-x^2+yz-xy & 0 \\ z & z^3 & 1 \end{vmatrix}$$

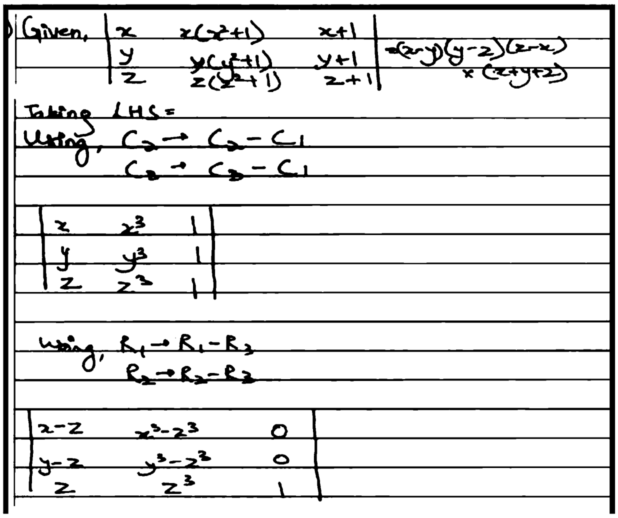
$$(x-y)(y-z)\begin{vmatrix} 1 & x^2+y^2+xy & 0 \\ 0 & (z-x)(z+x)+y(z-x) & 0 \\ z & z^3 & 1 \end{vmatrix}$$

$$(x-y)(y-z)(z-x)(x+y+z)\begin{vmatrix} 1 & x^2+y^2+xy & 0 \\ 0 & 1 & 0 \\ z & z^3 & 1 \end{vmatrix}$$

$\therefore (x-y)(y-z)(z-x)(x+y+z)$

[ISC Marking Scheme, 2019]

OR



Given, $\begin{vmatrix} x & x(x^2+1) & x+1 \\ y & y(y^2+1) & y+1 \\ z & z(z^2+1) & z+1 \end{vmatrix} = (x-y)(y-z)(z-x) \times (x+y+z)$

Taking LHS =

Using, $C_2 \to C_2 - C_1$

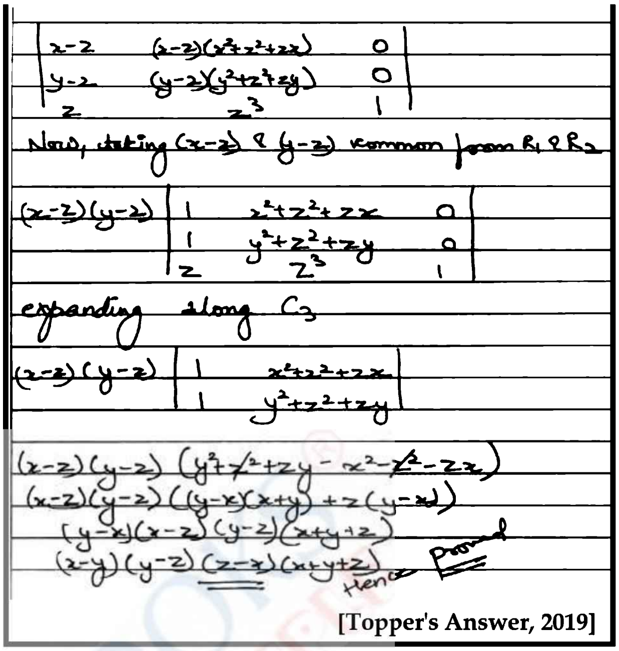
$C_3 \to C_3 - C_1$

$$\begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix}$$

Using, $R_1 \to R_1 - R_3$

$R_2 \to R_2 - R_3$

$$\begin{vmatrix} x-z & x^3-z^3 & 0 \\ y-z & y^3-z^3 & 0 \\ z & z^3 & 1 \end{vmatrix}$$



$$\begin{vmatrix} x-z & (x-z)(x^2+z^2+zx) & 0 \\ y-z & (y-z)(y^2+z^2+zy) & 0 \\ z & z^3 & 1 \end{vmatrix}$$

Now, taking $(x-z)$ & $(y-z)$ common from R_1 & R_2

$$(x-z)(y-z)\begin{vmatrix} 1 & x^2+z^2+zx & 0 \\ 1 & y^2+z^2+zy & 0 \\ z & z^3 & 1 \end{vmatrix}$$

expanding along C_3

$$(x-z)(y-z)\begin{vmatrix} 1 & x^2+z^2+zx \\ 1 & y^2+z^2+zy \end{vmatrix}$$

$(x-z)(y-z)(y^2+\cancel{z^2}+zy-x^2-\cancel{z^2}-zx)$

$(x-z)(y-z)((y-x)(x+y)+z(y-x))$

$(y-x)(x-z)(y-z)(x+y+z)$

$(x-y)(y-z)\underline{\underline{(z-x)}}(x+y+z)$ Hence Proved

[Topper's Answer, 2019]

Examiner's Comment

- Several candidates did not apply the properties of determinants in the correct order to reduce further and did not take common factors in a row/column. Some candidates solved this problem by expanding the determinant directly.

Answering Tip

- Give adequate practice in solving questions based on different types of properties. Emphasise on revision of basic concepts of Algebra.

Q. 3. Use properties of determinants to solve for x :

$$\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0 \text{ and } x \neq 0$$ **[ISC, 2018]**

Sol. Given, $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix} = 0$$

Taking common $(x + a + b + c)$ from C_1

$$\Rightarrow (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_3$

$$\Rightarrow \quad (x+a+b+c)\begin{vmatrix} 0 & 0 & -x \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix} = 0$$

Expanding along R_1, we get

$\Rightarrow \quad [x + a + b + c]\,[-x\,(b - (x - b)] = 0$

$\Rightarrow \quad x^2\,(x + a + b + c) = 0 \qquad [\because x \neq 0]$

$\Rightarrow \quad x = -(a + b + c)$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Most of the candidates made errors while applying the properties of determinants in the correct order. Hence, they failed to take common factor $(x + a + b + c)$ and could not simplify the determinant.

Answering Tip

- Learn every property of the determinants with proper examples, Logical and reasoning skills should be used to apply the correct property.

Q. 4. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & a & a \\ b & a+c & b \\ c & c & a+b \end{vmatrix} = 4abc$$

[ISC, 2016]

Sol. Let, $\Delta = \begin{vmatrix} b+c & a & a \\ b & a+c & b \\ c & c & a+b \end{vmatrix}$

$R_1 \to R_1 - R_2 - R_3$,

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & a+c & b \\ c & c & a+b \end{vmatrix}$$

$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & a+c-b & 0 \\ c & 0 & a+b-c \end{vmatrix}$$

Expanding along 1st row,

$$= 2c(ab + b^2 - bc) - 2b(0 - ca - c^2 + bc)$$

$$= 2abc + 2b^2c - 2bc^2 + 2abc + 2bc^2 - 2b^2c$$

$$= 4abc$$

Hence Proved.

[ISC Marking Scheme, 2016]

Examiner's Comment

- Many candidates were not able to attempt this part correctly. Several candidates expanded directly. At times, correct co-factors were not used.

Answering Tip

- Plenty of practice must be done in using determinant properties. Understand the idea of obtaining two zeroes in a row or a column for easiest simplification :

Q. 5. Using properties of determinants, prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

[ISC, 2015]

Sol. LHS $= \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

Applying $C_1 \to C_1 - bC_3$,

$$= \begin{vmatrix} 1+a^2+b^2 & 2ab & -2b \\ 0 & 1-a^2+b^2 & 2a \\ b(1+a^2+b^2) & -2a & 1-a^2-b^2 \end{vmatrix}$$

Apply $C_2 \to C_2 + aC_3$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

Taking common $(1 + a^2 + b^2)$ from C_1 and C_2,

$$D = (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

Replace R_3 by $R_3 - bR_1$ to get

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

Expanding by C_1, we get

$$= (1+a^2+b^2)^2(1-a^2+b^2+2a^2) = (1+a^2+b^2)^3$$

[ISC Marking Scheme, 2015]

AI **Q. 6. Using properties of determinants, prove that**

$$\begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = a^2+b^2+c^2+1$$

[ISC 2014, 2005]

Sol. LHS $= \begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$

Taking a, b, c common from R_1, R_2, R_3

$$= abc \begin{vmatrix} a+1/a & b & c \\ a & b+1/b & c \\ a & b & c+1/c \end{vmatrix}$$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= abc \begin{vmatrix} 1/a & -1/b & 0 \\ 0 & 1/b & -1/c \\ a & b & c+1/c \end{vmatrix}$$

Taking $1/a, 1/b, 1/c$ common from C_1, C_2, C_3 respectively.

$$= \frac{abc}{abc} \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= (c^2+1+b^2)+a^2$$

$$= a^2+b^2+c^2+1$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- Determinant properties were not correctly implemented by a number of candidates. Some also expanded the determinant directly, thereby going against the directions, rows and columns not correctly identified by several candidates. Some candidates used irrelevant properties.

Answering Tip

- Students need to understand that determinant properties are to be used in order to simplify the given determinant. Determinant properties and their effective utilisation need to be learnt properly with numerous examples. Plenty of practice will give the students an idea of how to obtain two zeroes in a line for easiest simplification.

Q. 7. Using properties of determinants, prove that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

[ISC, 2013]

Sol. $\text{LHS} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix}$

Apply $R_3 \rightarrow R_3 + R_1$, we get

$$= \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$$

Taking $(x + y + z)$ common from R_3,

$$= (x+y+z)\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$, we get

$$= (x+y+z)\begin{vmatrix} x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \\ 0 & 0 & 1 \end{vmatrix}$$

Taking common $(x - y)$ from C_1 and $(y - z)$ from C_2

$$= (x+y+z)(x-y)(y-z)\begin{vmatrix} 1 & 1 & z \\ x+y & y+z & z^2 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we get

$= (x + y + z)(x - y)(y - z)(y + z - x - y)$

$= (x - y)(y - z)(z - x)(x + y + z) = \text{RHS}$

Hence Proved.

Q. 8. Using properties of determinants, prove that

$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$$

[ISC, 2012]

Sol. $\text{LHS} = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

Using $C_2 \rightarrow C_2 - C_1$,

$$= \begin{vmatrix} a & b & a+b+c \\ 2a & a+2b & 4a+3b+2c \\ 3a & 3a+3b & 10a+6b+3c \end{vmatrix}$$

Using $C_3 \rightarrow C_3 - C_1 - C_2$,

$$= \begin{vmatrix} a & b & c \\ 2a & a+2b & a+b+2c \\ 3a & 3a+3b & 4a+3b+3c \end{vmatrix}$$

Using $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$,

$$= \begin{vmatrix} a & b & c \\ 0 & a & a+b \\ 0 & 3a & 4a+3b \end{vmatrix}$$

Expanding with respect to C_1,

$= a[a(4a + 3b) - 3a(a + b)]$

$= a[4a^2 + 3ab - 3a^2 - 3ab]$

$= a^3 = \text{RHS}$

Hence Proved.

Q. 9. By using properties of determinants, prove that the determinant $\begin{vmatrix} a & \sin x & \cos x \\ -\sin x & -a & 1 \\ \cos x & 1 & a \end{vmatrix}$ **is independent of** x. **[ISC, 2010]**

Sol. Let $\Delta = \begin{vmatrix} a & \sin x & \cos x \\ -\sin x & -a & 1 \\ \cos x & 1 & a \end{vmatrix}$

Apply $C_1 \rightarrow C_1 + C_2$

$$\Delta = \begin{vmatrix} a+\sin x & \sin x & \cos x \\ -(a+\sin x) & -a & 1 \\ 1+\cos x & 1 & a \end{vmatrix}$$

Apply $R_1 \rightarrow R_1 + R_2$

$$\Delta = \begin{vmatrix} 0 & -a+\sin x & 1+\cos x \\ -(a+\sin x) & -a & 1 \\ 1+\cos x & 1 & a \end{vmatrix}$$

Expanding along R_1, we get

$\Delta = -(-a + \sin x)[-a^2 - a\sin x - 1 - \cos x]$

$\quad + (1 + \cos x)[-a - \sin x + a + a\cos x]$

$= -[a^3 + a^2\sin x + a + a\cos x - a^2\sin x$

$\quad - a\sin^2 x - \sin x - \sin x\cos x]$

$\quad + [a\cos x - \sin x - \sin x\cos x$

$\quad + a\cos^2 x]$

$= -a^3$, which is independent of x.

Q. 10. By using properties of determinants, prove that:

$$\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix} = 2 + 4\sin 2x$$

[ISC, 2009]

Sol. $\text{LHS} = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$

OSWAAL BOOKS LEARNING MADE SIMPLE

Apply $C_1 \to C_1 + C_2$,

$$= \begin{vmatrix} 2 & \cos^2 x & 4\sin 2x \\ 2 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$= \begin{vmatrix} 2 & \cos^2 x & 4\sin 2x \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$= 1(2 - (-4\sin 2x))$ (Expanding along R_2)

$= 2 + 4\sin 2x =$ R.H.S. **Hence Proved.**

AI Q. 11. Prove that : $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}$

$= (b^2 - ac)(ax^2 + 2bxy + cy^2)$

[ISC, 2008]

Sol. LHS $= \begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}$

Apply $R_3 \to R_3 - xR_1 - yR_2$, we get

$$= \begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ 0 & 0 & -ax^2 - bxy - bxy - cy^2 \end{vmatrix}$$

Expanding along R_3, we get

$= -(ax^2 + 2bxy + cy^2)(ac - b^2)$

$= (b^2 - ac)(ax^2 + 2bxy + cy^2) =$ RHS

Hence Proved.

Q. 12. Using properties of determinants, solve for x :

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

Sol. Given, $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Apply $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3a-x & 3a-x & 3a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

Apply $C_2 \to C_2 - C_1$, we get

$$\begin{vmatrix} 3a-x & 0 & 3a-x \\ a-x & 2x & a-x \\ a-x & 0 & a+x \end{vmatrix} = 0$$

Apply $C_3 \to C_3 - C_1$, we get

$$\begin{vmatrix} 3a-x & 0 & 0 \\ a-x & 2x & 0 \\ a-x & 0 & 2x \end{vmatrix} = 0$$

Expanding along R_1, we get

$4x^2(3a - x) = 0$

$\therefore \quad x = 0, 3a$

Q. 13. Using properties of determinants, prove that :

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab.$$

Sol. Using $R_1 \to \frac{1}{a}R_1$, $R_2 \to \frac{1}{b}R_2$, $R_3 \to \frac{1}{c}R_3$,

$$\therefore \quad \text{LHS} = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Using $R_1 \to R_1 + R_2 + R_3$,

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from R_1,

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$,

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

Expanding along R_1,

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1)$$

$= abc + bc + ca + ab =$ RHS **Hence Proved.**

AI Q. 14. If $a + b + c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then using properties of determinants, prove that $a = b = c$.

Sol. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Using $C_1 \to C_1 + C_2 + C_3$,

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$$

Taking $(a + b + c)$ common from C_1,

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Using $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$,

$$= (a+b+c)\begin{vmatrix}1 & b & c\\ 0 & c-b & a-c\\ 0 & a-b & b-c\end{vmatrix}$$

Expanding along C_1,

$$\Delta = (a+b+c)[(c-b)(b-c) - (a-c)(a-b)]$$

$$\Rightarrow \Delta = (a+b+c)[ab+bc+ca-a^2-b^2-c^2]$$

Multiplying and dividing by 2,

$$\Delta = \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$

Given, $\Delta = 0$ and $(a+b+c) \neq 0$

Thus $[(a-b)^2+(b-c)^2+(c-a)^2] = 0$

$\therefore a-b = 0, b-c = 0, c-a = 0$ *i.e.*, $a = b = c$

Hence Proved.

Q. 15. Using the properties of determinants, prove the following :

$$\begin{vmatrix}a & b-c & c+b\\ a+c & b & c-a\\ a-b & b+a & c\end{vmatrix} = (a+b+c)(a^2+b^2+c^2).$$

Sol. LHS $= \begin{vmatrix}a & b-c & c+b\\ a+c & b & c-a\\ a-b & b+a & c\end{vmatrix}$

Multiplying and divide C_1 by a,

$$= \frac{1}{a}\begin{vmatrix}a^2 & b-c & c+b\\ a^2+ac & b & c-a\\ a^2-ab & b+a & c\end{vmatrix}$$

$C_1 \to C_1 + bC_2 + cC_3$,

$$= \frac{1}{a}\begin{vmatrix}a^2+b^2+c^2 & b-c & c+b\\ a^2+b^2+c^2 & b & c-a\\ a^2+b^2+c^2 & b+a & c\end{vmatrix}$$

Taking $(a^2+b^2+c^2)$ common from C_1

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix}1 & b-c & c+b\\ 1 & b & c-a\\ 1 & b+a & c\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix}1 & b-c & c+b\\ 0 & c & -(a+b)\\ 0 & a+c & -b\end{vmatrix}$$

Now expanding along C_1, we get

$$= \frac{1}{a}\times a(a^2+b^2+c^2)(a+b+c)$$

$= (a^2+b^2+c^2)(a+b+c) =$ RHS

Hence Proved.

Q. 16. If $A + B + C = \pi$. then find the value of :

$$\begin{vmatrix}\sin(A+B+C) & \sin B & \cos C\\ -\sin B & 0 & \tan A\\ \cos(A+B) & -\tan A & 0\end{vmatrix}$$

Sol. $\begin{vmatrix}\sin(A+B+C) & \sin B & \cos C\\ -\sin B & 0 & \tan A\\ \cos(A+B) & -\tan A & 0\end{vmatrix}$

$= \begin{vmatrix}0 & \sin B & \cos C\\ -\sin B & 0 & \tan A\\ -\cos C & -\tan A & 0\end{vmatrix}$ (As, $A + B + C = \pi$) ... (i)

Now, expanding along R_1

$$= 0\times\begin{vmatrix}0 & \tan A\\ -\tan A & 0\end{vmatrix} - \sin B\times\begin{vmatrix}-\sin B & \tan A\\ -\cos C & 0\end{vmatrix} + \cos C\times\begin{vmatrix}-\sin B & 0\\ -\cos C & -\tan A\end{vmatrix}$$

$= 0 - \sin B \tan A \cos C + \cos C \sin B \tan A$

$= 0$

Hence, value of given determinant is zero.

Q. 17. Given that $x = -9$ is a root of $\begin{vmatrix}x & 3 & 7\\ 2 & x & 2\\ 7 & 6 & x\end{vmatrix} = 0$. Find the other roots.

Sol. Given, $\begin{vmatrix}x & 3 & 7\\ 2 & x & 2\\ 7 & 6 & x\end{vmatrix} = 0$ ...(i)

Now, expanding along R_1

$$\Rightarrow \quad x\begin{vmatrix}x & 2\\ 6 & x\end{vmatrix} - 3\begin{vmatrix}2 & 2\\ 7 & x\end{vmatrix} + 7\begin{vmatrix}2 & x\\ 7 & 6\end{vmatrix} = 0$$

$$\Rightarrow x(x^2-12) - 3(2x-14) + 7(12-7x) = 0$$

$$\Rightarrow \quad x^3 - 12x - 6x + 42 + 84 - 49x = 0$$

$$x^3 - 67x + 126 = 0 \quad \text{...(ii)}$$

$x = -9$ is a root of (ii).

$x+9$ is a factor of $x^3 - 67x + 126$

Dividing $x^3 - 67x + 126$ by $x + 9$, the quotient is $x^2 - 9x + 14$.

$\therefore$ *Eq.* (ii) can be written as $(x+9)(x^2-9x+14) = 0$

Solving $x^2 - 9x + 14 = 0$, we get

$$x = \frac{9 \pm \sqrt{81-56}}{2}$$

$$= \frac{9 \pm 5}{2} = 2, 7.$$

The other roots of eq. (i) are 2 and 7.

Q. 18. If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix}p & b & c\\ a & q & c\\ a & b & r\end{vmatrix} = 0$, show that

$$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2.$$

Sol. Given, $\begin{vmatrix}p & b & c\\ a & q & c\\ a & b & r\end{vmatrix} = 0$

Apply $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$

$$\begin{vmatrix}p-a & b-q & 0\\ a & q & c\\ 0 & b-q & r-c\end{vmatrix} = 0$$

Expand along R_1,

$$(p-a)[q(r-c) - c(b-q)] - (b-q)[a(r-c) - 0] + 0 = 0$$

OSWAAL BOOKS LEARNING MADE SIMPLE

$$\Rightarrow \quad q(p-a)(r-c)-c(p-a)(b-q)-a(b-q)(r-c)=0$$

Now, divide throughout by $(p–a)\,(q–b)\,(r–c)$, we get

$$\Rightarrow \quad \frac{q}{q-b}+\frac{c}{r-c}+\frac{a}{p-a}=0$$

$$\Rightarrow \quad \frac{q}{q-b}+\left(\frac{c}{r-c}+1\right)+\left(\frac{a}{p-a}+1\right)=2$$

$$\Rightarrow \quad \frac{q}{q-b}+\frac{c+r-c}{r-c}+\frac{a+p-a}{p-a}=2$$

$$\frac{q}{q-b}+\frac{r}{r-c}+\frac{p}{p-a}=2$$

$$\Rightarrow \quad \frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$$

Hence Proved.

Q. 19. Using properties of determinants, prove that :

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 3 & 4+3p & 2+4p+3q \\ 4 & 7+4p & 2+7p+4q \end{vmatrix} = 1$$

Sol. Taking $\quad \text{LHS} = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 3 & 4+3p & 2+4p+3q \\ 4 & 7+4p & 2+7p+4q \end{vmatrix}$

Applying $R_2 \to R_2 - 3R_1$,

$$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 4 & 7+4p & 2+7p+4q \end{vmatrix}$$

Applying $R_3 \to R_3 - 4R_1$,

$$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 3 & -2+3p \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$,

$$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3,

= 1 = RHS **Hence Proved.**

Q. 20. Using properties of determinants, prove the following :

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(bc+ca+ab).$$

Sol. $\text{LHS} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Using $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$,

$$\text{LHS} = \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

Taking abc common from C_3,

$$\text{LHS} = \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$,

$$\text{LHS} = \begin{vmatrix} a^2 & a^3 & 1 \\ b^2-a^2 & b^3-a^3 & 0 \\ c^2-a^2 & c^3-a^3 & 0 \end{vmatrix}$$

Taking common $(b-a)$ from R_2, $(c-a)$ from R_3,

$$\text{LHS} = (b-a)(c-a)\begin{vmatrix} a^2 & a^3 & 1 \\ b+a & b^2+ab+a^2 & 0 \\ c+a & c^2+ac+a^2 & 0 \end{vmatrix}$$

Expanding along C_3, we get

$$\text{LHS} = (b-a)(c-a)\begin{vmatrix} b+a & b^2+ab+a^2 \\ c+a & c^2+ac+a^2 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$,

$$= (b-a)(c-a)\begin{vmatrix} b+a & b^2+ab+a^2 \\ c-b & c^2-b^2+ac-ab \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} b+a & b^2+ab+a^2 \\ -(b-c) & -(b-c)(a+b+c) \end{vmatrix}$$

Taking $-(b-c)$ common from R_2,

$$= -(b-a)(c-a)(b-c)\begin{vmatrix} a+b & a^2+b^2+ab \\ 1 & a+b+c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\big(a^2+ab+ac+ab+b^2 + bc-a^2-b^2-ab\big)$$

$$= (a-b)(b-c)(c-a)(ab+bc+ac) = \text{RHS}$$

Hence Proved.

Q. 21. Using properties of determinants, prove that

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z).$$

Sol. $\text{LHS} = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

Using $C_1 \to C_1 + C_2 + C_3$,

$$= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix}$$

Taking common $(a+x+y+z)$ from C_1,

$$= (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}$$

Using $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$,

$$= (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we get

$$\text{LHS} = (a + x + y + z)(a^2 - 0)$$
$$= a^2(a + x + y + z) = \text{RHS}$$

Hence Proved.

AI Q. 22. Using properties of determinants, prove that

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2.$$

Sol. $\text{LHS} = \begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$

Using $C_1 \to C_1 + C_2 + C_3$,

$$= \begin{vmatrix} 5x+\lambda & 2x & 2x \\ 5x+\lambda & x+\lambda & 2x \\ 5x+\lambda & 2x & x+\lambda \end{vmatrix}$$

Taking $(5x + \lambda)$ common from C_1,

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+\lambda & 2x \\ 1 & 2x & x+\lambda \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x+\lambda & 0 \\ 0 & 0 & -x+\lambda \end{vmatrix}$$

Expanding along C_1, we get

$$\text{LHS} = (5x + \lambda)\{(\lambda - x)^2 - 0\}$$
$$= (5x + \lambda)(\lambda - x)^2 = \text{RHS}$$

Hence Proved.

Q. 23. Prove the following :

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Sol. $\text{LHS} = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Taking a, b and c common from C_1, C_2 and C_3 respectively,

$$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 - C_3$,

$$= abc\begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$$

Using $R_2 \to R_2 - R_3$,

$$= abc\begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 2b & b+c & c \end{vmatrix}$$

Expanding along C_1,

$$= abc[2b(ca - c^2 + ca + c^2)]$$
$$= abc(4abc) = 4a^2b^2c^2 = \text{RHS}$$

Hence Proved.

Q. 24. Using properties of determinants, prove that :

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3.$$

Sol. $\text{LHS} = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$

Using $R_1 \to R_1 + R_2 + R_3$,

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$,

$$= \begin{vmatrix} (x+y+z) & 0 & 0 \\ 2z & 0 & -(x+y+z) \\ x-y-z & (x+y+z) & (x+y+z) \end{vmatrix}$$

Expanding with respect to R_1,

$$= (x + y + z)\{0(x + y + z) + (x + y + z)^2\}$$
$$= (x + y + z)^3 = \text{RHS}$$

Hence Proved.

Q. 25. Using properties of determinants, prove that :

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$$

Sol. $\text{LHS} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Using $C_1 \to C_1 + C_2 + C_3$,

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

Expanding along C_1,

$$\text{LHS} = 2(a + b + c)\{(a + b + c)^2 - 0\}$$
$$= 2(a + b + c)^3 = \text{RHS}$$

Hence Proved.

Q. 26. Using properties of determinants, prove that :

$$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1 + x^2 + y^2 + z^2.$$

Sol. $\text{LHS} = \begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix}$

Using $R_1 \to xR_1$, $R_2 \to yR_2$, $R_3 \to zR_3$,

$$= \frac{1}{xyz}\begin{vmatrix} x(x^2+1) & x^2y & x^2z \\ xy^2 & y(y^2+1) & y^2z \\ xz^2 & yz^2 & z(z^2+1) \end{vmatrix}$$

OSWAAL BOOKS LEARNING MADE SIMPLE

Taking common x from C_1, y from C_2 & z from C_3,

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2+1 & x^2 & x^2 \\ y^2 & y^2+1 & y^2 \\ z^2 & z^2 & z^2+1 \end{vmatrix}$$

Using $R_1 \to R_1 + R_2 + R_3$ and taking common $(x^2 + y^2 + z^2 + 1)$ from R_1

$$= \left(x^2+y^2+z^2+1\right)\begin{vmatrix} 1 & 1 & 1 \\ y^2 & y^2+1 & y^2 \\ z^2 & z^2 & z^2+1 \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$,

$$= (1+x^2+y^2+z^2)\begin{vmatrix} 1 & 0 & 0 \\ y^2 & 1 & 0 \\ z^2 & 0 & 1 \end{vmatrix}$$

Expanding along C_1,

LHS = $(1 + x^2 + y^2 + z^2)$ = RHS **Hence Proved.**

Q. 27. Using properties of determinants, prove that :

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$$

Sol. Applying $C_1 \to C_1 - (C_2 + C_3)$, we get

$$\text{LHS} = \begin{vmatrix} -2a & c+a & a+b \\ -2p & r+p & p+q \\ -2x & z+x & x+y \end{vmatrix}$$

Taking (– 2) common from C_1

$$= -2\begin{vmatrix} a & c+a & a+b \\ p & r+p & p+q \\ x & z+x & x+y \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= -2\begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix}$$

Using $C_2 \leftrightarrow C_3$,

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \text{RHS}$$

AI Q. 28. Using properties of determinants, prove that :

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

Sol.

$$\text{LHS} = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$= \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix}$$

Taking $(b - a)$ and $(c - a)$ common from R_2 and R_3 respectively,

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & b^2+a^2+ab \\ 0 & 1 & c^2+a^2+ac \end{vmatrix}$$

Using $R_3 \to R_3 - R_2$,

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & b^2+a^2+ab \\ 0 & 0 & c^2-b^2+ac-ab \end{vmatrix}$$

Taking $(c - b)$ from R_3,

$$= (b-a)(c-a)(c-b)\begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & b^2+a^2+ab \\ 0 & 0 & a+b+c \end{vmatrix}$$

Expanding with respect to C_1,

$= (a - b)(b - c)(c - a)(a + b + c)$ = RHS

Hence Proved.

Q. 29. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2.$$

Sol.

$$\text{LHS} = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$,

$$= \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix}$$

Taking $(1 + x + x^2)$ common from C_1,

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 0 & x^2-x & 1-x^2 \end{vmatrix}$$

Taking $(1 - x)$ common from R_2 and R_3,

$$= (1+x+x^2)(1-x)^2\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & 1+x \end{vmatrix}$$

Expanding with respect to C_1,

$= (1 + x + x^2)(1 - x)^2(1 + x + x^2)$

$= \{(1 - x)(1 + x + x^2)\}^2$

$= (1 - x^3)^2$ = RHS **Hence Proved.**

Q. 30. Using properties of determinants, prove the following :

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$

Sol.

$$\text{LHS} = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$,

$$= \begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$$

Taking $3x + 3y$ common from C_1,

$$= (3x+3y)\begin{vmatrix}1 & x+y & x+2y\\1 & x & x+y\\1 & x+2y & x\end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$= 3(x+y)\begin{vmatrix}1 & x+y & x+2y\\0 & -y & -y\\0 & y & -2y\end{vmatrix}$$

Taking y common from R_2 and R_3,

$$= 3(x+y)(y)(y)\begin{vmatrix}1 & x+y & x+2y\\0 & -1 & -1\\0 & 1 & -2\end{vmatrix}$$

Expanding with respect to C_1,

LHS $= 3y^2(x + y)(2 + 1)$

$= 9y^2(x + y) =$ RHS **Hence Proved.**

Q. 31. Using properties of determinants, prove the following :

$$\begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \gamma+\alpha & \alpha+\beta\end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma).$$

Sol. LHS $= \begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \gamma+\alpha & \alpha+\beta\end{vmatrix}$

Using $R_1 \to R_1 + R_3$,

$$= \begin{vmatrix}\alpha+\beta+\gamma & \gamma+\alpha+\beta & \alpha+\beta+\gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \gamma+\alpha & \alpha+\beta\end{vmatrix}$$

Taking $\alpha + \beta + \gamma$ common from R_1,

$$= (\alpha+\beta+\gamma)\begin{vmatrix}1 & 1 & 1\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \gamma+\alpha & \alpha+\beta\end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$,

$$= (\alpha+\beta+\gamma)\begin{vmatrix}1 & 0 & 0\\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2\\ \beta+\gamma & \alpha-\beta & \alpha-\gamma\end{vmatrix}$$

Taking $(\beta - \alpha)$ and $(\gamma - \alpha)$ common from C_2 and C_3,

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix}1 & 0 & 0\\ \alpha^2 & \beta+\alpha & \gamma+\alpha\\ \beta+\gamma & -1 & -1\end{vmatrix}$$

Expanding with respect to R_1,

LHS $= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(- \beta - \alpha + \gamma + \alpha)$

$= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma) =$ RHS

Hence Proved.

Q. 32. Using properties of determinants, prove the following :

$$\begin{vmatrix}b+c & a-b & a\\ c+a & b-c & b\\ a+b & c-a & c\end{vmatrix} = 3abc - a^3 - b^3 - c^3.$$

Sol. LHS $= \begin{vmatrix}b+c & a-b & a\\ c+a & b-c & b\\ a+b & c-a & c\end{vmatrix}$

Using $R_1 \to R_1 + R_2 + R_3$,

$$= \begin{vmatrix}2(a+b+c) & 0 & a+b+c\\ c+a & b-c & b\\ a+b & c-a & c\end{vmatrix}$$

Taking $(a + b + c)$ common from R_1,

$$= (a+b+c)\begin{vmatrix}2 & 0 & 1\\ c+a & b-c & b\\ a+b & c-a & c\end{vmatrix}$$

Using $C_1 \to C_1 - 2C_3$,

$$= (a+b+c)\begin{vmatrix}0 & 0 & 1\\ c+a-2b & b-c & b\\ a+b-2c & c-a & c\end{vmatrix}$$

Using $C_1 \to C_1 + C_2$,

$$= (a+b+c)\begin{vmatrix}0 & 0 & 1\\ a-b & b-c & b\\ b-c & c-a & c\end{vmatrix}$$

Expanding with respect to R_1,

LHS $= (a + b + c)[(c - a)(a - b) - (b - c)^2]$

$= (a + b + c)(- a^2 - b^2 - c^2 + ab + bc + ca]$

$= (3abc - a^3 - b^3 - c^3) =$ RHS **Hence Proved.**

Q. 33. Using properties of determinants, prove the following :

$$\begin{vmatrix}a & b & c\\ a-b & b-c & c-a\\ b+c & c+a & a+b\end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

Sol. LHS $= \begin{vmatrix}a & b & c\\ a-b & b-c & c-a\\ b+c & c+a & a+b\end{vmatrix}$

Using $C_1 \to C_1 + C_2 + C_3$,

$$= \begin{vmatrix}a+b+c & b & c\\ 0 & b-c & c-a\\ 2(a+b+c) & c+a & a+b\end{vmatrix}$$

Taking $(a + b + c)$ common from C_1,

$$= (a+b+c)\begin{vmatrix}1 & b & c\\ 0 & b-c & c-a\\ 2 & c+a & a+b\end{vmatrix}$$

Using $R_3 \to R_3 - 2R_1$,

$$= (a+b+c)\begin{vmatrix}1 & b & c\\ 0 & b-c & c-a\\ 0 & c+a-2b & a+b-2c\end{vmatrix}$$

Using $R_3 \to R_3 + R_2$,

$$= (a+b+c)\begin{vmatrix}1 & b & c\\ 0 & b-c & c-a\\ 0 & a-b & b-c\end{vmatrix}$$

Expanding with respect to C_1,

LHS $= (a + b + c)[(b - c)^2 - (c - a)(a - b)]$

OSWAAL BOOKS LEARNING MADE SIMPLE

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= (a^3+b^3+c^3-3abc) = \text{RHS}$$

Hence Proved.

Q. 34. Using properties of determinants, prove the following :

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\,(a+b+c)(a^2+b^2+c^2).$$

Sol. $\text{LHS} = \begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$

Using $C_2 \to C_2 - 2C_1 - 2C_3$,

$$= \begin{vmatrix} a^2 & -a^2-b^2-c^2 & bc \\ b^2 & -a^2-b^2-c^2 & ca \\ c^2 & -a^2-b^2-c^2 & ab \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$ common from C_2,

$$= (a^2+b^2+c^2)\begin{vmatrix} a^2 & -1 & bc \\ b^2 & -1 & ca \\ c^2 & -1 & ab \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$= (a^2+b^2+c^2)\begin{vmatrix} a^2 & -1 & bc \\ b^2-a^2 & 0 & c(a-b) \\ c^2-a^2 & 0 & b(a-c) \end{vmatrix}$$

Taking $(a-b)$ and $(c-a)$ common from R_2 and R_3 and multiplying C_2 by (-1),

$$= (a-b)(c-a)(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ -a-b & 0 & c \\ c+a & 0 & -b \end{vmatrix}$$

Using $R_2 \to R_2 + R_3$,

$$= (a-b)(c-a)(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ c-b & 0 & c-b \\ c+a & 0 & -b \end{vmatrix}$$

Expanding with respect to C_2,

$$\text{LHS} = (a-b)(c-a)(a^2+b^2+c^2)$$

$$[1\,\{-b\times(c-b)-(c-b)(c+a)\}]$$

$$= (a-b)(b-c)(c-a)(a^2+b^2+c^2)(a+b+c)$$

$$= \text{RHS}$$

Hence Proved.

6 Marks Questions

Q. 1. Using properties of determinants, prove

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)\,(x-y)\,(y-z)\,(z-x),$$

where p is any scalar

[ISC Specimen Paper, 2018, 2021]

Sol. Let $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

(Taking common x, y, z and p from R_1, R_2, R_3 and C_1, respectively)

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (pxyz)\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

(Interchanging columns)

$$= (1+pxyz)\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\therefore \quad \Delta = (1+pxyz)\begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 0 \end{vmatrix}$$

$$= (1+pxyz)\,(y-x)\,(z-x)\begin{vmatrix} x & x^2 & 1 \\ 1 & y+x & 0 \\ 1 & z+x & 0 \end{vmatrix}$$

(Taking common $(y-x)$ and $(z-x)$ from R_1 and R_2 respectively)

Expanding along C_1, we get

$$\Delta = (1+pxyz)\,(y-x)\,(z-x)\,[(z+x)-(y+x)]$$

$$= (1+pxyz)\,(x-y)\,(y-z)\,(z-x)$$

Hence Proved.

Q. 2. Using properties of determinants, prove that :

$$\begin{vmatrix} a & b & b+c \\ c & a & c+a \\ b & c & a+b \end{vmatrix} = (a+b+c)\,(a-c)^2$$

[ISC, 2017]

Sol. LHS $= \begin{vmatrix} a & b & b+c \\ c & a & c+a \\ b & c & a+b \end{vmatrix}$

Apply $R_1 \to R_1 + R_2 + R_3$,

$$= \begin{vmatrix} a+b+c & a+b+c & 2(a+b+c) \\ c & a & c+a \\ b & c & a+b \end{vmatrix}$$

Taking common $(a + b + c)$ from R_1,

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 2 \\ c & a & c+a \\ b & c & a+b \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$ and $C_3 \to C_3 - 2C_2$

$$= (a+b+c)\begin{vmatrix} 0 & 1 & 0 \\ c-a & a & c-a \\ b-c & c & a+b-2c \end{vmatrix}$$

Expanding along R_1, we get

LHS $= (a + b + c)\, [-1\{(c - a)\,(a + b - 2c)$

$- (c - a)\,(b - c)\}]$

$= (a + b + c)\,(a - c)\,(a + b - 2c - b + c)$

$= (a + b + c)\,(a - c)^2$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Some candidates did not apply the properties of determinants in the right order. Many candidates used $R_1 \to R_1 + R_2 + R_3$ but failed to take common $(a + b + c)$. In addition, candidates used row/column operation second time but failed to simplify the determinant.

Answering Tip

- Practice adequate question based on properties of determinants. Logical and reasoning skill should be used to apply correct property.

Q. 3. Using properties of determinants, show that $p\alpha^2+2q\alpha+r=0$, given that : p, q and r are not in G.P. and

$$\begin{vmatrix} 1 & \dfrac{q}{p} & \alpha+\dfrac{q}{p} \\ 1 & \dfrac{r}{q} & \alpha+\dfrac{r}{q} \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$ **[ISC, 2011]**

Sol. $$\begin{vmatrix} 1 & \dfrac{q}{p} & \alpha+\dfrac{q}{p} \\ 1 & \dfrac{r}{q} & \alpha+\dfrac{r}{q} \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$

Apply $R_1 \to R_1 - R_2$, we get

$$\begin{vmatrix} 0 & \dfrac{q^2-pr}{pq} & \dfrac{q^2-pr}{pq} \\ 1 & \dfrac{r}{q} & \alpha+\dfrac{r}{q} \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$

Apply $C_2 \to C_2 - C_3$, we get

$$\begin{vmatrix} 0 & 0 & \dfrac{q^2-pr}{pq} \\ 1 & -\alpha & \alpha+\dfrac{r}{q} \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{q^2-pr}{pq}[(q\alpha+r)+(p\alpha+q)\alpha] = 0$$

(Expand along R_1)

$$\Rightarrow \quad \frac{q^2-pr}{pq}[q\alpha+r+p\alpha^2+q\alpha] = 0$$

or $$\frac{q^2-pr}{pq}[p\alpha^2+2q\alpha+r] = 0$$

Since p, q and r are not in G.P.,

$$\Rightarrow \frac{p}{q} \neq \frac{q}{r} \text{ i.e., } pr \neq q^2 \text{ or } q^2 - pr \neq 0$$

$$\therefore \quad p\alpha^2 + 2q\alpha + r = 0.$$

Q. 4. Using the properties of determinants, show that:

$$\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x+y+z)^3$$

[ISC, 2007]

Sol. LHS $= \begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$

Apply $R_1 \to R_1 + R_2 + R_3$,

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Taking $(x + y + z)$ common from R_1

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$= (x+y+z)\begin{vmatrix} 0 & 0 & 1 \\ y+z+x & -(x+y+z) & 2y \\ 0 & x+y+z & z-x-y \end{vmatrix}$$

Taking $(x + y + z)$ common from C_1 and C_2

$$= (x+y+z)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2y \\ 0 & 1 & z-x-y \end{vmatrix}$$

OSWAAL BOOKS LEARNING MADE SIMPLE

$= (x + y + z)^3\{1(1 - 0)\}$ (Expand along R_1)

$= (x + y + z)^3 = \text{RHS}$ **Hence Proved.**

Q. 5. Show that :

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha\beta + \beta\gamma + \gamma\alpha)$$

[ISC, 2006]

Sol. Here,

$$\text{LHS} = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix}$$

Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$,

$$= \begin{vmatrix} 1 & 0 & 0 \\ \alpha^2 & \beta^2 - \alpha^2 & \gamma^2 - \alpha^2 \\ \alpha^3 & \beta^3 - \alpha^3 & \gamma^3 - \alpha^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \alpha^2 & (\beta+\alpha)(\beta-\alpha) & (\gamma+\alpha)(\gamma-\alpha) \\ \alpha^3 & (\beta-\alpha)(\beta^2+\alpha\beta+\alpha^2) & (\gamma-\alpha)(\gamma^2+\gamma\alpha+\alpha^2) \end{vmatrix}$$

Taking $(\beta - \alpha)$ common from C_2 and $(\gamma - \alpha)$ common from C_3,

$$= (\beta-\alpha)(\gamma-\alpha) \begin{vmatrix} 1 & 0 & 0 \\ \alpha^2 & (\beta+\alpha) & (\gamma+\alpha) \\ \alpha^3 & (\beta^2+\alpha\beta+\alpha^2) & (\gamma^2+\gamma\alpha+\alpha^2) \end{vmatrix}$$

$= (\beta - \alpha)(\gamma - \alpha)\, 1[(\beta + \alpha)(\gamma^2 + \gamma\alpha + \alpha^2)$
$\quad - (\gamma + \alpha)(\beta^2 + \alpha\beta + \alpha^2)]$

(Expanding along R_1)

$= (\beta - \alpha)(\gamma - \alpha)[\beta\gamma^2 + \alpha\beta\gamma + \beta\alpha^2 + \alpha\gamma^2$
$\quad + \gamma\alpha^2 + \alpha^3 - \gamma\beta^2 - \alpha\beta\gamma - \gamma\alpha^2 - \alpha\beta^2$
$\quad - \alpha^2\beta - \alpha^3]$

$= (\beta - \alpha)(\gamma - \alpha)[\beta\gamma^2 - \gamma\beta^2 + \alpha\gamma^2 - \alpha\beta^2]$

$= (\beta - \alpha)(\gamma - \alpha)[\beta\gamma(\gamma - \beta) + \alpha(\gamma^2 - \beta^2)]$

$= (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)[\beta\gamma + \alpha(\gamma+\beta)]$

$= (\beta - \alpha)(\gamma - \alpha)(\gamma - \beta)[\beta\gamma + \alpha\gamma+\alpha\beta)]$

$= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= \text{RHS}$ **Hence Proved.**

Q. 6. If $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ **then show that $xyz = -1$.** **[ISC, 2004]**

Sol. Given that $\begin{vmatrix} x^2 & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = 0$$

Apply $C_1 \leftrightarrow C_2$ in second determinant

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (-1)xyz \begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} = 0$$

Apply $C_2 \leftrightarrow C_3$ in second determinant

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (-1)(-1)xyz \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \qquad (1+xyz) \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = 0$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$(1+xyz) \begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 0 \end{vmatrix} = 0$$

Expand along C_3,

$(1+xyz)1[(z^2-x^2)(y-x)-(y^2-x^2)(z-x)] = 0$

$\Rightarrow (1+xyz)[(z+x)(z-x)(y-x)$
$\qquad -(y+x)(y-x)(z-x)] = 0$

$\Rightarrow \qquad (1+xyz)(z-x)(y-x)[z+x-y-x] = 0$

$\Rightarrow \qquad (1+xyz)(z-x)(y-x)(z-y) = 0$

Since $x \neq y \neq z$, $z - x \neq 0$, $y - x \neq 0$ and $z - y \neq 0$

$\therefore \qquad 1+xyz = 0$

$\Rightarrow \qquad xyz = -1$ **Hence Proved.**

Q. 7. Using properties of determinants, prove that

$$\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & yzx & zxy \end{vmatrix} = xyz(x-y)(y-z)(z-x)(xy+yz+zx)$$

Sol. $\text{LHS} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & yzx & zxy \end{vmatrix}$

Taking xyz common from R_3,

$$= xyz \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= xyz\begin{vmatrix} x^2 & y^2-x^2 & z^2-x^2 \\ x^3 & y^3-x^3 & z^3-x^3 \\ 1 & 0 & 0 \end{vmatrix}$$

Taking common $(y-x)$ from C_2 and $(z-x)$ from C_3,

$$xyz(y-x)(z-x)\begin{vmatrix} x^2 & (y+x) & (z+x) \\ x^3 & (y^2+x^2+xy) & (x^2+z^2+xz) \\ 1 & 0 & 0 \end{vmatrix}$$

Now expanding along C_1

$$= xyz(y-x)(z-x)[x^2(0-0)+(y+x)(z^2+x^2+xz) - (z+x)(y^2+x^2+xy)]$$

$$= xyz(y-x)(z-x)[yz^2+yx^2+xyz+xz^2+x^3+x^2z - zy^2-zx^2-xyz-xy^2-x^3-x^2y]$$

$$= xyz(y-x)(z-x)[yz^2+xz^2-zy^2-xy^2]$$

$$= xyz(y-x)(z-x)[yz^2-zy^2+xz^2-xy^2]$$

$$= xyz(y-x)(z-x)(z-y)(xy+yz+zx)$$

$$= xyz(x-y)(y-z)(z-x)(xy+yz+zx)$$

= RHS **Hence Proved.**

Q. 8. Using properties of determinants, show that ΔABC is isosceles if :

$$\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1+\cos A} & \mathbf{1+\cos B} & \mathbf{1+\cos C} \\ \mathbf{\cos^2 A+\cos A} & \mathbf{\cos^2 B+\cos B} & \mathbf{\cos^2 C+\cos C} \end{vmatrix} = 0$$

Sol. Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$,

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & \cos B-\cos A & \cos C-\cos A \\ \cos^2 A+\cos A & \begin{pmatrix} \cos^2 B+\cos B \\ -\cos^2 A-\cos A \end{pmatrix} & \begin{pmatrix} \cos^2 C+\cos C \\ -\cos^2 A-\cos A \end{pmatrix} \end{vmatrix} = 0$$

$R_3 \to R_3 - R_2$,

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & \cos B-\cos A & \cos C-\cos A \\ \cos^2 A-1 & \cos^2 B-\cos^2 A & \cos^2 C-\cos^2 A \end{vmatrix} = 0$$

Taking $(\cos B - \cos A)$ common from C_2 and $(\cos C - \cos A)$ common from C_3,

$(\cos B - \cos A) \times (\cos C - \cos A)$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & 1 & 1 \\ \cos^2 A-1 & \cos B+\cos A & \cos C+\cos A \end{vmatrix} = 0$$

$C_3 \to C_3 - C_2$,

$(\cos B - \cos A) \times (\cos C - \cos A)$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & 1 & 0 \\ \cos^2 A-1 & \cos B+\cos A & \cos C-\cos B \end{vmatrix} = 0$$

$$\therefore\ (\cos B-\cos A)\times(\cos C-\cos A)1\,[(\cos C-\cos B)-0]=0$$

(Expanding along C_1)

$$\therefore\ \cos B=\cos A \text{ or } \cos C=\cos A \text{ or } \cos C=\cos B$$

$$\Rightarrow\ \angle B=\angle A \text{ or } \angle C=\angle A \text{ or } \angle C=\angle B$$

$\Rightarrow$ ΔABC is an isosceles triangle.

Q. 9. If $a+b+c=0$ and $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0$, then show that $x = 0$ or $x=\pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$.

Sol. Given, $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0$

Apply $C_1 \to C_1 + C_2 + C_3$,

$$\begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix}=0$$

Take $(a+b+c-x)$ common from C_1,

$$\therefore\quad (a+b+c-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix}=0$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$(a+b+c-x)\begin{vmatrix} 1 & c & b \\ 0 & b-x-c & a-b \\ 0 & a-c & c-x-b \end{vmatrix}=0$$

Expand along C_1, we get

$$(a+b+c-x)1\,[(b-x-c)(c-x-b)-(a-c)(a-b)]=0$$

$$\Rightarrow (a+b+c-x)\begin{bmatrix} bc-bx-b^2-xc+x^2+xb \\ -c^2+cx+bc-(a^2-ab-ac+bc) \end{bmatrix}=0$$

$$\Rightarrow\ (a+b+c-x)\left[x^2-(a^2+b^2+c^2-ab-bc-ca)\right]=0$$

Since $a+b+c = 0$,

$$-x[x^2-\{(a^2+b^2-c^2)-(ab+bc+ca)\}]=0$$

Now, since

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$

$$\Rightarrow\quad a^2+b^2+c^2=-2(ab+bc+ca) \qquad \text{...(i)}$$

(as $a+b+c = 0$)

$$\therefore\quad -x\left[x^2-\{-2(ab+bc+ca)-(ab+bc+ca)\}\right]=0$$

$$\Rightarrow\quad -x[x^2+3(ab+bc+ca)]=0$$

$$\Rightarrow\quad -x\left[x^2+\left(-\frac{3}{2}(a^2+b^2+c^2)\right)\right]=0 \qquad \text{(from (i))}$$

$$\Rightarrow\quad x=0 \text{ or } x^2=\frac{3}{2}(a^2+b^2+c^2)$$

$$\Rightarrow\quad x=0 \text{ or } x=\pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$$

OSWAAL BOOKS LEARNING MADE SIMPLE

Q. 10. Prove that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$ **is divisible by $(x + y + z)$, and hence find the quotient.**

Sol. $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$= \begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ (x-y)(x+y+z) & (y-z)(x+y+z) & (z-x)(x+y+z) \\ (x-z)(x+y+z) & (y-x)(x+y+z) & (z-y)(x+y+z) \end{vmatrix}$$

Taking $(x + y + z)$ common from R_2 and R_3,

$$= (x+y+z)^2 \begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ x-y & y-z & z-x \\ x-z & y-x & z-y \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$,

$= (x + y + z)^2$

$$\begin{vmatrix} (xy+yz+zx)-(x^2+y^2+z^2) & zx-y^2 & xy-z^2 \\ 0 & y-z & z-x \\ 0 & y-x & z-y \end{vmatrix}$$

Expanding along C_1,

$= (x + y + z)^2[xy + yz + zx - (x^2 + y^2 + z^2)]$

$[(y - z)(z - y) - (z - x)(y - x)]$

$= (x + y + z)^2[xy + yz + zx - (x^2 + y^2 + z^2)]$

$[yz - y^2 - z^2 + yz - yz + xz + xy - x^2]$

$= (x + y + z)^2[xy + yz + zx - (x^2 + y^2 + z^2)]^2$

Hence, it is divisible by $(x + y + z)$ and the quotient is $(x + y + z)[xy + yz + zx - (x^2 + y^2 + z^2)]^2$.

Q. 11. Using properties of determinants, prove that :

$$\begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3.$$

Sol. LHS $= \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix}$

Apply $(R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3)$,

$$= \frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & zx^2 \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$,

$$= \begin{vmatrix} (y+z)^2 & x^2-(y+z)^2 & x^2-(y+z)^2 \\ y^2 & (x+z)^2-y^2 & 0 \\ z^2 & 0 & (x+y)^2-z^2 \end{vmatrix}$$

Taking $(x + y + z)$ common from C_2 and C_3,

$$= (x+y+z)^2 \begin{vmatrix} (y+z)^2 & x-y-z & x-y-z \\ y^2 & x+z-y & 0 \\ z^2 & 0 & x+y-z \end{vmatrix}$$

Apply, $R_1 \to R_1 - R_2 - R_3$,

$$= (x+y+z)^2 \begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x+z-y & 0 \\ z^2 & 0 & x+y-z \end{vmatrix}$$

Apply, $C_2 \to yC_2, C_3 \to zC_3$,

$$= \frac{(x+y+z)^2}{yz} \begin{vmatrix} 2yz & -2yz & -2yz \\ y^2 & xy+zy-y^2 & 0 \\ z^2 & 0 & zx+yz-z^2 \end{vmatrix}$$

Apply, $C_2 \to C_2 + C_1$ & $C_3 \to C_3 + C_1$,

$$= \frac{(x+y+z)^2}{yz} \begin{vmatrix} 2yz & 0 & 0 \\ y^2 & xy+zy & y^2 \\ z^2 & z^2 & zx+yz \end{vmatrix}$$

Taking yz common from R_1, y from R_2 and z from R_3

$$= (x+y+z)^2 yz \begin{vmatrix} 2 & 0 & 0 \\ y & x+z & y \\ z & z & x+y \end{vmatrix}$$

Expanding along R_1

$= (x + y + z)^2 yz\ [2\{x + z)(x + y) - zy\}]$

$= (x + y + z)^2 yz\ [2x\ (x + y + z)]$

$= 2xyz\ (x + y + z)^3$

= RHS **Hence, Proved.**

Q. 12. Using properties of determinants, prove that

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ac)^3.$$

Sol. $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$,

$$\text{LHS} = \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$,

$$= (ab+bc+ac)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$C_2 \to C_2 - C_1$, $C_3 \to C_2 - C_1$,

$$= (ab+bc+ac)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -(ab+bc+ac) & 0 \\ ac+bc & 0 & -(ab+bc+ca) \end{vmatrix}$$

Expanding along R_1,

$= (ab + bc + ca)^3 =$ RHS **Hence Proved.**

Q. 13. Using properties of determinants, prove that

$$\begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (z+y)^2 & xy \\ zy & xy & (z+x)^2 \end{vmatrix} = 2xyz(x+y+z)^3.$$

Sol. Try yourself, similar to Q. 11 in 6 marks questions.

Q. 14. Using properties of determinants, prove that :

$$\begin{vmatrix} \frac{(a+b)^2}{c} & c & c \\ a & \frac{(b+c)^2}{a} & a \\ b & b & \frac{(c+a)^2}{b} \end{vmatrix} = 2(a+b+c)^3.$$

Sol. LHS = $\begin{vmatrix} \frac{(a+b)^2}{2} & c & c \\ b & \frac{(b+c)^2}{a} & a \\ b & b & \frac{(c+a)^2}{b} \end{vmatrix}$

Taking common $\frac{1}{c}, \frac{1}{a}$ and $\frac{1}{b}$ from R_1, R_2 and R_3 respectively,

$$= \frac{1}{abc}\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}$$

Apply $C_1 \rightarrow C_1 - C_3$, $C_2 \rightarrow C_2 - C_3$,

$$= \frac{1}{abc}\begin{vmatrix} (a+b+c)(a+b-c) & 0 & c^2 \\ 0 & (b+c+a)(b+c-a) & a^2 \\ (b+c+a)(b-c-a) & (b+c+a)(b-c-a) & (c+a)^2 \end{vmatrix}$$

Taking $(a + b + c)$ common from C_1 and C_2 respectively

$$= \frac{(a+b+c)^2}{abc}\begin{vmatrix} (a+b-c) & 0 & c^2 \\ 0 & (b+c-a) & a^2 \\ (b-c-a) & (b-c-a) & (c+a)^2 \end{vmatrix}$$

$R_3 \rightarrow R_3 - R_1 - R_2$,

$$= \frac{(a+b+c)^2}{abc}\begin{vmatrix} (a+b-c) & 0 & c^2 \\ 0 & (b+c-a) & a^2 \\ (-2a) & (-2c) & 2ca \end{vmatrix}$$

Multiply c to C_1 and a to C_2

$$= \frac{(a+b+c)^2}{abcca}\begin{vmatrix} (ac+bc-c^2) & 0 & c^2 \\ 0 & (ba+ca-a^2) & a^2 \\ (-2ac) & (-2ca) & 2ca \end{vmatrix}$$

$C_1 \rightarrow C_1 + C_3$, $C_2 \rightarrow C_2 + C_3$,

$$= \frac{(a+b+c)^2}{abcca}\begin{vmatrix} (ac+bc) & c^2 & c^2 \\ a^2 & (ba+ca) & a^2 \\ 0 & 0 & 2ca \end{vmatrix}$$

Taking common c from R_1, a from R_2 and $2ac$ from R_3

$$= \frac{(a+b+c)^2 2c^2a^2}{abcca}\begin{vmatrix} (a+b) & c & c \\ a & (b+c) & a \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3

$$= \frac{2(a+b+c)^2}{b}(ab+ac+b^2+bc-ac)$$

$= 2(a + b + c)^3$ = RHS **Hence Proved.**

Q. 15. If $p \neq 0, q \neq 0$ and $\begin{vmatrix} p & q & p\alpha+q \\ q & r & q\alpha+r \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$, then using properties of determinants, prove that at least one of the following statements is true :

(a) p, q, r are in G.P.,

(b) α is a root of the equation $px^2 + 2qx + r = 0$.

Sol. Given $\begin{vmatrix} p & q & p\alpha+q \\ q & r & q\alpha+r \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$

Multiply R_1 by q and R_2 by p

$$\Rightarrow \quad \frac{1}{pq}\begin{vmatrix} pq & q^2 & pq\alpha+q^2 \\ pq & pr & pq\alpha+pr \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$

$R_1 \rightarrow R_1 - R_2$,

$$\Rightarrow \quad \frac{1}{pq}\begin{vmatrix} 0 & q^2-pr & q^2-pr \\ pq & pr & pq\alpha+pr \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$

Taking $q^2 - pr$ common from R_1

$$\Rightarrow \frac{q^2-pr}{pq}\begin{vmatrix} 0 & 1 & 1 \\ pq & pr & pq\alpha+pr \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$

Taking p common from R_2

$$\Rightarrow \quad \frac{q^2-pr}{pq}p\begin{vmatrix} 0 & 1 & 1 \\ q & r & q\alpha+r \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$

$C_2 \rightarrow C_2 - C_3$,

$$\Rightarrow \quad \frac{q^2-pr}{pq}p\begin{vmatrix} 0 & 0 & 1 \\ q & -q\alpha & q\alpha+r \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$$

$$\frac{q^2-pr}{q}(q^2\alpha + rq + pq\alpha^2 + q^2\alpha) = 0$$

$$\Rightarrow \quad (q^2 - pr)(2q\alpha + r + p\alpha^2) = 0$$

(a) Now, if $q^2 - pr = 0$

Then p, q, r are in GP

(b) Now, if, $2q\alpha + r + p\alpha^2 = 0$

Then α is root of the equation

$$2qx + r + px^2 = 0$$

OSWAAL BOOKS LEARNING MADE SIMPLE

TOPIC-2

Applications of Determinants

Scan to know more about this topic

Applications of determinants

Scan to know more about this topic

Cramer's rule

Scan to know more about this topic

Area of triangle and collinearity of three points

Revision Notes

- **Area of Triangle**

  The area of triangle having vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- **Condition of collinearity of Three Points**

  Let three given points be $A(x_1, y_1,)$, $B(x_2, y_2,)$ and $C(x_3, y_3,)$.

  If A, B, C are collinear, then area of triangle ABC is zero.

  i.e., A, B, C are collinear $\Leftrightarrow$ Area of triangle $ABC = 0$

  or

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

- **Solution of a nonhomogeneous system linear equations**

  Cramer's Rule :

  Let the three nonhomogeneous linear equations be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

  The solution of the system of linear equations is given by

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

  Where,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},\ \text{Provided that } D \neq 0$$

  Property I : If every element in a row or a column of a determinant is multiplied by the same non-zero constant k, then the value of the determinant gets multiplied by k, *i.e.,*

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

  Property II : A determinant can be expressed as the sum of several determinants of the same order, *i.e.,*

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

  Property III : The value of the determinant is not affected if the elements of a row or column are increased or diminished by the same multiple of the corresponding elements of any other row or column, *i.e.,*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1+qc_1 & b_1 & c_1 \\ a_2+pb_2+qc_2 & b_2 & c_2 \\ a_3+pb_3+qc_3 & b_3 & c_3 \end{vmatrix}$$

  Property IV : If each elements of a row or column of a determinant is zero, its value is zero, *i.e.,*

$$\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

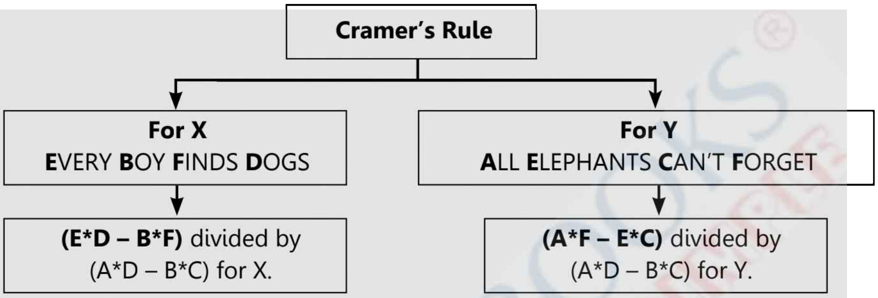
- **Condition for consistency**
  (a) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

  (b) If $D = 0$ and $D_x = D_y = D_z = 0$, the given system of equations is consistent has infinitely many solutions.
  (c) If $D = 0$ and atleast one of the determinants D_x, D_y, D_z is non-zero, then the given system of equations is inconsistent.

Mnemonics

Mnemonic for Cramer's Rule



Cramer's Rule
For X
EVERY BOY FINDS DOGS
For Y
ALL ELEPHANTS CAN'T FORGET
(E*D – B*F) divided by (A*D – B*C) for X.
(A*F – E*C) divided by (A*D – B*C) for Y.

Interpretation

$2x - 3y = 9$

$x + 5y = -2$

Comparing the above set of equations by

$Ax + By = E$

$Cx + Dy = F$

We, get $A = 2$, $B = -3$, $C = 1$, $D = 5$, $E = 9$ and $F = -2$

So, $x = \dfrac{9\times5-(-3)\times(-2)}{2\times5-(-3)\times1} = \dfrac{45-6}{10+3} = \dfrac{39}{13} = 3$

and $y = \dfrac{2\times(-2)-9\times1}{2\times5-(-3)\times1} = \dfrac{-4-9}{10+3} = \dfrac{-13}{13} = -1$

So, the solution set is (3, –1).

OSWAAL BOOKS® LEARNING MADE SIMPLE

1 Mark Questions

AI Q. 1. Cramer's Rule fails for ....

(a) Determinant > 0 **(b) Determinant < 0**

(c) Determinant = 0 **(d) Determinant = non-real**

Sol. Correct Option : (c)

Explanation: This is because Cramer's rule involves division by determinant which should never be equal to 0 leading to not defined numbers.

Q. 2. Cramer's Rule is not suitable for which type of problems?

(a) Small systems with 4 unknowns

(b) Systems with 2 unknowns

(c) Large systems

(d) Systems with 3 unknowns

Sol. Correct Option : (c)

Explanation: Generally, in large systems, excessive multiplicative operations are required which becomes very cumbersome to solve.

AI Q. 3. If the points $(x, -2)$, $(5, 2)$ and $(8, 8)$ are collinear, find x using determinants.

Sol. Given points $(x, -2)$, $(5, 2)$ and $(8, 8)$ are collinear *i.e.*, area of triangle is zero.

$$\therefore \quad \Delta = 0 \Rightarrow \begin{vmatrix} x & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = 0$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$\Rightarrow \quad \begin{vmatrix} x & -2 & 1 \\ 5-x & 4 & 0 \\ 8-x & 10 & 0 \end{vmatrix} = 0$$

Expanding along C_3, we get

$$\Rightarrow \quad 10(5-x) - 4(8-x) = 0$$
$$\Rightarrow \quad -10x + 4x + 50 - 32 = 0$$
$$\Rightarrow \quad -6x = -18$$
$$\Rightarrow \quad x = 3$$

Q. 4. Using determinants, find the area of the triangle whose vertices are (– 2, 4), (2, – 6) and (5, 4). Are given points collinear ?

Sol. Let Δ be the area of triangle, Then,

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix}$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & 4 & 1 \\ 4 & -10 & 0 \\ 7 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta = \frac{1}{2}[7\{0 + 10\}]$$

$$= \frac{1}{2}(70) = 35 \text{ sq. units.}$$

Clearly, $\Delta \neq 0$ therefore given points are not collinear.

2 Marks Questions

Q 1. Using determinants, find the values of k, if the area of triangle with vertices (–2, 0), (0, 4) and (0, k) is 4 square units. **[ISC, 2019]**

Sol.

$$\text{Area of } \Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & 4 \\ 1 & 0 & k \end{vmatrix} = 4$$

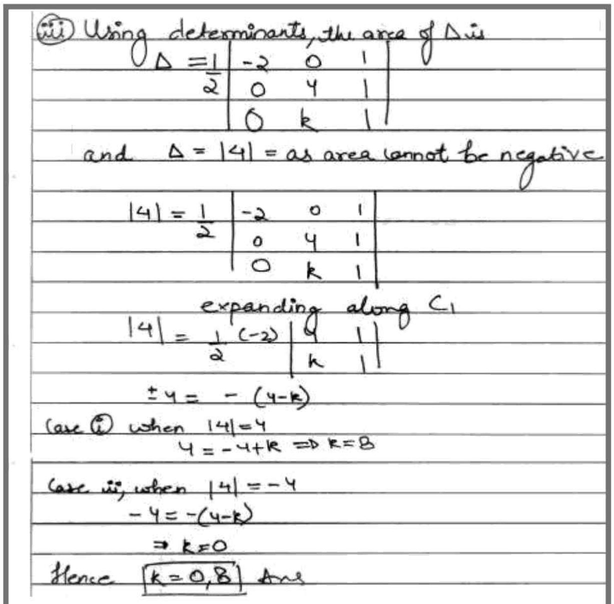
$$= |1(0) + 2(k-4) + 0| = 8$$

$$= |k-4| = 4$$

$$\Rightarrow k - 4 = \pm 4$$

$k = 8$ or 0 **[ISC Marking Scheme, 2019]**

OR



(iii) Using determinants, the area of Δ is

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

and Δ = |4| = as area cannot be negative

$$|4| = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

expanding along C_1

$$|4| = \frac{1}{2}(-2)\begin{vmatrix} 4 & 1 \\ k & 1 \end{vmatrix}$$

$$\pm 4 = -(4-k)$$

Case (i) when $|4| = 4$

$$4 = -4 + k \Rightarrow k = 8$$

Case (ii), when $|4| = -4$

$$-4 = -(4-k)$$

$$\Rightarrow k = 0$$

Hence $\boxed{k = 0, 8}$ Ans

[Topper's Answer, 2019]

Q. 2. Show that the points (a, $b + c$), (b, $c + a$) and (c, $a + b$) are collinear.

Sol. We have,

$$\Delta = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$,

$$= \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix}$$

Taking common $(a + b + c)$ from C_1,

$$= (a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix}$$

$$= (a + b + c) \times 0$$

[$\because$ C_1 and C_3 are identical]

$$= 0$$

Since, Δ = 0, therefore given points are collinear.

AI Q. 3. Show the following system of equations is inconsistent :

$2x + y = 3$, $4x + 2y = 5$

Sol. Given, system of equations :

$$2x + y = 3$$

$$4x + 2y = 5$$

$$\therefore \quad D = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4 - 4 = 0$$

$$\text{and} \quad D_x = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6 - 5 = 1 \neq 0$$

Since, $D = 0$ and $D \neq 0$, hence, given system of equations is inconsistent.

Q. 4. Find the equation of the line joining A(1, 3) and B(0, 0) using determinants and find the value of k if D(k, 0) is a point such that area of ΔABD is 3 square units.

Sol. Using determinants, the line joining A(1, 3) and B(0, 0) is given by $\begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$

Expanding along R_3, we get

$$1(3x - y) = 0 \Rightarrow y = 3x$$

Now, $D(k, 0)$ is a point such that area ΔABD = 3 sq. units.

$$\therefore \quad \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

Expanding along R_2, we get

$$(0 - 3k) = \pm 6$$

$$\Rightarrow \quad k = \pm 2$$

AI **Q. 5. If the points (a_1, b_1), (a_2, b_2) and (a_1+a_2, b_1+b_2) are collinear, show that $a_1b_2 = a_2b_1$.**

Sol. Given points are collinear,

$$\therefore \quad \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_1+a_2 & b_1+b_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow a_1(b_2 - b_1 - b_2) - b_1(a_2 - a_1 - a_2) + 1(a_2b_1 + a_2b_2 - a_1b_2 - a_2b_2) = 0$$

$$\Rightarrow \quad -a_1b_1 + b_1a_1 + a_2b_1 - a_1b_2 = 0$$

$$\Rightarrow \quad a_2b_1 - a_1b_2 = 0$$

$$\Rightarrow \quad a_1b_2 = a_2b_1$$

Hence proved.

4 Marks Questions

AI **Q. 1. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of an equilateral triangle whose each side is equal to a, then prove that**

$$\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = 3a^4$$

Sol. Let Δ the are of triangle ABC, then

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad 4\Delta = 2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad 4\Delta = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}$$

Squaring both sides, we get

$$\Rightarrow \quad 16\Delta^2 = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 \quad \text{...(i)}$$

We know that, area of an equilateral triangle with each side equal to 'a' is $\frac{\sqrt{3}}{4}a^2$.

$$\therefore \quad \Delta = \frac{\sqrt{3}}{4}a^2$$

$$\Rightarrow \quad 16\Delta^2 = 3a^4 \quad \text{...(ii)}$$

From eqs. (i) and (ii), we get

$$\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = 3a^4$$

Hence Proved.

Q. 2. Solve the following system of equations by Cramer's Rule :

$$\mathbf{2x - y = 17}$$
$$\mathbf{3x + 5y = 6}$$

Sol. Given system of equations :

$$2x - y = 17$$
$$3x + 5y = 6$$

$$\therefore \quad D = \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix} = 10 + 3 = 13 \neq 0$$

Now, $$D_x = \begin{vmatrix} 17 & -1 \\ 6 & 5 \end{vmatrix} = 85 + 6 = 91$$

and $$D_y = \begin{vmatrix} 2 & 17 \\ 3 & 6 \end{vmatrix} = 12 - 51 = -39$$

Hence, by **Cramer's** rule, we get

$$x = \frac{D_x}{D} = \frac{91}{13} = 7$$

and $$y = \frac{D_y}{D} = \frac{-39}{13} = 3$$

Hence, $x = 7$ and $y = -3$

Q. 3. Solve the system of equations $x + 2y = 3$ and $4x + 8y = 12$ by using determinants.

Sol. Given, system of equations :

$$x + 2y = 3$$
$$4x + 8y = 12$$

$$\therefore \quad D = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = 8 - 8 = 0$$

$$D_x = \begin{vmatrix} 3 & 2 \\ 12 & 8 \end{vmatrix} = 24 - 24 = 0$$

$$D_y = \begin{vmatrix} 1 & 3 \\ 4 & 12 \end{vmatrix} = 12 - 12 = 0$$

Thus, $D = D_x = D_y = 0$

Hence, given system has infinite number of solutions. Let $y = k$, then

$$x + 2y = 3$$
$$\Rightarrow \quad x + 2k = 3$$
$$\Rightarrow \quad x = 3 - 2k$$

Therefore, $x = 3 - 2k$, $y = k$ is the solution of the given system of equations, where k is an arbitrary real number.

AI **Q. 4. Determine the value of λ for which the following system of equation fail to have a unique solution :**

$$\mathbf{\lambda x + 3y - z = 1}$$
$$\mathbf{x + 2y + z = 2}$$
$$\mathbf{-\lambda x + y + 2z = -1}$$

Does it have any solution for this value of λ ?

Sol. Given, system of equations :

$$\lambda x + 3y - z = 1$$
$$x + 2y + z = 2$$
$$-\lambda x + y + 2z = -1$$

$$\therefore \quad D = \begin{vmatrix} \lambda & 3 & -1 \\ 1 & 2 & 1 \\ -\lambda & 1 & 2 \end{vmatrix}$$

OSWAAL BOOKS® LEARNING MADE SIMPLE

If the given system of equations will fail to have a unique solution, then

$$D = 0$$

$$\therefore \quad \begin{vmatrix} \lambda & 3 & -1 \\ 1 & 2 & 1 \\ -\lambda & 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \lambda(4-1) - 3(2+\lambda) - 1(1+2\lambda) = 0$$

$$\Rightarrow \quad 3\lambda - 6 - 3\lambda - 1 - 2\lambda = 0$$

$$\Rightarrow \quad -2\lambda - 7 = 0 \Rightarrow \lambda = -\frac{7}{2}$$

Now, $$D_x = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= 1(4-1) - 3(4+1) - 1(2+2)$$

$$= 3 - 15 - 4 = -16$$

Thus, for $\lambda = \frac{-7}{2}$, we obtain $D = 0$ and $D_1 \neq 0$

Hence, the given system of equations has no solution for $\lambda = \frac{-7}{2}$.

6 Marks Questions

Q. 1. Test the consistency of following system of equations :

$$x + y + z = 1$$
$$x + 2y + 3z = 4$$
$$x + 3y + 5z = 7$$

Sol. Given, system of equations :

$$x + y + z = 1 \quad \text{...(i)}$$
$$x + 2y + 3z = 4 \quad \text{...(ii)}$$
$$x + 3y + 5z = 7 \quad \text{...(iii)}$$

$$\therefore \; D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 1(10-9) - 1(5-3) + 1(3-2)$$

$$= 1 - 2 + 1 = 0$$

$$D_x = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 3 \\ 7 & 3 & 5 \end{vmatrix} = 1(10-9) - 1(20-21) + 1(12-14)$$

$$= 1 + 1 - 2 = 0$$

$$D_y = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 7 & 5 \end{vmatrix} = 1(20-21) - 1(5-3) + 1(7-4)$$

$$= -1 - 2 + 3 = 0$$

$$D_z = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \end{vmatrix} = 1(14-12) - 1(7-4) + 1(3-2)$$

$$= 2 - 3 + 1 = 0$$

Thus, $D = D_x = D_y = D_z = 0$

So, either the system is consistent with infinitely many solutions or it is inconsistent.

From eqs. (i) and (ii), we have

$$x + y = 1 - z \quad \text{...(iv)}$$

and $$x + 2y = 4 - 3z \quad \text{...(v)}$$

Solve the above eqs (iv) and (v), using Cramer's rule

Here, $$D = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 \neq 0$$

$$D_x = \begin{vmatrix} 1-z & 1 \\ 4-3z & 2 \end{vmatrix}$$

$$= 2(1-z) - 1(4-3z)$$

$$= 2 - 2z - 4 + 3z = z - 2$$

and $$D_y = \begin{vmatrix} 1 & 1-z \\ 1 & 4-3z \end{vmatrix} = 4 - 3z - (1-z)$$

$$= 4 - 3z - 1 + z = 3 - 2z$$

$$\therefore \quad x = \frac{D_x}{D} = \frac{z-2}{1} = z - 2$$

and $$y = \frac{D_y}{D} = \frac{3-2z}{1} = 3 - 2z$$

Now, let $z = k$, where k is any arbitrary real number.

$\therefore x = k - 2, y = 3 - 2k, z = k$

Hence, $x = k - 2$, $y = 3 - 2k$ and $z = k$ is the solution of the given system of equation for each value of k.

AI **Q. 2. For what values of a and b, the system of equations**

$$2x + ay + 6z = 8$$
$$x + 2y + bz = 5$$
$$x + y + 3z = 4,$$

has (i) unique solution (ii) no solution.

Sol. For the given system of equations, we have

$$D = \begin{vmatrix} 2 & a & 6 \\ 1 & 2 & b \\ 1 & 1 & 3 \end{vmatrix} = 2(6-b) - a(3-b) + 6(1-2)$$

$$= 12 - 2b - 3a + ab - 6$$
$$= 6 - 3a - 2b + ab$$
$$= (b-3)(a-2)$$

$$D_x = \begin{vmatrix} 8 & a & 6 \\ 5 & 2 & b \\ 4 & 1 & 3 \end{vmatrix} = 8(6-b) - a(15-4b) + 6(5-8)$$

$$= 48 - 8b - 15a + 4ab - 18$$
$$= 30 - 15a - 8b + 4ab$$
$$= (a-2)(4b-15)$$

$$D_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & b \\ 1 & 4 & 3 \end{vmatrix} = 2(15-4b) - 8(3-b) + 6(4-5)$$

$$= 30 - 8b - 24 + 8b - 6$$
$$= 0$$

$$D_z = \begin{vmatrix} 2 & a & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = 2(8-5) - a(4-5) + 8(1-2)$$

$$= 6 + a - 8 = a - 2$$

(i) For unique solution, we must have

$D \neq 0 \Rightarrow (a-2)(b-3) \neq 0 \Rightarrow a \neq 2$ or $b \neq 3$

(ii) For no solution, we must have $D = 0$ and at least one of D_x, D_y and D_z is non-zero.

Clearly, $b = 3$, we have

$D = 0$ and $D_z \neq 0$

Hence, the system has no solution for $b = 3$.

Q. 3. Solve the following system of equations by Cramer's rule :

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4,\ \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1 \text{ and } \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Sol. Let $\frac{1}{x}=p$, $\frac{1}{y}=q$ and $\frac{1}{z}=r$, then the above system of equations can be written as

$$2p+3q+10r=4$$
$$4p-6q+5r=1$$
$$6p+9q-20r=2$$

Here, $D=\begin{vmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{vmatrix}$

$$=2(120-45)-3(-80-30)+10(36+36)$$
$$=150+330+720=1200$$

$$D_1=\begin{vmatrix}4 & 3 & 10\\ 1 & -6 & 5\\ 2 & 9 & -20\end{vmatrix}$$

$$=4(120-45)-3(-20-10)+10(9+12)$$
$$=300+90+210=600$$

$$D_2=\begin{vmatrix}2 & 4 & 10\\ 4 & 1 & 5\\ 6 & 2 & -20\end{vmatrix}$$

$$=2(-20-120)-4(-80-30)+10(8-6)$$
$$=-60+440+20=400$$

and $D_3=\begin{vmatrix}2 & 3 & 4\\ 4 & -6 & 1\\ 6 & 9 & 2\end{vmatrix}$

$$=2(-12-9)-3(8-6)+4(36+36)$$
$$=-42-6+288=240$$

$\therefore\quad p=\frac{D_1}{D}=\frac{600}{1200}=\frac{1}{2}\Rightarrow\frac{1}{x}=\frac{1}{2}\Rightarrow x=2,$

$q=\frac{D_2}{D}=\frac{400}{1200}=\frac{1}{3}\Rightarrow\frac{1}{y}=\frac{1}{3}\Rightarrow y=3,$

$r=\frac{D_3}{D}=\frac{240}{1200}=\frac{1}{5}\Rightarrow\frac{1}{z}=\frac{1}{5}\Rightarrow z=5$

Hence, $x=2$, $y=3$ and $z=5$.

Q. 4. Using determinants, show that the following system of linear equation is inconsistent :

$$x-3y+5z=4$$
$$2x-6y+10z=11$$
$$3x-9y+15z=12$$

Sol. For the given system of equations, we have

$$D=\begin{vmatrix}1 & -3 & 5\\ 2 & -6 & 10\\ 3 & -9 & 15\end{vmatrix}$$

Taking (– 3) common from C_2, we get

$$D=\begin{vmatrix}1 & 1 & 5\\ 2 & 2 & 10\\ 3 & 3 & 15\end{vmatrix}=0$$

[$\because$ C_1 and C_2 are identical]

Now, $D_x=\begin{vmatrix}4 & -3 & 5\\ 11 & -6 & 10\\ 12 & -9 & 15\end{vmatrix}$

Taking common (– 3) and 5 from C_2 and C_3, we get

$$D_x=-15\begin{vmatrix}4 & 1 & 1\\ 11 & 2 & 2\\ 12 & 3 & 3\end{vmatrix}=0$$

[$\therefore$ C_2 and C_3 are identical]

$$D_y=\begin{vmatrix}1 & 4 & 5\\ 2 & 11 & 10\\ 3 & 12 & 15\end{vmatrix}$$

Taking common (5) from C_3, we get

$$D_y=5\begin{vmatrix}1 & 4 & 1\\ 2 & 11 & 2\\ 3 & 12 & 3\end{vmatrix}=0$$

[$\because$ C_1 and C_3 are identical]

$$D_z=\begin{vmatrix}1 & -3 & 4\\ 2 & -6 & 11\\ 3 & -9 & 12\end{vmatrix}$$

Taking Common (– 3) from C_2, we get

$$D_z=(-3)\begin{vmatrix}1 & 1 & 4\\ 2 & 2 & 11\\ 3 & 3 & 12\end{vmatrix}=0$$

[$\because$ C_1 and C_2 are identical]

Thus, $D=D_x=D_y=D_z=0$

So, given system of equations may or many not be consistent.

If we put $z=k$ is any two of three equations, we find that the two equations obtained are inconsistent as they represent a pair of parallel lines.

i.e., $\quad x-3y=4-5k$

$$2x-6y=11-10k$$

[Taking first two equations]

or $\quad x-3y=4-5k$

$$x-3y=\frac{11-10k}{2},$$

which represent pair of parallel lines.

Hence, the given system equations is inconsistent.

OSWAAL BOOKS® LEARNING MADE SIMPLE

SELF ASSESSMENT TEST

UNIT–II

Maximum Marks : 23 **Timing : 1 Hour**

Q. 1. **If $A = \begin{pmatrix} 0 & 3 \\ 2 & -5 \end{pmatrix}$ and $kA = \begin{pmatrix} 0 & 4a \\ -8 & 5b \end{pmatrix}$ find the value of k and a.**

Q. 2. **If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A - I)^3 + (A + I)^3 - 7A$.**

Q. 3. **If A is a square matrix of order 3 such that $|adj\, A| = 64$, find (A).**

Q. 4. **If $\begin{bmatrix} x^2 & 3 & 4 \\ 1 & 9 & 8 \end{bmatrix} + \begin{bmatrix} -3x & 1 & -5 \\ -3 & -2 & -6 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -1 \\ -2 & 7 & 2 \end{bmatrix}$, then find the value of x.**

Q. 5. **If $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$, write the value of $a - 2b$.**

Q. 6. **Write the inverse of the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.**

Q. 7. **Using properties of determinants, find the value of the determinant $\begin{vmatrix} x^3 & x^2 & x \\ y^3 & y^2 & y \\ z^3 & z^2 & z \end{vmatrix}$.**

Q. 8. **On her birthday Seema decided to donate some money to children of an orphanage home. If there were 8 children less, everyone would have got ₹10 more. However if there were 16 children more, everyone would have got ₹10 less. Using matrix method, find the number of children and the amount distributed by Seema.**

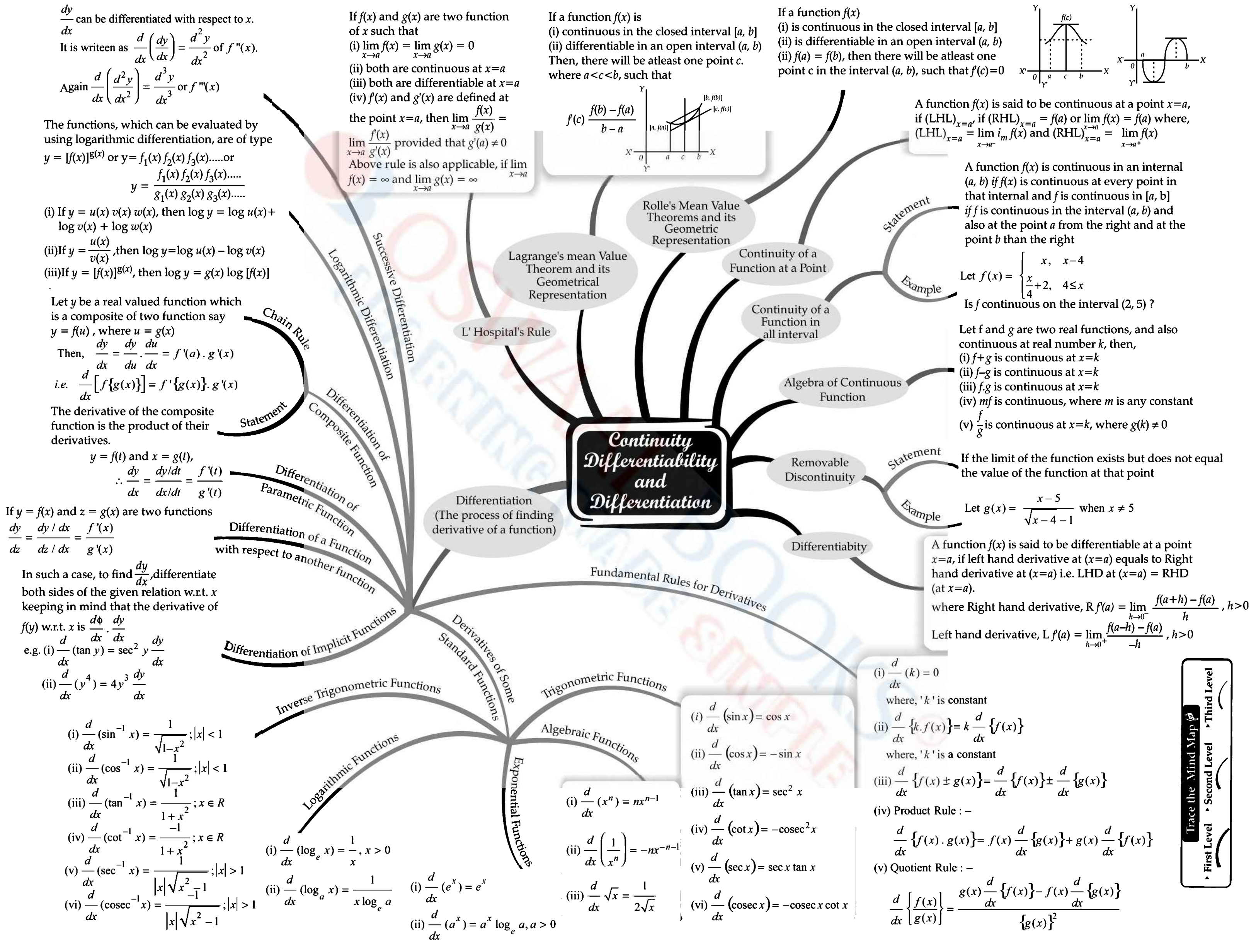
Q. 9. **Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$, then find A^{-1}.**

OSWAAL EXAM TOOLS

OSWAAL PREPARATION TOOLS

Solutions for Self-assessment Test

Visit : *http://qr.w69b.com/g/pi8gdC0qk* OR Scan the Code



Continuity Differentiability and Differentiation

Rolle's Mean Value Theorems and its Geometric Representation

If a function $f(x)$
(i) is continuous in the closed interval $[a, b]$
(ii) is differentiable in an open interval (a, b)
(ii) $f(a) = f(b)$, then there will be atleast one point c in the interval (a, b), such that $f'(c)=0$

Lagrange's mean Value Theorem and its Geometrical Representation

If a function $f(x)$ is
(i) continuous in the closed interval $[a, b]$
(ii) differentiable in an open interval (a, b)
Then, there will be atleast one point c.
where $a<c<b$, such that

$$f'(c)\ \frac{f(b)-f(a)}{b-a}$$

L' Hospital's Rule

If $f(x)$ and $g(x)$ are two function of x such that
(i) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$
(ii) both are continuous at $x=a$
(iii) both are differentiable at $x=a$
(iv) $f'(x)$ and $g'(x)$ are defined at the point $x=a$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$
Above rule is also applicable, if $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$

Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x=a$, if $(\text{LHL})_{x=a}$, if $(\text{RHL})_{x=a} = f(a)$ or $\lim\limits_{x\to a} f(x) = f(a)$ where, $(\text{LHL})_{x=a} = \lim\limits_{x\to a^-} i_m f(x)$ and $(\text{RHL})_{x=a} = \lim\limits_{x\to a^+} f(x)$

Continuity of a Function in all interval

Statement

A function $f(x)$ is continuous in an internal (a, b) if $f(x)$ is continuous at every point in that internal and f is continuous in $[a, b]$ if f is continuous in the interval (a, b) and also at the point a from the right and at the point b than the right

Example

$$\text{Let } f(x) = \begin{cases} x, & x-4 \\ \frac{x}{4}+2, & 4\le x \end{cases}$$

Is f continuous on the interval (2, 5) ?

Algebra of Continuous Function

Let f and g are two real functions, and also continuous at real number k, then,
(i) $f+g$ is continuous at $x=k$
(ii) $f-g$ is continuous at $x=k$
(iii) $f.g$ is continuous at $x=k$
(iv) mf is continuous, where m is any constant
(v) $\frac{f}{g}$ is continuous at $x=k$, where $g(k) \neq 0$

Removable Discontinuity

Statement

If the limit of the function exists but does not equal the value of the function at that point

Example

Let $g(x) = \dfrac{x-5}{\sqrt{x-4}-1}$ when $x \neq 5$

Differentiabity

A function $f(x)$ is said to be differentiable at a point $x=a$, if left hand derivative at $(x=a)$ equals to Right hand derivative at $(x=a)$ i.e. LHD at $(x=a)$ = RHD (at $x=a$).
where Right hand derivative, $\text{R}f'(a) = \lim\limits_{h\to 0^-}\dfrac{f(a+h)-f(a)}{h}$, $h>0$
Left hand derivative, $\text{L}f'(a) = \lim\limits_{h\to 0^+}\dfrac{f(a-h)-f(a)}{-h}$, $h>0$

Differentiation (The process of finding derivative of a function)

Fundamental Rules for Derivatives

(i) $\frac{d}{dx}(k) = 0$
where, 'k' is constant
(ii) $\frac{d}{dx}\{k.f(x)\} = k\frac{d}{dx}\{f(x)\}$
where, 'k' is a constant
(iii) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$
(iv) Product Rule : –

$$\frac{d}{dx}\{f(x).g(x)\} = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$$

(v) Quotient Rule : –

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}\{f(x)\} - f(x)\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$$

Derivatives of Some Standard Functions

Trigonometric Functions

(i) $\frac{d}{dx}(\sin x) = \cos x$
(ii) $\frac{d}{dx}(\cos x) = -\sin x$
(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$
(iv) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$
(v) $\frac{d}{dx}(\sec x) = \sec x \tan x$
(vi) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

Algebraic Functions

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$
(ii) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -nx^{-n-1}$
(iii) $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

Exponential Functions

(i) $\frac{d}{dx}(e^x) = e^x$
(ii) $\frac{d}{dx}(a^x) = a^x \log_e a, a > 0$

Logarithmic Functions

(i) $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$
(ii) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

Inverse Trigonometric Functions

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; |x| < 1$
(ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}; |x| < 1$
(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}; x \in R$
(iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}; x \in R$
(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}; |x| > 1$
(vi) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}; |x| > 1$

Differentiation of Implicit Functions

In such a case, to find $\frac{dy}{dx}$, differentiate both sides of the given relation w.r.t. x keeping in mind that the derivative of $f(y)$ w.r.t. x is $\frac{d\phi}{dx}.\frac{dy}{dx}$
e.g. (i) $\frac{d}{dx}(\tan y) = \sec^2 y\frac{dy}{dx}$
(ii) $\frac{d}{dx}(y^4) = 4y^3\frac{dy}{dx}$

Differentiation of a Function with respect to another function

If $y = f(x)$ and $z = g(x)$ are two functions

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$$

Differentiation of Parametric Function

$y = f(t)$ and $x = g(t)$,

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}$$

Differentiation of Composite Function

Chain Rule

Let y be a real valued function which is a composite of two function say $y = f(u)$, where $u = g(x)$
Then, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = f'(a).g'(x)$
i.e. $\frac{d}{dx}\left[f\{g(x)\}\right] = f'\{g(x)\}.g'(x)$

Statement

The derivative of the composite function is the product of their derivatives.

Logarithmic Differentiation

The functions, which can be evaluated by using logarithmic differentiation, are of type $y = [f(x)]^{g(x)}$ or $y = f_1(x) f_2(x) f_3(x)$.....or

$$y = \frac{f_1(x) f_2(x) f_3(x).....}{g_1(x) g_2(x) g_3(x).....}$$

(i) If $y = u(x)\, v(x)\, w(x)$, then $\log y = \log u(x) + \log v(x) + \log w(x)$
(ii) If $y = \frac{u(x)}{v(x)}$, then $\log y = \log u(x) - \log v(x)$
(iii) If $y = [f(x)]^{g(x)}$, then $\log y = g(x) \log [f(x)]$

Successive Differentiation

$\frac{dy}{dx}$ can be differentiated with respect to x.
It is writeen as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ of $f''(x)$.
Again $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ or $f'''(x)$

Trace the Mind Map
- First Level
- Second Level
- Third Level

OSWAAL BOOKS LEARNING MADE SIMPLE

UNIT-3
Calculus

CHAPTER
5
CONTINUITY, DIFFERENTIABILITY & DIFFERENTIATION

SYLLABUS

Continuity, Differentiability and Differentiation, derivative of composite functions, chain rule, derivatives of inverse trigonometric functions, derivatives of implicit functions. Concept of exponential and logarithmic functions.

Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivatives of functions expressed in parametric forms. Second order derivatives. Rolle's and Lagrange's Mean value Theorems (without proof) and their geometric interpretation.

- ➢ *Continuity*
  - *Continuity of a function at a point $x = a$.*
  - *Continuity of a function in an interval.*
  - *Algebra of continues function.*
  - *Removable discontinuity.*
- ➢ *Differentiation*
  - *Concept of continuity and differentiability of $|x|$, $[x]$, etc.*
  - *Derivatives of trigonometric functions.*
  - *Derivatives of exponential functions.*
  - *Derivatives of logarithmic functions.*
  - *Derivatives of inverse trigonometric functions - differentiation by means of substitution.*
  - *Derivatives of implicit functions and chain rule.*
  - *e for composite functions.*
  - *Derivatives of Parametric functions.*
  - *Differentiation of a function with respect to another function e.g. differentiation of $\sin x^3$ with respect to x^3.*
  - *Logarithmic Differentiation - Finding dy/dx when $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$.*
  - *Successive differentiation up to 2nd order.*

Note 1: *Derivatives of composite functions using chain rule.*

Note 2: *Derivatives of determinants to be covered.*

- *L'Hospital's theorem.*
  - $\frac{0}{0}$ *form,* $\frac{\infty}{\infty}$ *form,* 0^0 *form,* ∞^∞ *form etc.*
- *Rolle's Mean Value Theorem - tis geometrical interpretation.*
- *Lagrange's Mean Value Theorem - its geometrical interpretation*

Trend Analysis

List of Concepts	2016	2017	2018	2019	2020
Continuity and Differentiability			1 Q (2 marks) 1 Q (4 marks)	1 Q (2 marks) 1 Q (4 marks)	
Differentiation	1 Q (6 marks)	1 Q (4 marks)	1 Q (4 marks)	1 Q (4 marks)	1 Q (2 marks)
L'Hospital Rule, Rolle's Theorem and Lagrange's mean value Theorem	1 Q (4 marks) 1 Q (6 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (4 marks)	1 Q (2 marks)	1 Q (4 marks)

TOPIC-1
Continuity and Differentiability

TOPIC - 1
Continuity and Differentiability .... **P. 113**
TOPIC - 2
Differentiation .... **P. 121**
TOPIC - 3
L' Hospital Rule,
Rolle's Theorem and Lagrange's
Mean Value Theorem .... **P. 136**

Revision Notes

- **Limit :** For a function $f(x)$, $\lim_{x\to a} f(x)$ exists iff

  $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

- **Continuity at a point**

  A function $f(x)$ is continuous at a point $x = a$, were $a \in$ domain of $f(x)$, if

  $$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

  where, $\lim_{x\to a^-} f(x)$ is **Left Hand Limit** (LHL) of $f(x)$ at $x = a$

  $\lim_{x\to a^+} f(x)$ is **Right Hand Limit** (RHL) of $f(x)$ at $x = a$

  and $f(a)$ is the value of $f(x)$ at $x = a$.

- **Continuity on an interval**
  - **Continuity on an open interval :** A function $f(x)$ is said to be continuous on an open interval (a, b) iff it is continuous at every point on the interval (a, b).
  - **Continuity on a closed interval :** A function $f(x)$ is said to be continuous on closed interval $[a, b]$ iff it is continuous on (a, b) and it is continuous at 'a' from the right side and at 'b' from the left.

    i.e., for $f(x)$ to be continuous on $[a, b]$ iff

    (i) f is continuous on (a, b)

    (ii) $\lim_{x\to a^+} f(x) = f(a)$

    (iii) $\lim_{x\to b^-} f(x) = f(b)$

Scan to know more about this topic

Continuity of a function

- **Algebra of continuity**

  Let f and g be two real and continuous function at $x = a$, then

  (i) $f \pm g$ is continuous at $x = a$

  (ii) $f\,g$ is continuous at $x = a$

  (iii) $\frac{f}{g}$ is continuous at $x = a$, provided $g(a) \neq 0$

  (iv) $f(x)$ is continuous at $x = a$, where a is a real number.

- **Discontinuous function**

  A point where $f(x)$ is not continuous is called a **point of discontinuity** of $f(x)$ and the function is said to be **discontinuous** at that point. In particular, if either or both of the two limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$.

  do not exist or if both of them exist but are unequal, or even if the two limits are equal but their common value is not equal to $f(a)$, the function $f(x)$ is said to be discontinuous at $x = a$.

- **Removable discontinuity :** In case, $\lim_{x\to b} f(x)$ exist but not equal to $f(a)$, then the function is said to have a removable discontinuity or discontinuity.

- **Differentiability**

  Let $f(x)$ be defined at any point c in the interval (a, b) Then $f(x)$ is said to be differentiable at $x = c$ if the function has a derivative at this point, *i.e.*, if $f'(c)$ exists. Hence, if $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists, then the function $f(x)$ is called differentiable at point $x = c$.

  For the existence of this limit it is necessary that when $h \to 0$, the *left-hand* and *right-hand* limits both must exist and they must be equal.

Scan to know more about this topic

Differentiability of a function

OSWAAL BOOKS LEARNING MADE SIMPLE

The **right-hand derivative** (RHD) of $f(x)$ at the point $x = c$ is defined as

$$Rf'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h},\ h > 0$$

Similarly, the **left hand derivative** (LHD) of $f(x)$ at $x = c$ is defined as

$$Lf'(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h},\ h > 0$$

or

$$Lf'(c) = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h},\ h > 0$$

Hence, function $f(x)$ is differentiable at $x = c$ iff

$$Rf'(c) = Lf'(c)$$

Important Facts

- If a function is differentiable at a point, it is continuous at that point as well.
- If a function is not differentiable at a point, it may or may not be continuous at that point.
- If a function is continuous at a point, it may or may not differentiable at that point.
- If a function is discontinuous at a point, it is not differentiable at that point.

Know the Formulae

➢ Formulae For Limits

(a) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin x}$

(b) $\lim_{x\to 0} \frac{\tan x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan x}$

(c) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin^{-1} x}$

(d) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan^{-1} x}$

(e) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, a > 0$

(f) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(g) $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

(h) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(i) $\lim_{x\to 0}(1+kx)^{1/x} = e^k$, where k is any constant

(j) $\lim_{x\to\infty} \frac{\log_a(1+x)}{x} = \log_a e,\ a > 0 \neq 1$

(k) $\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{x\to\infty} \frac{\cos x}{x} = 0$

(l) $\lim_{x\to\infty} \cos x = 1$

1 Mark Questions

Q. 1. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ then which of the following can be a discontinuous function?

(a) $f(x) + g(x)$ **(b)** $f(x) - g(x)$

(c) $f(x).g(x)$ **(d)** $\frac{g(x)}{f(x)}$

Sol. Correct option : (d)

Explanation : Since $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ are continuous functions, then by using the algebra of continuous functions , the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).g(x)$ are also continuous functions but $\frac{g(x)}{f(x)}$ is discontinuous function at $x = 0$.

AI **Q. 2. The function $f(x) = \frac{4 - x^2}{4x - x^3}$**

(a) discontinuous at only one point

(b) discontinuous at exactly two points

(c) discontinuous at exactly three points

(d) none of these

Sol. Correct option : (c)

Explanation : Given that,

$f(x) = \frac{4 - x^2}{4x - x^3}$, then it is discontinuous if

$$\Rightarrow \quad 4x - x^3 = 0$$
$$\Rightarrow \quad x(4 - x^2) = 0$$
$$\Rightarrow \quad x(2+x)(2-x) = 0$$
$$\Rightarrow \quad x = 0, -2, 2$$

Thus, the given function is discontinuous at exactly three points.

Q. 3. The function $f(x)$ = cot x is discontinuous on the set

(a) $\{x = n\pi : n \in Z\}$

(b) $\{x = 2n\pi : n \in Z\}$

(c) $\left\{x = (2n+1)\frac{\pi}{2} ; n \in Z\right\}$

(d) $\left\{x = \frac{n\pi}{2} ; n \in Z\right\}$

Sol. Correct option : (a)

Explanation : Given that,

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

It is discontinuous at

$\sin x = 0$

$\Rightarrow x = n\pi, n \in Z$

Thus, the given function is discontinuous at $\{x = n\pi : n \in Z\}$.

AI **Q. 4. Determine the value of the constant 'k' so that the function $f(x) = \begin{cases} \dfrac{kx}{|x|} & , \text{ if } x < 0 \\ 3 & , \text{ if } x \geq 0 \end{cases}$ is**

Sol. Since f is continuous at $x = 0$,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Here $f(0) = 3$,

$$\text{LHL} = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} \frac{kx}{|x|} = \lim_{x \to 0^-} \frac{k\,x}{-x} = -k$$

$\therefore \quad -k = 3$ or $k = -3$.

Q. 5. The set of points where the function f given by $f(x) = |2x - 1| \sin x$ is differentiable is

(a) R

(b) $R - \left\{\frac{1}{2}\right\}$

(c) $(0, \infty)$

(d) none of these

Sol. Correct option : (b)

Explanation : Given that,

$f(x) = |2x - 1| \sin x$

The function $\sin x$ is differentiable.

The function $|2x - 1|$ is differentiable, except

$2x - 1 = 0$

$\Rightarrow x = \frac{1}{2}$

Thus, the given function is differentiable $R - \left\{\frac{1}{2}\right\}$.

Q. 6. The function $f(x) = e^{|x|}$ is

(a) continuous everywhere but not differentiable at $x = 0$

(b) continuous and differentiable everywhere

(c) not continuous at $x = 0$

(d) none of these

Sol. Correct option : (a)

Explanation : Given that,

$f(x) = e^{|x|}$

The functions e^x and $|x|$ are continuous functions for all real value of x.

Since e^x is differentiable everywhere but $|x|$ is non-differentiable at $x = 0$.

Thus, the given functions $f(x) = e^{|x|}$ is continuous everywhere but not differentiable at $x = 0$.

Q. 7. If the following function $f(x)$ is continuous at $x = 0$, then write the value of k.

$$f(x) = \begin{cases} \dfrac{\sin \frac{3x}{2}}{x} & , \quad x \neq 0 \\ k & , \quad x = 0 \end{cases}$$

Sol.
$$\lim_{x \to 0} \frac{\sin \frac{3x}{2}}{x} = \lim_{x \to 0} \frac{3}{2} \cdot \frac{\sin \frac{3x}{2}}{\frac{3x}{2}} = \frac{3}{2}$$

or
$$k = \frac{3}{2} \quad \left[\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

Q. 8. If the function $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & \text{when } x \neq 1 \\ k & \text{when } x = 1 \end{cases}$ is given to be continuous at $x = 1$, then the value of k is

Sol. ***Explanation :***

$$k = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} \qquad (x \neq 1)$$

$$= \lim_{x \to 1} x + 1$$

$$= 1 + 1$$

$$= 2$$

2 Marks Questions

AI **Q. 1. $f(x) = \dfrac{x^2 - 9}{x - 3}$ is not defined at $x = 3$. What value should be assigned to $f(3)$ for continuity of $f(x)$ at $x = 3$? [ISC, Specimen Paper, 2021] [ISC, 2019]**

Sol.
$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)} = 3 + 3 = 6$$

$f(x)$ will be continuous $x = 6$, if

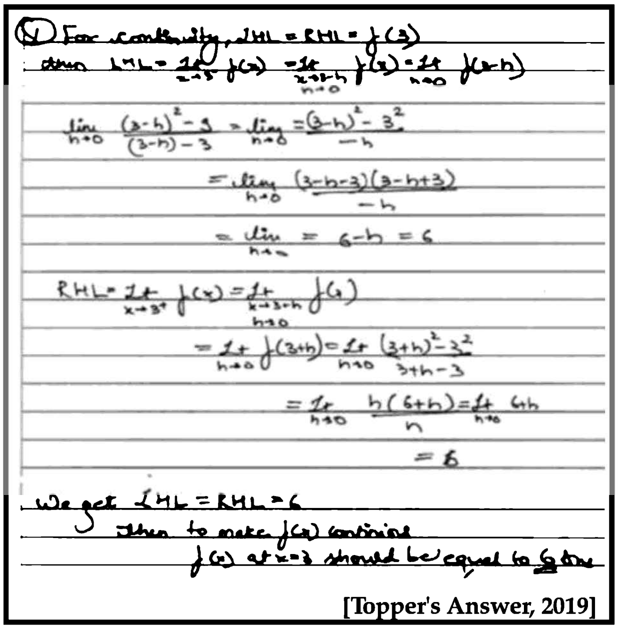
$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = f(3)$$

$\therefore \quad f(3) = 6$

[ISC Marking Scheme, 2019]

OR

OSWAAL BOOKS LEARNING MADE SIMPLE



① For continuity, LHL = RHL = f(3)

then LHL = $\lim_{x\to 3^-} f(x) = \lim_{\substack{x\to 3-h\\ h\to 0}} f(x) = \lim_{h\to 0} f(3-h)$

$\lim_{h\to 0}\frac{(3-h)^2-9}{(3-h)-3} = \lim_{h\to 0} = \frac{(3-h)^2-3^2}{-h}$

$= \lim_{h\to 0}\frac{(3-h-3)(3-h+3)}{-h}$

$= \lim_{h\to} = 6-h = 6$

RHL = $\lim_{x\to 3^+} f(x) = \lim_{\substack{x\to 3+h\\ h\to 0}} f(x)$

$= \lim_{h\to 0} f(3+h) = \lim_{h\to 0}\frac{(3+h)^2-3^2}{3+h-3}$

$= \lim_{h\to 0}\frac{h(6+h)}{h} = \lim_{h\to 0} 6+h$

$= 6$

We get LHL = RHL = 6

then to make f(x) continious

f(x) at x=3 should be equal to 6 too

[Topper's Answer, 2019]

Q. 2. Find the value of constant '*k*' so that function *f*(*x*) defined as :

$$f(x) = \begin{cases} \dfrac{x^2-2x-3}{x+1}, & x \neq -1 \\ k, & x = -1 \end{cases}$$

is continuous at $x = -1$ **[ISC, 2018]**

Sol. Given, the function $f(x)$ is continuous,

$$\therefore \quad \lim_{x\to -1} f(x) = f(-1)$$

$$\Rightarrow \quad \lim_{x\to -1}\frac{x^2-2x-3}{x+1} = k$$

$$\Rightarrow \quad \lim_{x\to -1}\frac{(x+1)(x-3)}{x+1} = k$$

$$\Rightarrow \quad \lim_{x\to -1}(x-3) = k$$

$$\Rightarrow \quad k = -4$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Many candidates made errors in calculating LHL and RHL and some of the candidates could not solve it further.

Answering Tip

- Understand the concept of continuous and discontinues function with suitable examples.

AI Q. 3. Find the value of the constant *k*, so that the function *f*(*x*) is defined below is continuous at $x = 0$,

$$\text{where } f(x) = \begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

Sol. Given, the function $f(x)$ is continuous,

$$\therefore \quad \lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{1-\cos 4x}{8x^2} = k$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{2\sin^2 2x}{8x^2} = k \quad [\because \cos 4x = 1 - 2\sin^2 2x]$$

$$\Rightarrow \quad \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = k$$

$$\Rightarrow \quad k = 1 \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

Q. 4. Find the value of *k* for which the function

$$f(x) = \begin{cases} \dfrac{\sin x - \cos x}{4x-\pi}, & x \neq \dfrac{\pi}{4} \\ k, & x = \dfrac{\pi}{4} \end{cases} \textbf{ is continuous at } x = \frac{x}{4}$$

Sol.

$$\lim_{x\to\frac{\pi}{4}} f(x) = f\left(\frac{\pi}{4}\right)$$

$$\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)}{4\left(x-\frac{\pi}{4}\right)} = k \quad \therefore k = \frac{\sqrt{2}}{4}$$

4 Marks Questions

Q. 1. Show that the function $f(x) = |x-4|$, $x \in R$ is continuous, but not differentiable at $x = 4$. **[ISC, 2019]**

Sol.

$f(x) = |x-4| \ \forall x \in$ R

$$f(x) = \begin{cases} x-4 & x > 4 \\ 4-x & x < 4 \\ 0 & x = 4 \end{cases}$$

$$\text{LHL} = \lim_{x\to 4^-} f(4-h)$$

$$= \lim_{h\to 0} f(4-h) = \lim_{h\to 0} 4-(4-h) = 0$$

$$\text{RHL} = \lim_{x\to 4^+} f(x) = \lim_{h\to 0} f(4+h) = \lim_{h\to 0} f(4+h)-4 = 0$$

$f(4) = 0$

$f(4)$ = LHL = RHL

$\therefore$ $f(x)$ is continuous at $x = 4$

$$\text{LHD} = \lim_{x\to 4^-} f^1(x) = \lim_{h\to 0}\frac{f(4-h)-f(4)}{-h}$$

$$= \lim_{h\to 0}\frac{[4-(4-h)]-0}{-h} = -1$$

$$\text{RHD} = \lim_{x\to 4^+} f^1(x) = \lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$$

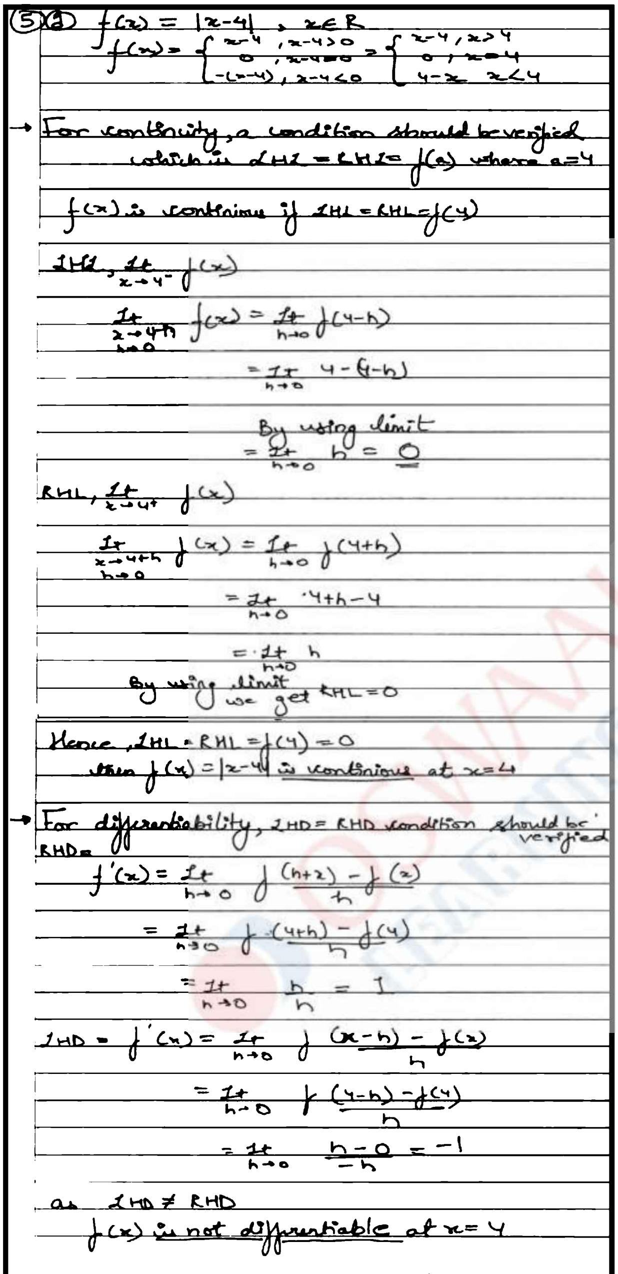
$$= \lim_{h\to 0}\frac{(4+h)-4-0}{h} = 1$$

LHD ≠ RHD

∴ f is not differentiable at $x = 4$.

[ISC Marking Scheme, 2019]

OR



⑤ ② $f(x) = |x-4|$, $x \in R$

$f(x) = \begin{cases} x-4, & x-4>0 \\ 0, & x-4=0 \\ -(x-4), & x-4<0 \end{cases} \Rightarrow \begin{cases} x-4, & x>4 \\ 0, & x=4 \\ 4-x & x<4 \end{cases}$

→ For continuity, a condition should be verified which is LHL = RHL = $f(a)$ where $a=4$

$f(x)$ is continious if LHL = RHL = $f(4)$

LHL, $\underset{x\to 4^-}{lt} f(x)$

$\underset{\substack{x\to 4-h\\ h\to 0}}{lt} f(x) = \underset{h\to 0}{lt} f(4-h)$

$= \underset{h\to 0}{lt}\ 4-(4-h)$

By using limit

$= \underset{h\to 0}{lt}\ h = \underline{0}$

RHL, $\underset{x\to 4^+}{lt} f(x)$

$\underset{\substack{x\to 4+h\\ h\to 0}}{lt} f(x) = \underset{h\to 0}{lt} f(4+h)$

$= \underset{h\to 0}{lt}\ 4+h-4$

$= \underset{h\to 0}{lt}\ h$

By using limit we get RHL = 0

Hence, LHL = RHL = $f(4) = 0$

then $f(x) = |x-4|$ is continious at $x=4$

→ For differentiability, LHD = RHD condition should be verified

RHD =

$f'(x) = \underset{h\to 0}{lt}\ f\frac{(h+x)-f(x)}{h}$

$= \underset{h\to 0}{lt}\ f\frac{(4+h)-f(4)}{h}$

$= \underset{h\to 0}{lt}\ \frac{h}{h} = 1$

LHD = $f'(x) = \underset{h\to 0}{lt}\ f\frac{(x-h)-f(x)}{h}$

$= \underset{h\to 0}{lt}\ f\frac{(4-h)-f(4)}{h}$

$= \underset{h\to 0}{lt}\ \frac{h-0}{-h} = -1$

as LHD ≠ RHD

$f(x)$ is not differentiable at $x = 4$

[Topper's Answer, 2019]

AI Q. 2. Show that the function $f(x) = \begin{cases} x^2, & x \le 1 \\ \frac{1}{x}, & x > 1 \end{cases}$ is continuous at $x = 1$ but not differentiable.

[ISC, 2018]

Sol. Given, $f(x) = \begin{cases} x^2, & x \le 1 \\ \frac{1}{x}, & x > 1 \end{cases}$

Now, L.H.L = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2) = 1$

and R.H.L. = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(\frac{1}{x}\right) = 1$

∴ L.H.L. = R.H.L.

Hence, $f(x)$ is continuous at $x = 1$

Now, L.H.D. = $\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0^-}\frac{(1+h)^2-1}{h} = 2$$

and R.H.D. = $\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0^+}\frac{\left(\frac{1}{1+h}\right)-1}{h} = \frac{-h}{h(1+h)} = -1$$

∴ L.H.D. ≠ R.H.D.

Hence, $f(x)$ is not differentiable at $x = 1$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Many candidates successfully attempted continuity of function correctly but failed to calculate left hand derivative and R.H. derivative.

Answering Tip

- Practice more and more examples of continuity and differentiability to under the concept deeply.

Q. 3. Prove that the function $f(x) = |x-1|$, $x \in R$, is continuous at $x = 1$ but not differentiable.

[ISC Specimen Paper, 2018]

Sol. Given, $f(x) = |x-1|$, $x \in$ R as $x \ge 0$

$$f(x) = \begin{cases} (x-1), & x \ge 1 \\ -(x-1), & x < 1 \end{cases}$$

Here, $f(1) = 0$

L.H.L. = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}[-(x-1)] = 0$

R.H.L. = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x-1) = 0$

∴ L.H.L. = R.H.L. = $f(1)$

∴ $f(x)$ is continuous.

L.H.D. = $Lf'(1) = \lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0^-}\frac{-(1+h-1)-0}{h}$$

$$= \lim_{h\to 0^-}\frac{-h}{h} = -1$$

R.H.D. = $Rf'(1)$

$$= \lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+}\frac{(1+h-1)-0}{h}$$

$$= \lim_{h\to 0^+} \frac{h}{h} = 1$$

i.e., L.H.D. ≠ R.H.D.

∴ $f(x)$ is not differentiable at $x = 1$.

Q. 4. If $f(x) = \begin{cases} 1, & \text{if } x \le 3 \\ ax+b, & \text{if } 3 < x < 5 \\ 7, & \text{if } x \ge 5 \end{cases}$

Find *a* and *b*, so that $f(x)$ is a continuous function.

Sol. Given, $f(x) = \begin{cases} 1, & x \le 3 \\ ax+b, & 3 < x < 5 \\ 7, & x \ge 5 \end{cases}$

Since, $f(x)$ is continuous at $x = 3$ and $x = 5$,

∵ At $x = 3$, L.H.L. = R.H.L.

$$\Rightarrow \quad \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\lim_{x\to 3}(1) = \lim_{x\to 3} ax+b$$

$$1 = a \times 3 + b$$

$$\Rightarrow \quad 3a + b = 1 \qquad \ldots\text{(i)}$$

Similarly, at $x = 5$,

L.H.L. = R.H.L.

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x)$$

$$\lim_{x\to 5}(ax+b) = \lim_{x\to 5}(7)$$

$$\Rightarrow \quad a(5) + b = 7$$

$$\Rightarrow \quad 5a + b = 7 \qquad \ldots\text{(ii)}$$

Solving equations (i) and (ii), we get

$$a = 3 \text{ and } b = -8$$

AI Q. 5. Find the value of *k*, for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

is continuous at $x = 0$.

Sol. Given, $f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \textit{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \textit{if } 0 \le x < 1 \end{cases}$

Since, $f(x)$ is continuous at $x = 0$,

i.e., L.H.L. = $f(0)$ = R.H.L.

Take L.H.L. = $f(0)$

$$\Rightarrow \quad \lim_{x\to 0^-} f(x) = \frac{2(0)+1}{0-1} = -1$$

Take R.H.L. = $f(0)$

$$\lim_{x\to 0} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x} = -1$$

$$\Rightarrow \lim_{x\to 0} \frac{(\sqrt{1+kx})^2-(\sqrt{1-kx})^2}{x(\sqrt{1+kx}+\sqrt{1-kx})} = -1$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{1+kx-1+kx}{x(\sqrt{1+kx}+\sqrt{1-kx})} = -1$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{2kx}{x(\sqrt{1+kx}+\sqrt{1-kx})} = -1$$

$$\Rightarrow \quad \frac{2k}{\sqrt{1+0}+\sqrt{1-0}} = -1$$

$$\Rightarrow \quad \frac{2k}{1+1} = -1$$

$$\Rightarrow \quad \frac{2k}{2} = -1$$

$$\therefore \quad k = -1$$

Thus, $f(x)$ is continuous at $x = 0$, if $k = -1$.

Q. 6. If the function $f(x)$ is given by

$$f(x) = \begin{cases} 3ax+b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax-2b, & \text{if } x < 1 \end{cases}$$

is continuous at $x = 1$, find the values of *a* and *b*.

Sol. Given, $f(x) = \begin{cases} 3ax+b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax-2b, & \text{if } x < 1 \end{cases}$

Since, $f(x)$ is continuous at $x = 1$

L.H.L. = $f(1)$ = R.H.L.

Take L.H.L. = $f(1)$

$$\Rightarrow \quad \lim_{x\to 1^-} f(x) = 11$$

$$\Rightarrow \lim_{x\to 1}(5ax-2b) = 11$$

$$\Rightarrow \quad 5a(1) - 2b = 11$$

$$\Rightarrow \quad 5a - 2b = 11 \qquad \ldots\text{(i)}$$

Also, $f(1)$ = R.H.L.

$$\Rightarrow \quad 11 = \lim_{x\to 1^+} f(x)$$

$$\Rightarrow \quad 11 = \lim_{x\to 1}(3ax+b)$$

$$\Rightarrow \quad 11 = 3a + b \qquad \ldots\text{(ii)}$$

Solving (i) and (ii), we get

$$a = 3 \text{ and } b = 2$$

Q. 7. The function $f(x)$ is defined as

$$f(x) = \begin{cases} x^2+ax+b, & 0 \le x < 2 \\ 3x+2, & 2 \le x \le 4 \\ 2ax+5b, & 4 < x \le 8 \end{cases}$$

If $f(x)$ is continuous on [0, 8], find the values of *a* and *b*.

Sol. Given, $f(x) = \begin{cases} x^2+ax+b, & 0 \le x < 2 \\ 3x+2, & 2 \le x \le 4 \\ 2ax+5b, & 4 < x \le 8 \end{cases}$

Since, $f(x)$ is continuous on [0, 8]

i.e., $f(x)$ is continuous at $x = 2$ and $x = 4$

∴ At $x = 2$, L.H.L. = $f(x)$ = R.H.L.

Take L.H.L = $f(x)$

$$\Rightarrow \lim_{x\to 2^-}(x^2+ax+b)=3x+2$$

$$\Rightarrow \lim_{x\to 2^-}(x^2+ax+b)=3\times 2+2$$

$$\Rightarrow \quad 4 + 2a + b = 6 + 2$$

$$\Rightarrow \quad 2a + b = 4 \qquad \ldots\text{(i)}$$

Also, at $x = 4$, L.H.L. $= f(4) =$ R.H.L.

Take $f(x) =$ R.H.L.

$$f(4)=\lim_{x\to 4^+} f(x)$$

$$\Rightarrow \quad 3(4)+2=\lim_{x\to 4}(2ax+5b)$$

$$\Rightarrow \quad 14=8a+5b \qquad \ldots\text{(ii)}$$

Solving equations (i) and (ii),

$$a = 3 \text{ and } b = -2$$

Q. 8. If the following function is differentiable at $x = 2$, then find the values of a and b.

$$f(x)=\begin{cases} x^2, & \text{if } x\le 2\\ ax+b, & \text{if } x>2\end{cases}$$

Sol. Given, $f(x) = \begin{cases} x^2, & x\le 2\\ ax+b & x>2\end{cases}$

Since, f is differentiable at $x = 2$, therefore f is continuous at $x = 2$

$$\Rightarrow \lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$

$$\Rightarrow \lim_{x\to 2}(x^2)=\lim_{x\to 2}(ax+b)=4$$

$$\Rightarrow \quad 4 = 2a + b \qquad \ldots\text{(i)}$$

Since, f is differentiable at $x = 2$

$$\therefore \quad Lf'(2) = Rf'(2)$$

$$\Rightarrow \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\ (h>0)$$

$$\Rightarrow \lim_{h\to 0}\frac{(2-h)^2-4}{-h}=\lim_{h\to 0}\frac{\{a(2+h)+b\}-4}{h}$$

$$=\lim_{h\to 0}\frac{2a+a\cdot h+b-4}{h}$$

by (i) $\quad 2a + b = 4$

$$\Rightarrow \lim_{h\to 0}(-h+4)=\lim_{h\to 0}\frac{4+ah-4}{h}$$

$$\therefore \quad 4 = a$$

$$b = 4 - 2a = -4 \qquad \text{[By (i)]}$$

Hence, $a = 4$ and $b = -4$.

AI **Q. 9. For what value of λ the function defined by**

$$f(x)=\begin{cases}\lambda(x^2+2), & \text{if } x\le 0\\ 4x+6, & \text{if } x>0\end{cases}$$

is continuous at $x = 0$?

Hence, check the differentiability of $f(x)$ at $x = 0$.

Sol. Given, $f(x) = f(x)=\begin{cases}\lambda(x^2+2), & \text{if } x\le 0\\ 4x+6, & \text{if } x>0\end{cases}$

Taking $\quad$ L.H.L. $=\lim_{x\to 0^-} f(x)=2\lambda$

and $\quad$ R.H.L. $=\lim_{x\to 0^+} f(x)=6$

$$f(0) = 2\lambda$$

As given function is continuous at $x = 0$

$$\therefore \quad 2\lambda = 6 \Rightarrow \lambda = 3$$

Differentiability Test :

$$\text{L.H.D.} = \lim_{h\to 0}\frac{f(0)-f(0-h)}{-h}$$

$$=\lim_{h\to 0}\frac{3(2)-3[(-h)^2+2]}{-h}$$

$$= 0$$

and

$$\text{R.H.D.} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{(4h+6)-3(2)}{h}$$

$$= 4$$

$$\text{L.H.D.} \ne \text{R.H.D.}$$

$\therefore$ $f(x)$ is not differentiable at $x = 0$.

6 Marks Questions

AI **Q. 1. Find the values of p and q, for which**

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\dfrac{\pi}{2}\\ p, & \text{if } x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\dfrac{\pi}{2}\end{cases}$$

is continuous at $x=\dfrac{\pi}{2}$.

Sol. Given, $\quad f(x) = \begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & x<\dfrac{\pi}{2}\\ p, & x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & x>\dfrac{\pi}{2}\end{cases}$

LHL $= \lim f(x)$

$$\underset{\left(\text{at } x=\frac{\pi}{2}\right)}{\text{LHL}} = \lim_{x\to \frac{\pi}{2}^-} f(x)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin^3 x)}{3\cos^2 x}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{(1^3-\sin^3 x)}{3(1^2-\sin^2 x)}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)(1+\sin x+\sin^2 x)}{3(1-\sin x)(1+\sin x)}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{1+\sin x+\sin^2 x}{3(1+\sin x)}$$

Put $x=\frac{\pi}{2}-h$, as $x\to\frac{\pi}{2}$, $h\to 0$

$$=\lim_{h\to0}\frac{1+\sin\left(\frac{\pi}{2}-h\right)+\sin^2\left(\frac{\pi}{2}-h\right)}{3\left[1+\sin\left(\frac{\pi}{2}-h\right)\right]}$$

$$=\lim_{h\to0}\frac{1+\cos h+\cos^2 h}{3[1+\cos h]}=\frac{1+1+1}{3(1+1)}=\frac{1}{2}$$

$$\underset{\left(\text{at } x=\frac{\pi}{2}\right)}{\text{RHL}}=\lim_{x\to\frac{\pi}{2}^+}f(x)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{q(1-\sin x)}{(\pi-2x)^2}$$

Put $x=\frac{\pi}{2}+h$, as $x\to\frac{p}{2}$, $h\to 0$

$$=q\lim_{h\to0}\frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\left\{\pi-2\left(\frac{\pi}{2}+h\right)\right\}^2}$$

$$=q\lim_{h\to0}\frac{1-\cos h}{4h^2}=q\lim_{h\to0}\frac{2\sin^2\frac{h}{2}}{4h^2}$$

$$=\frac{q}{2}\lim_{h\to0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\times\frac{1}{4}=\frac{q}{8}\qquad\left[\because\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1\right]$$

Also, $f\left(\frac{\pi}{2}\right)=p$

Since, $f(x)$ is continuous at $x=f\left(\frac{\pi}{2}\right)$

$\therefore$ LHL = RHL $=f\left(\frac{\pi}{2}\right)$

$\therefore\quad \frac{1}{2}=\frac{q}{8}=p$

$\Rightarrow\quad q=4$ and $p=\frac{1}{2}$

Q. 2. Let $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & \text{if } x<0\\ a, & \text{if } x=0\\ \dfrac{\sqrt{x}}{16+\sqrt{x}-4}, & \text{if } x>0\end{cases}$

If $f(x)$ is continuous at $x=0$, determine the value of a.

Sol. Given, $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & \text{if } x<0\\ a, & \text{if } x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{if } x>0\end{cases}$

Since, $f(x)$ is continuous at $x=0$

$$\Rightarrow\quad \lim_{x\to0}f(x)=\lim_{x\to0^-}f(x)=\lim_{x\to0^+}f(x)$$

Take L.H.L. $=\lim_{x\to0^-}f(x)$

$$=\lim_{x\to0}\frac{1-\cos 4x}{x^2}$$

$$=\lim_{x\to0}\frac{1-(1-2\sin^2 2x)}{x^2}$$

$$=\lim_{x\to0}2\frac{\sin^2 2x}{\frac{4x^2}{4}}$$

$$=8\cdot\lim_{x\to0}\left(\frac{\sin 2x}{2x}\right)^2$$

$$=8\times1=8$$

and $\qquad f(0)=a$

As $f(x)$ is continuous at $x=0$

$\therefore\qquad f(0)=a=$ L.H.L.

$\Rightarrow\qquad a=8$

Take R.H.L. $=\lim_{x\to0^+}f(x)$

$$=\lim_{x\to0}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}\times\frac{(\sqrt{16+\sqrt{x}}+4)}{(\sqrt{16+\sqrt{x}}+4)}$$

$$=\lim_{x\to0}\frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16}$$

$$=\lim_{x\to0}\frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{\sqrt{x}}$$

$$=\lim_{x\to0}\sqrt{16+\sqrt{x}}+4$$

$$=\lim_{x\to0}\sqrt{16+\sqrt{0}}+4$$

$$=\sqrt{16}+4=4+4=8$$

$\therefore\qquad \underset{x\to0^-}{f(x)}=f(0)=\underset{x\to0^+}{f(x)}=8$

$\qquad a=8$

Q. 3. Find the value of k, so that the following function is continuous at $x=2$.

$$f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2}, & x\neq2\\ k, & x=2\end{cases}$$

Sol. Given, $f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2}, & x\neq2\\ k, & x=2\end{cases}$

Since, $f(x)$ is continuous at $x=2$.

i.e., L.H.L. $=f(2)=$ R.H.L.

$$\Rightarrow\quad \lim_{x\to2^-}f(x)=k$$

$$\Rightarrow\quad \lim_{x\to2^-}\frac{x^3+x^2-16x+20}{(x-2)^2}=k$$

$$\Rightarrow\quad \lim_{h\to0}\frac{(2-h)^3+(2-h)^2-16(2-h)+20}{(2-h-2)^2}=k$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{(8-12h+6h^2-h^3+4-4h+h^2-32+16h+20)}{h^2}=k$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{-h^3+7h^2}{h^2}=k$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{h^2(-h+7)}{h^2}=k$$

$$\Rightarrow \quad 0+7=k$$

$$\Rightarrow \quad k=7$$

Thus, $f(x)$ is continuous at $x = 2$ if $k = 7$.

AI **Q. 4. Find the value of k, so that the function f is continuous at $x = \pi/2$.**

$$f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\[2mm] 5, & \text{if } x=\dfrac{\pi}{2}\end{cases}$$

Sol. Given, $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\[2mm] 5, & \text{if } x=\dfrac{\pi}{2}\end{cases}$

Since, $f(x)$ is continuous at $x=\dfrac{\pi}{2}$

i.e., L.H.L. $=f\left(\dfrac{\pi}{2}\right)=$ R.H.L.

Take L.H.L. $=f\left(\dfrac{\pi}{2}\right)$

$$\Rightarrow \quad \lim_{x\to\frac{\pi}{2}^-} f(x)=5$$

$$\Rightarrow \quad \lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=5$$

Substitute $x=\dfrac{\pi}{2}-h$, as $h\to 0, x\to\dfrac{\pi}{2}$

$$\therefore \quad \lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)}=5$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{k\sin h}{2h}=5$$

$$\Rightarrow \quad \frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=5 \qquad \left[\because \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$\Rightarrow \quad \frac{k}{2}(1)=5$$

$$\Rightarrow \quad k=10$$

Thus, $f(x)$ is continuous at $x=\dfrac{\pi}{2}$ if $k = 10$.

TOPIC-2

Differentiation

Revision Notes

- **Derivative :** The derivative of a given function $f(x)$ at $x = 0$ of its domain is defined as :

  $f'(a) = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$, provided the limit exists.

- **Derivative of Some Standard Functions**

  (a) $\dfrac{d}{dx}(x^n)=nx^{n-1}$

  (b) $\dfrac{d}{dx}(k)=0$, where k is any constant

  (c) $\dfrac{d}{dx}(a^x)=a^x\log_e a, a>0$

  (d) $\dfrac{d}{dx}(e^x)=e^x$

  (e) $\dfrac{d}{dx}(\log_a x)=\dfrac{1}{x\log_e a}=\dfrac{1}{x}\log_a e$

  (f) $\dfrac{d}{dx}(\log_e x)=\dfrac{1}{x}$

  (g) $\dfrac{d}{dx}(\sin x)=\cos x$

  (h) $\dfrac{d}{dx}(\cos x)=-\sin x$

  (i) $\dfrac{d}{dx}(\tan x)=\sec^2 x$

  (j) $\dfrac{d}{dx}(\sec x)=\sec x\tan x$

  (k) $\dfrac{d}{dx}(\cot x)=-\text{cosec}^2 x$

  (l) $\dfrac{d}{dx}(\text{cosec}\, x)=-\text{cosec}\, x\cot x$

  (m) $\dfrac{d}{dx}(\sin^{-1}x)=\dfrac{1}{\sqrt{1-x^2}}, x\in(-1,1)$

  (n) $\dfrac{d}{dx}(\cos^{-1}x)=-\dfrac{1}{\sqrt{1-x^2}}, x\in(-1,1)$

  (o) $\dfrac{d}{dx}(\tan^{-1}x)=\dfrac{1}{1+x^2}, x\in R$

  (p) $\dfrac{d}{dx}(\cot^{-1}x)=-\dfrac{1}{1+x^2}, x\in R$

Scan to know more about this topic

Differentiation

(q) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

(r) $\frac{d}{dx}(\text{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

- **Product rule of derivatives**

$$\frac{d}{dx}(u.v) = u.\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

- **Quotient rule of derivatives**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2} = \frac{vu' - uv'}{v^2}$$

- **Chain Rule**

If $y = f(u)$, $u = g(w)$, $w = h(x)$

then, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dw}.\frac{dw}{dx}$ or $\frac{dy}{dx} = f'(u).g'(w).h'(x)$

Important Facts

- **Following derivatives should also be memorized by you for quick use :**
  - $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
  - $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
  - $\frac{d}{dx}(x^x) = x^x(1 + \log x)$

- **Application of Logarithmic operation**

If $y = \{f,(x)\}^{f_2(x)}$ or $y = f_1(x).f_3(x)....$ or $y = \frac{f_1(x).f_2(x).f_3(x)....}{g_1(x).g_2(x).g_3(x)....}$

then it is convenient to take the logarithm of the function first and then differentiate.

- **Derivatives of Parametric functions**

If $y = f(t)$ and $x = g(t)$, where t is a parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

- Derivatives of a function w.r.t to another function

Let $y = f(x)$; $z = g(x)$, then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

- **Derivatives and Implicit functions**

If the variables x (independent variable) and y (dependent variable) are connected by a relation of the form $f(x, y) = 0$, then y is said to be an implicit function of x.

To find $\frac{dy}{dx}$ in such cases, we differentiate both sides of given function w.r.t x, keeping in mind that derivative of $\phi(y)$ w.r.t. x is $\frac{d\phi}{dy}.\frac{dy}{dx}$.

e.g. $\therefore$ $\frac{dy}{dx}$ of implicit function $xy = 1$ is obtained as following.

$$xy = 1$$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1)$$

$$\Rightarrow \quad \frac{d}{dx}(x).y + x\frac{d}{dx}(y) = 0$$

$$\Rightarrow \quad 1.y + x.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$$

1 Mark Questions

Q. 1. If $y=\log\left(\dfrac{1-x^2}{1+x^2}\right)$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{4x^3}{1-x^4}$ (b) $\dfrac{-4x^3}{1-x^4}$

(c) $\dfrac{1}{4-x^4}$ (d) $\dfrac{-4x^3}{1-x^4}$

Sol. Correct option : (b)

Explanation : Given that,

$$y=\log\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow y=\log\left(1-x^2\right)-\log\left(1+x^2\right)$$

Differentiate with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(1-x^2\right)\right]-\frac{d}{dx}\left[\log\left(1+x^2\right)\right]$$

$$=\frac{-2x}{1-x^2}-\frac{2x}{1+x^2}$$

$$=-2x\left(\frac{2}{\left(1-x^2\right)\left(1+x^2\right)}\right)$$

$$=-\frac{4x}{1-x^4}$$

AI **Q. 2. If $y=\sqrt{\sin x+y}$, then $\dfrac{dy}{dx}$ is equal to**

(a) $\dfrac{\cos x}{2y-1}$ (b) $\dfrac{\cos x}{1-2y}$

(c) $\dfrac{\sin x}{1-2y}$ (d) $\dfrac{\sin x}{2y-1}$

Sol. Correct option : (a)

Explanation : Given that,

$$y=\sqrt{\sin x+y}$$

$$\Rightarrow y^2=\sin x+y$$

Differentiate with respect to x, we have

$$\Rightarrow \quad 2y\frac{dy}{dx}=\cos x+\frac{dy}{dx}$$

$$\Rightarrow (2y-1)\frac{dy}{dx}=\cos x$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{\cos x}{2y-1}$$

Q. 3. The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1}x$ is

(a) 2 (b) $\dfrac{-1}{2\sqrt{1-x^2}}$

(c) $\dfrac{2}{x}$ (d) $1-x^2$

Sol. Correct option : (a)

Explanation : Let

$$u=\cos^{-1}\left(2x^2-1\right)$$

$$\Rightarrow \frac{du}{dx}=-\frac{4x}{\sqrt{1-\left(2x^2-1\right)^2}}$$

$$\Rightarrow \frac{du}{dx}=-\frac{4x}{\sqrt{1-4x^4+4x^2-1}}$$

$$\Rightarrow \frac{du}{dx}=-\frac{4x}{\sqrt{-4x^4+4x^2}}$$

$$\Rightarrow \frac{du}{dx}=-\frac{2}{\sqrt{1-x^2}}$$

And, $\quad v=\cos^{-1}x$

$$\frac{dv}{dx}=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

Thus, $\dfrac{du}{dv}=2.$

Q. 4. If $x=t^2$ and $y=t^3$ then $\dfrac{d^2y}{dx^2}$ is

(a) $\dfrac{3}{2}$ (b) $\dfrac{3}{4t}$

(c) $\dfrac{3}{2t}$ (d) $\dfrac{3}{4}$

Sol. Correct option : (b)

Explanation : Given that,

$x=t^2$ and $y=t^3$

Then, $\dfrac{dx}{dt}=2t$ and $\dfrac{dy}{dt}=3t^2$

Thus,

$$\frac{dy}{dx}=\frac{3t^2}{2t}=\frac{3t}{2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{3}{2}\frac{dt}{dx}$$

$$\frac{3}{2}.\frac{1}{2t}=\frac{3}{4t}$$

AI **Q. 5. Write the derivative of sin x with respect to cos x.**

Sol. Let $\quad u=\sin x$

$\therefore \quad \dfrac{du}{dx}=\cos x$

and $\quad v=\cos x$

$\therefore \quad \dfrac{dv}{dx}=-\sin x$ or $\dfrac{du}{dv}=-\cot x$

Q. 6. If $f(x)=x+7$ and $g(x)=x-7$, $x\in R$, then find $\dfrac{d}{dx}$ $(fog)(x)$.

Sol. Given, $\quad f(x)=x+7$ and $g(x)=x-7$

Now, $\quad fog=f(g(x))$

$$=f(x-7)=(x-7)+7=x$$

$$\therefore \quad \frac{d}{dx}\left[(fog)(x)\right]=\frac{d}{dx}(x)$$

$$=1$$

Q. 7. If $y = x|x|$, find $\dfrac{dy}{dx}$ for $x < 0$.

Sol. Given, $y = x|x|$

Since, $x < 0$, then $|x| = -x$

$$\therefore \quad y = x(-x)$$
$$y = -x^2$$
$$\therefore \quad \frac{dy}{dx} = -2x$$

2 Marks Questions

AI **Q. 1. Find $\dfrac{dx}{dt}$, if $x = at^2$ and $y = 2at$.**

[ISC Specimen Paper, 2018]

Sol.
$$x = at^2$$
$$\therefore \quad \frac{dx}{dt} = 2at$$
$$y = 2at$$
$$\therefore \quad \frac{dy}{dt} = 2a$$
$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

Q 2. Find $\dfrac{dx}{dt}$, if $x^3 + y^3 = 3axy$. **[ISC, 2020]**

Sol.
$$x^3 + y^3 = 3axy$$
$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(3axy)$$
$$3x^2 + 3y^2\frac{dy}{dx} = 3a\frac{d}{dx}(xy)$$
$$x^2 + y^2\frac{dy}{dx} = a\left\{x\frac{d}{dx}y + y\frac{d}{dx}x\right\}$$
$$x^2 + y^2\frac{dy}{dx} = a\left\{x\frac{dy}{dx} + y(1)\right\}$$
$$y^2\frac{dy}{dx} - ax\frac{dy}{dx} = ay - x^2$$
$$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$
$$\therefore \quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Some candidates apply wrong formula for product. Some leave the question in the middle as they do not combined the terms having *dy*/*dx*.

Answering Tip

- Learn the all the formulae and combined the terms of *dy*/*dx* together before giving the final answer.

Q. 3. If $y = e^{\sin x^2}$, then $\dfrac{dy}{dx}$.

Sol. Given $y = e^{\sin x^2}$

Let $y = e^z$, where $z = \sin t$ and $t = x^2$

For, $t = x^2$,

On differentiating both sides of t w.r.t. x, we get

$$\frac{dt}{dx} = 2x \qquad \ldots\text{(i)}$$

For, $z = \sin t$,

On differentiating both sides of z w.r.t. t, we get

$$\frac{dz}{dt} = \cos t \qquad \ldots\text{(ii)}$$

For, $y = e^z$,

On differentiating both sides of y w.r.t. z, we get

$$\frac{dy}{dz} = e^z \qquad \ldots\text{(iii)}$$

Using chain rule of derivative, $\dfrac{dy}{dx}$

$$= \frac{dy}{dz} \times \frac{dz}{dt} \times \frac{dt}{dx} = e^z \times \cos t \times 2x$$

[from eqs. (i), (ii) and (iii)]

$$= 2xe^{\sin t}\cos t = 2xe^{\sin x^2}\cos x^2$$

AI **Q. 4. Find the derivative of $f(e^{\tan x})$ w.r.t. x at $x = 0$. It is given that $f'(1) = 5$.**

Sol. Let $y = f(e^{\tan x})$

$$\frac{dy}{dx} = f'(e^{\tan x})\frac{d}{dx}(e^{\tan x})$$
$$\Rightarrow \quad \frac{dy}{dx} = f'(e^{\tan x})e^{\tan x}\sec^2 x$$

Now,
$$\left[\frac{dy}{dx}\right]_{x=0} = f'(e^{\tan 0})e^{\tan 0}\sec^2 0$$
$$= f'(e^0)e^0 1$$
$$= f'(1) = 5$$

Examiner's Comment

- Use of the rule for composite function as well as chain rule is forgotten by several students.

Answering Tip

- Derivatives of all forms of functions require practice and review from time to time. Sufficient time needs to be spent on this topic.

Q. 5. If $f(x) = x + 1$, find $\dfrac{d}{dx}$ $(fof)(x)$.

Sol. Given, $f(x) = x + 1$

$$\therefore \quad (fof)(x) = fof(x)$$
$$= f(x + 1)$$
$$= x + 1 + 1$$
$$= x + 2$$

Now, $$\frac{d}{dx}(fof)(x) = \frac{d}{dx}(x + 2)$$

$$= \frac{d}{dx}(x)+\frac{d}{dx}(2)$$

$$= 1$$

Hence, $\frac{d}{dx}\ (fof)(x) = 1$

AI Q. 6. Differentiate $e^{\sqrt{3x}}$, with respect to x.

Sol. Let $y = e^{\sqrt{3x}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sqrt{3x}})$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\sqrt{3x}}\frac{d}{dx}\sqrt{3x}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\sqrt{3x}}\sqrt{3}\frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{3}e^{\sqrt{3x}}\frac{1}{2\sqrt{x}}\frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3}}{2}e^{\sqrt{3x}}.\frac{1}{\sqrt{x}}.1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3}}{2\sqrt{x}}e^{\sqrt{3x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3x}}{2x}e^{\sqrt{3x}}$$

Q. 7. Differentiate $\sin^2(x^2)$ w.r.t. x^2.

Sol. Let $x^2 = t$

So, $$\frac{d\left[\sin^2(x)^2\right]}{dx^2} = \frac{d(\sin^2 t)}{dt}$$

$$= 2\sin t \times (\sin t)'$$

$$= 2\sin t \times \cos t$$

$$= 2\sin(x^2)\cos(x^2)$$

4 Marks Questions

Q. 1. If $y = e^{\sin^{-1}x}$ and $z = e^{\cos^{-1}x}$, prove that $\frac{dy}{dz} = e^{\frac{\pi}{2}}$.

[ISC, 2019]

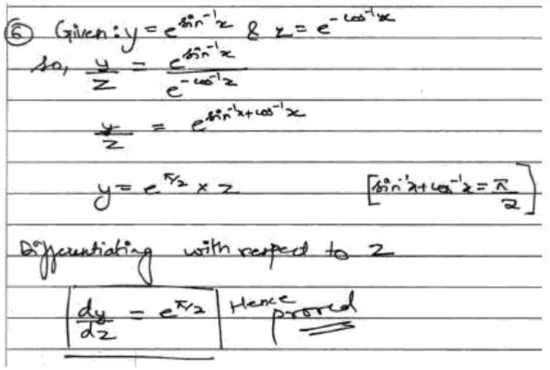
Sol. $$\frac{dy}{dz} = e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}, \frac{dz}{dx} = e^{-\cos^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$\frac{dz}{dx} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}}{e^{-\cos^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}}$$

$$\cos^{-1}x = e^{\frac{\pi}{2}}$$

[ISC Marking Scheme, 2019]

OR



⑥ Given: $y = e^{\sin^{-1}x}$ & $z = e^{-\cos^{-1}x}$

so, $\frac{y}{z} = \frac{e^{\sin^{-1}x}}{e^{-\cos^{-1}x}}$

$\frac{y}{z} = e^{\sin^{-1}x+\cos^{-1}x}$

$y = e^{\pi/2} \times z$ $\left[\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}\right]$

Differentiating with respect to z

$\boxed{\frac{dy}{dz} = e^{\pi/2}}$ Hence proved

[Topper's Answer 2019]

Examiner's Comment

- Some common errors made by many candidates in this question were:
  - differentiation of exponential functions
  - elimination of the parameter
  - errors in simplification.

Answering Tip

- Understand the concept of derivatives of exponential functions and parametric functions. Do enough practice for solving different types of problems on the topic.

AI Q. 2. If $x = \tan\left(\frac{1}{a}\log y\right)$, prove that $(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$ **[ISC, 2018]**

Sol. Given, $x = \tan\left(\frac{1}{a}\log y\right)$

$$\Rightarrow \quad \frac{1}{a}\log y = \tan^{-1}x$$

Differentiating w.r.t x, we get

$$\frac{1}{y}.\frac{dy}{dx} = a.\frac{1}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = a.y$$

Differentiating again w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2}+2x.\frac{dy}{dx} = a\frac{dy}{dx}$$

$$(1+x^2)\frac{d^2y}{dx^2}+(2x-a)\frac{dy}{dx} = 0$$ **Hence Proved.**

[ISC Marking Scheme, 2018]

Examiner's Comment

- Most of the candidates successfully attempted the first derivative part but made errors in differentiating second time by applying chain rule. Some candidates could not simplify to get the required result.

Answering Tip

- Do adequate practice to solve different types of questions and providing the result in the process of differentiation and simplifying equation.

AI Q. 3. If $y = e^{a\cos^{-1}x}$, where $-1 \le x \le 1$ then show that :

$$(1-x^2)\,y_2 - xy_1 - a^2y = 0$$

[ISC Specimen Paper, 2018]

Sol. Given, $y = e^{a\cos^{-1}x}$, $-1 \le x \le 1$ ...(i)

Taking log on both sides,

$\log y = a\cos^{-1}x$ $(\because \log_e e = 1)$

Differentiate w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = a\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-a\,e^{a\cos^{-1}}x}{\sqrt{1-x^2}} \qquad \text{...(ii)}$$

Differentiating again eq. (ii), w.r.t. x,

$$\frac{d^2y}{dx^2} = -a\left[\frac{\sqrt{1-x^2}\,\frac{d}{dx}e^{a\cos^{-1}x} - e^{a\cos^{-1}x}.\frac{d}{dx}\sqrt{1-x^2}}{(1-x^2)}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -a\left[\frac{\sqrt{1-x^2}\,e^{a\cos^{-1}x}a\left(\frac{-1}{\sqrt{1-x^2}}\right) - e^{a\cos^{-1}x}\,\frac{1.(-2x)}{2\sqrt{1-x^2}}}{(1-x^2)}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -a\left[\frac{-a\,e^{a\cos^{-1}x} + \frac{x\,e^{a\cos^{-1}x}}{\sqrt{1-x^2}}}{(1-x^2)}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{a^2\,e^{a\cos^{-1}x} - \frac{ax\,e^{a\cos^{-1}x}}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = a^2y + x\frac{dy}{dx} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

$(1-x^2)y_2 - xy_1 - a^2y = 0$ **Hence Proved.**

Q. 4. If $y = \cos(\sin x)$, show that :

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$$

[ISC, 2017]

Sol. Given, $y = \cos(\sin x)$ ...(i)

Differentiating w.r.t. x,

$$\frac{dy}{dx} = -\sin(\sin x).\cos x \qquad \text{...(ii)}$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = -[\cos^2 x\cos(\sin x) - \sin(\sin x).\sin x]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y\cos^2 x. + \sin x.\sin(\sin x) \quad \text{[From (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y.\cos^2 x - \sin x.\frac{1}{\cos x}.\frac{dy}{dx} \quad \text{[From(ii)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y.\cos^2 x - \tan x\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \tan x.\frac{dy}{dx} + y\times\cos^2 x = 0$$

Hence Proved.

[ISC Marking Scheme, 2017]

Q. 5. Using a suitable substitution, find the derivative of $\tan^{-1}\sqrt{\frac{a-x}{a+x}}$ with respect to x. [ISC, 2010]

Sol. Let $y = \tan^{-1}\sqrt{\frac{a-x}{a+x}}$

Put $x = a\cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1}\frac{x}{a}$

$$\therefore \quad y = \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan^{-1}(\tan\theta)$$

$$\Rightarrow \quad y = \theta = \frac{1}{2}\cos^{-1}\frac{x}{a}$$

$$\frac{dy}{dx} = -\frac{1}{2}\frac{1}{\sqrt{1-x^2/a^2}}\times\frac{1}{a} = -\frac{1}{2\sqrt{a^2-x^2}}$$

Q. 6. Using a suitable substitution, find the derivative of $\tan^{-1}\frac{4\sqrt{x}}{1-4x}$ with respect to x. [ISC, 2009]

Sol. Let $y = \tan^{-1}\frac{4\sqrt{x}}{1-4x}$

Put $2\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}(2\sqrt{x})$

$$\therefore \quad y = \tan\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) = 2\theta$$

$$= 2\tan^{-1}(2\sqrt{x})$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{1+4x}\cdot\frac{2}{2\sqrt{x}} = \frac{2}{\sqrt{x}(1+4x)}$$

AI Q. 7. Find the derivative of $\sin x^2$ with respect to x^3. [ISC, 2009]

Sol. Let $u = \sin x^2$ and $v = x^2$

$$\Rightarrow \quad \frac{du}{dx} = \frac{d}{dx}(\sin x^2) \text{ and } \frac{dv}{dx} = \frac{d}{dx}(x^3)$$

$$\Rightarrow \quad \frac{du}{dx} = \cos x^2\frac{d}{dx}(x^2) \text{ and } \frac{dv}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{du}{dx} = 2x\cos x^2 \text{ and } \frac{dv}{dx} = 3x^2$$

Now, $$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{2x\cos x^2}{3x^2}$$

$$= \frac{2\cos x^2}{3x}$$

Q. 8. Find $\frac{dy}{dx}$, if $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$. **[ISC, 2008]**

Sol. Given, $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Put, $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\therefore \quad y = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$= \tan^{-1}\left[\frac{\sec\theta - 1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\tan\frac{\theta}{2}\right]$$

$$\Rightarrow \quad y = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

AI Q. 9. If $y = \sqrt{\frac{1-\cos x}{1+\cos x}}$, then find $\frac{dy}{dx}$. **[ISC, 2008]**

Sol. Given, $y = \sqrt{\frac{1-\cos x}{1+\cos x}}$

On multiplying and dividing by $\sqrt{1-\cos x}$ on RHS, we get

$$y = \sqrt{\frac{(1-\cos x)(1-\cos x)}{(1+\cos x)(1-\cos x)}}$$

$$\Rightarrow \quad y = \sqrt{\frac{(1-\cos x)^2}{1-\cos^2 x}} = \sqrt{\frac{(1-\cos x)^2}{\sin^2 x}}$$

[$\because 1 - \cos^2 x = \sin^2 x$]

$$\Rightarrow \quad y = \frac{(1-\cos x)}{\sin x}$$

$$\Rightarrow \quad y = \frac{1}{\sin x} - \frac{\cos x}{\sin x}$$

$$\Rightarrow \quad y = \operatorname{cosec} x - \cot x$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = -\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x$$

Q. 10. If $\sin y = x\sin(a+y)$, show that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$ **[ISC, 2007]**

Sol. Given, $\sin y = x\sin(a+y)$

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiate both the sides w.r.t. y,

$$\frac{dx}{dy} = \frac{\sin(a+y)\cdot\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Q. 11. If $e^{x+y} = xy$, then show that

$$\frac{dy}{dx} = \frac{y(1-x)}{x(y-1)}$$ **[ISC, 2007]**

Sol. Given, $e^{x+y} = xy$

Taking logarithm on both sides, we get

$$x + y = \log_e(xy) = \log_e x + \log_e y$$

On differentiating both sides w.r.t. x, we get

$$1 + \frac{dy}{dx} = \frac{d}{dx}(\log_e x) + \frac{d}{dx}(\log_e y)$$

$$\Rightarrow \quad 1 + \frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \left(1 - \frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x} - 1$$

$$\Rightarrow \quad \frac{(y-1)}{y}\frac{dy}{dx} = \frac{1-x}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y(1-x)}{x(y-1)}$$

AI Q. 12. If $\sin(xy) + \cos(xy) = 1$ and $\tan(xy) \neq 1$, then show that $\frac{dy}{dx} = \frac{-y}{x}$.

Sol. Given, $\sin(xy) + \cos(xy) = 1$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}[\sin(xy) + \cos(xy)] = \frac{d(1)}{dx}$$

$$\Rightarrow \quad \frac{d}{dxy}(\sin xy)\frac{d}{dx}(xy) + \frac{d}{dxy}(\cos xy)\frac{d}{dx}(xy) = 0$$

$$\left[\frac{d\sin(xy)}{d(xy)} + \frac{d\cos(xy)}{d(xy)}\right] \times \frac{d}{dx}(xy) = 0$$

$$\Rightarrow \quad \left[\frac{d\sin(xy)}{d(xy)} + \frac{d\cos(xy)}{d(xy)}\right] \times \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$

$$\Rightarrow \quad (\cos xy - \sin xy)\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \quad \left(1 - \frac{\sin xy}{\cos xy}\right)\left(x\frac{dy}{dx} + y\right) = 0$$

[divide by cos xy]

$$\Rightarrow \quad (1 - \tan xy)\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0$$

$$[\because \tan xy \neq 1, \text{ given}]$$

$$\therefore \quad x\frac{dy}{dx} = -y$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

Q. 13. If $x^m y^n = (x + y)^{m+n}$, prove that : $\frac{dy}{dx} = \frac{y}{x}$

Sol. Given, $x^m y^n = (x + y)^{m+n}$

Taking log on both sides, we get

$$\Rightarrow \quad \log x^m y^n = \log (x + y)^{m+n}$$

$$\log x^m + \log y^n = (m + n) \log(x + y)$$

$$\Rightarrow m \log x + n \log y = (m + n) \log (x + y)$$

Differentiating w.r.t to x, we get

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} \times \left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{n}{y} - \frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{x(m+n) - m(x+y)}{x(x+y)}$$

$$\left(\frac{nx + ny - my - ny}{y(x+y)}\right)\frac{dy}{dx} = \frac{xm + xn - mx - my}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} \times \left(\frac{xn - my}{xn - my}\right) = \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

AI Q. 14. If $x = a \sin^3 t$ and $y = a \cos^3 t$, then find dy/dx.

Sol. Given, $x = a \sin^3 t$ and $y = a \cos^3 t$

Now, differentiating both sides of x w.r.t. t, we get

$$\frac{dx}{dt} = a\frac{d}{dt}(\sin^3 t)$$

$$= a\frac{d(\sin^3 t)}{d(\sin t)} \times \frac{d(\sin t)}{dt}$$

$$= a \times 3\sin^2 t \times \cos t$$

$$= 3a \sin^2 t \cos t \qquad \text{...(i)}$$

Now, differentiating both sides of y w.r.t. t, we get

$$\frac{dy}{dt} = a\frac{d(\cos^3 t)}{dt}$$

$$= a\frac{d(\cos^3 t)}{d(\cos t)} \times \frac{d(\cos t)}{dt}$$

$$= a \times 3\cos^2 t \times (-\sin t)$$

$$= -3a \cos^2 t \sin t \qquad \text{...(ii)}$$

$$\text{Now,} \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{-3a \sin t \cos^2 t}{3a \sin^2 t \cos t} \quad \text{[Using eqs. (i) and (ii)]}$$

$$= \frac{-\cos t}{\sin t} = -\cot t$$

Q. 15. If $x^y y^x = 5$, then show that

$$\frac{dy}{dx} = -\left(\frac{\log y + \frac{y}{x}}{\log x + \frac{x}{y}}\right)$$

Sol. Given, $x^y y^x = 5$

Taking logarithm on both sides, we get

$$\log x^y y^x = \log 5$$

$$\Rightarrow \quad \log x^y + \log y^x = \log 5$$

$$[\because \log(mn) = \log m + \log n]$$

$$\Rightarrow \quad y \log x + x \log y = \log 5$$

On differentiating both sides w.r.t. x, we get

$$y \cdot \frac{1}{x} + \log x\frac{dy}{dx} + \log y + \frac{x}{y} \cdot \frac{dy}{dx} = 0$$

[Using product rule of derivative]

$$\Rightarrow \quad \frac{dy}{dx}\left(\log x + \frac{x}{y}\right) = -\log y - \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\left(\log y + \frac{y}{x}\right)}{\left(\log x + \frac{x}{y}\right)}$$

Q. 16. If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then prove that :

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

Sol. Let $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$\text{Now,} \quad y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

$$\Rightarrow \quad y = 2\tan^{-1} x$$

On differentiating both sides w.r.t. x, we get

$$\therefore \quad \frac{dy}{dx} = 2\frac{d}{dx}(\tan^{-1} x) = \frac{2}{1+x^2}$$

Q. 17. If $y = e^x \log \tan 2x$, then find dy/dx.

Sol. Given, $y = e^x \log \tan 2x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^x\frac{d}{dx}(\log \tan 2x) + \log \tan 2x\frac{d(e^x)}{dx}$$

[product rule of derivative]

$$= e^x\frac{d(\log \tan 2x)}{d(\tan 2x)} \times \frac{d(\tan 2x)}{d(2x)} \times \frac{d(2x)}{dx}$$

$$+ e^x \log \tan 2x$$

[by chain rule of derivative]

$$= \frac{2e^x \sec^2 2x}{\tan 2x} + e^x \log \tan 2x$$

$$= \frac{2e^x}{\sin 2x \cos 2x} + e^x \log \tan 2x$$

$$= \frac{2 \times 2e^x}{2\sin 2x \cos 2x} + e^x \log \tan 2x$$

$$= \frac{4e^x}{\sin 4x} + e^x \log\tan 2x$$

[∵ sin 2x = 2sin x cos x]

$$= 4e^x \operatorname{cosec} 4x + e^x \log\tan 2x$$

$$= e^x(4\operatorname{cosec} 4x + \log(\tan 2x)]$$

Q. 18. If $y = (\cos x)^{\cos x}$, then find $\frac{dy}{dx}$.

Sol. Given $y = (\cos x)^{\cos x}$

Taking logarithm on both sides, we get

$\log y = \cos x \log(\cos x)$

On differentiating both sides *w.r.t.* x, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{d}{dx}\log(\cos x) + \log(\cos x)\frac{d}{dx}(\cos x)$$

[using product rule of derivative]

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x\frac{d\log(\cos x)}{d(\cos x)}\times\frac{d(\cos x)}{dx} - \sin x\log(\cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\cos x}\times(-\sin x) - \sin x\log(\cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\sin x - \sin x\log(\cos x)$$

$$\Rightarrow \frac{dy}{dx} = y[-\sin x - \sin x\log(\cos x)]$$

put $y = (\cos x)^{\cos x}$

$$\therefore \frac{dy}{dx} = (\cos x)^{\cos x}[-\sin x\{1 + \log(\cos x)\}]$$

AI Q. 19. If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, then prove that: $\frac{dy}{dx} = \frac{x+y}{x-y}$

Sol. Given, $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$

$$\therefore \frac{dx}{dt} = a[e^t(\cos t - \sin t) + e^t(\sin t + \cos t)]$$

$$= -y + x$$

and $$\frac{dy}{dt} = a[e^t(\sin t - \cos t) + e^t(\sin t + \cos t)]$$

$$= y + x$$

Therefore

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

Q. 20. If $(x-y)e^{\frac{x}{x-y}} = a$, prove that $y\frac{dy}{dx} + x = 2y$.

Sol. Given, $(x-y)e^{\frac{x}{x-y}} = a$

Taking log both sides

$$\Rightarrow \log(x-y) + \frac{x}{x-y} = \log a$$

Differentiate w.r.t. x

$$\Rightarrow \frac{1-y'}{x-y} + \frac{(x-y).1 - x(1-y')}{(x-y)^2} = 0, \quad \left(\because y' = \frac{dy}{dx}\right)$$

$$\Rightarrow (x-y)(1-y') + (x-y) - x(1-y') = 0$$

$$\Rightarrow yy' + x - 2y = 0$$

or $$y\frac{dy}{dx} + x = 2y$$

Hence Proved

Q. 21. If $y = \tan^{-1}\left(\frac{a}{x}\right) + \log\sqrt{\frac{x-a}{x+a}}$, prove that $\frac{dy}{dx} = \frac{2a^3}{x^4 - a^4}$

Sol. Given, $y = \tan^{-1}\left(\frac{a}{x}\right) + \log\sqrt{\frac{x-a}{x+a}}$

$$\therefore y = \tan^{-1}\left(\frac{a}{x}\right) + \frac{1}{2}\{\log(x-a) - \log(x+a)\}$$

Now, $$\frac{dy}{dx} = \frac{1}{1+\frac{a^2}{x^2}}\left(-\frac{a}{x^2}\right) + \frac{1}{2}\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\}$$

$$= \frac{-a}{x^2+a^2} + \frac{a}{x^2-a^2} = \frac{2a^3}{x^4-a^4}$$

Q. 22. If $x = \cos t(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

Sol. Given, $x = 3\cos t - 2\cos^3 t$ and $y = 3\sin t - 2\sin^3 t$

$$\therefore \frac{dx}{dt} = -3\sin t + 6\cos^2 t\sin t$$

and $$\frac{dy}{dt} = 3\cos t - 6\sin^2 t\cos t$$

Now, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t(1 - 2\sin^2 t)}{3\sin t(-1 + 2\cos^2 t)}$$

$$= \frac{\cos t\cos 2t}{\sin t\cos 2t} = \cot t$$

At $$t = \frac{\pi}{4}, \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

Q. 23. If $(\cos x)^y = (\cos y)^x$, find $\frac{dy}{dx}$.

Sol. Given, $(\cos x)^y = (\cos y)^x$

Taking log both sides, we get

$y\log\cos x = x\log\cos y$

On differentiating w.r.t. x, we get

$$\Rightarrow y\frac{-\sin x}{\cos x} + \log\cos x\cdot\frac{dy}{dx} = x\cdot\left(\frac{-\sin y}{\cos y}\right)\frac{dy}{dx} + \log\cos y$$

$$\Rightarrow (\log\cos x + x\tan y)\frac{dy}{dx} = \log\cos y + y\tan x$$

$$\frac{dy}{dx} = \frac{\log\cos y + y\tan x}{\log\cos x + x\tan y}$$

AI Q. 24. If $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Sol. Given, $x = \sqrt{a^{\sin^{-1} t}}$

Squaring on both sides, we get

$$x^2 = a^{\sin^{-1} t}$$

Taking log on both sides, we get

$$2\log x = \sin^{-1} t \log a$$

On differentiating w.r.t. 'x', we get

$$\Rightarrow \quad \frac{2}{x} = \log a . \frac{1}{\sqrt{1-t^2}} . \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{x}{2}\left[\log a \frac{1}{\sqrt{1-t^2}}\right]$$

Similarly, for $y = \sqrt{a^{\cos^{-1} t}}$

Squaring and taking log both sides, we get

$$2 \log y = \log a \cos^{-1} t$$

$$\therefore \quad \frac{dy}{dt} = -\frac{y}{2}\left[\log a \cdot \frac{1}{\sqrt{1-t^2}}\right]$$

Now, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{y}{2} \cdot \frac{2}{x} \frac{\sqrt{1-t^2}}{\sqrt{1-t^2}} = -\frac{y}{x}$$

Alternative Solution :

$$x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$$

$$\therefore \quad x \cdot y = \sqrt{a^{\sin^{-1} t + \cos^{-1} t}}$$

$$= \sqrt{a^{\pi/2}} = \text{Constant} \qquad \ldots\text{(i)}$$

Differentiating (i), we get

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

Q. 25. If $x = a \sec^3\theta$, $y = a \tan^3\theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$.

Sol. Given $x = a \sec^3\theta$

$$\therefore \quad \frac{dx}{d\theta} = 3a\sec^3\theta\tan\theta$$

and $y = a \tan^3\theta$

$$\therefore \quad \frac{dy}{d\theta} = 3a\tan^2\theta\sec^2\theta$$

Therefore, $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \sin\theta$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

6 Marks Questions

Q. 1. If $\log y = \tan^{-1}x$, prove that:

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$ **[ISC, 2016]**

Sol. Given, $\log y = \tan^{-1}x$

$$\therefore \quad y = e^{\tan^{-1}x}$$

$$\frac{dy}{dx} = e^{\tan^{-1}x} \times \frac{1}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = e^{\tan^{-1}x}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = y$$

Differentiating again, w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(2x) = \frac{dy}{dx}$$

$$\therefore \quad (1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$ **Hence Proved.**

[ISC Marking Scheme, 2016]

Examiner's Comment

- First differentiation was done by many candidates and errors were made in finding the second derivative and framing the required result. Use of the rule for composite function as well as chain rule was forgotten by several candidates.

Answering Tip

- Derivatives of all forms of functions require continuous practice. Sufficient time needs to be spent on this topic.

Q. 2. If $y = e^{m\cos^{-1}x}$, prove that :

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m^2y$$ **[ISC, 2015]**

Sol. Given, $y = e^{m\cos^{-1}x}$

$$\therefore \quad \frac{dy}{dx} = e^{m\cos^{-1}x} \times \frac{-m}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = -my \qquad \ldots\text{(i)}$$

Differentiating again w.r.t. x, we get

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} - \frac{2x}{2\sqrt{1-x^2}}\frac{dy}{dx} = -m\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m\frac{dy}{dx}\sqrt{1-x^2}$$

$$= -m \times (-my) \qquad \text{[from (i)]}$$

$$= m^2y$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- Some candidates wrote differentiation of $\cos^{-1} x$ incorrectly. Many candidates did not place $e^{m\cos^{-1} x}$ as y after first differentiation. Second order derivation was incorrectly shown by several candidates. Some made calculation mistakes while simplifying the equation.

Answering Tip

- Differentiation rules for different functions and terms need attention. A thorough revision is a must.

AI Q. 3. If $y = \dfrac{x\sin^{-1} x}{\sqrt{1-x^2}}$, **prove that :** $(1-x^2)\dfrac{dy}{dx} = x + \dfrac{y}{x}$

[ISC, 2014]

Sol. Given, $y = \dfrac{x\sin^{-1} x}{\sqrt{1-x^2}}$ ...(i)

$$\frac{dy}{dx} = \frac{\left(\sin^{-1} x + \dfrac{x}{\sqrt{1-x^2}}\right)\sqrt{1-x^2} - x\sin^{-1} x \times \dfrac{(-2x)}{2\sqrt{1-x^2}}}{1-x^2}$$

$$(1-x^2)\frac{dy}{dx} = \sqrt{1-x^2}\,\sin^{-1} x + x + xy \quad \text{[from (i)]}$$

$$(1-x^2)\frac{dy}{dx} = (1-x^2)\frac{y}{x} + x + xy$$

$$(1-x^2)\frac{dy}{dx} = \frac{y}{x} - xy + x + xy$$

$$(1-x^2)\frac{dy}{dx} = x + \frac{y}{x}$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- $\dfrac{u}{v}$ rule for differentiating the function was incorrectly done by a number of candidates. Some candidates erred while differentiating after cross-multiplication.

Answering Tip

- Differentiation rules for different functions and forms need continuous revision and practice. Obtaining the result in the required form needs simplification, cross multiplication, substitution, etc. hence enough revision and practice is required.

Q. 4. If $y = (\cot^{-1} x)^2$, **show that**

$$(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2 \qquad \textbf{[ISC, 2013]}$$

Sol. Given, $y = (\cot^{-1} x)^2$

$$\frac{dy}{dx} = 2\cot^{-1} x\,\frac{d}{dx}(\cot^{-1} x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\cot^{-1} x\left(\frac{-1}{1+x^2}\right)$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = -2\cot^{-1} x$$

Differentiating once again w.r.t. x

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = -2\cdot\frac{(-1)}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{2}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$$

Q. 5. If $e^y(x+1) = 1$, **then show that:**

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \qquad \textbf{[ISC, 2012]}$$

Sol. Given, $e^y(x + 1) = 1$

Taking log of both sides, we get

$y \log e + \log(x + 1) = \log 1$

$\Rightarrow \quad y + \log(x + 1) = 0$

$\Rightarrow \quad y = -\log(x + 1)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{1}{x+1} \qquad \text{...(i)}$$

Differentiating (i) again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} = \left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \qquad \text{[From (i)]}$$

Q. 6. If $y = x^x$, **prove that:** $\dfrac{d^2y}{dx^2} - \dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2 - \dfrac{y}{x} = 0$

[ISC, 2011]

Sol. Given, $y = x^x$

Taking log on both sides, we get

$\log y = x \log x$

Differentiating w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1+\log x) \qquad \text{...(i)}$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = y\left(0+\frac{1}{x}\right) + (1+\log x)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\frac{dy}{dx}\cdot\frac{dy}{dx} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$

Q. 7. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ **with respect to** x.

Sol. Let $y = x^{\sin x} + (\sin x)^{\cos x}$

or $y = u + v$

where, $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

Taking log both sides, we get

$$\log u = \sin x \log x \quad \text{and} \quad \log v = \cos x \log \sin x$$

Differentiate with respect to x,

$$\frac{1}{u}\frac{du}{dx} = \sin x \cdot \frac{1}{x} + \log x(\cos x)$$

$$= \frac{\sin x}{x} + \cos x \log x$$

$$\therefore \quad \frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

and $\dfrac{1}{v}\dfrac{dv}{dx} = \cos x \cdot \dfrac{1}{\sin x}(\cos x) + \log \sin x(-\sin x)$

$$= \frac{\cos^2 x}{\sin x} - \sin x \log \sin x$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \log \sin x\right]$$

Now, $y = u + v$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$+ (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \log \sin x\right]$$

AI Q. 8. If $x = a \sin 2t(1 + \cos 2t)$ and $y = b \cos 2t\,(1 - \cos 2t)$, find $\dfrac{dy}{dx}$, at $t = \dfrac{\pi}{4}$.

Sol. Here, $x = a \sin 2t(1 + \cos 2t)$ and $y = b \cos 2t(1 - \cos 2t)$

$$\frac{dx}{dt} = a[2\cos 2t(1 + \cos 2t) + \sin 2t(-2\sin 2t)]$$

$$= 2a[\cos 2t + (\cos^2 2t - \sin^2 2t)]$$

$$= 2a(\cos 2t + \cos 4t)$$

and $\dfrac{dy}{dt} = b[-2\sin 2t(1 - \cos 2t) + \cos 2t(2\sin 2t)]$

$$= 2b[-\sin 2t + 2\sin 2t \cos 2t]$$

$$= 2b(-\sin 2t + \sin 4t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2b(-\sin 2t + \sin 4t)}{2a(\cos 2t + \cos 4t)}$$

At $t = \dfrac{\pi}{4}$,

$$\frac{dy}{dx} = \frac{b\left(-\sin\frac{\pi}{2} + \sin\pi\right)}{a\left(\cos\frac{\pi}{2} + \cos\pi\right)}$$

$$= \frac{b(-1+0)}{a(0-1)} = \frac{b}{a}$$

AI Q. 9. Let $y = (\log x)^x + x^{x\cos x}$, then find $\dfrac{dy}{dx}$.

Sol. Let $\quad u = (\log x)^x$

Then $\quad \log u = x \log(\log x)$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = \frac{x}{x\log x} + \log(\log x)$$

$$= \frac{1}{\log x} + \log(\log x)$$

$$\Rightarrow \quad \frac{du}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right]$$

Let $\quad v = x^{x\cos x}$

Then, $\quad \log v = x \cos x \log x$

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = \frac{x\cos x}{x} + \cos x(\log x) - x\sin x \log x$$

$$\Rightarrow \quad \frac{dv}{dx} = x^{x\cos x}[\cos x + \cos x(\log x) - x\sin x \log x]$$

$$\therefore \quad y = u + v$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right]$$

$$+ x^{x\cos x}[\cos x + \cos x(\log x) - x\sin x \log x]$$

Q. 10. If $y = \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}} - \log\sqrt{1-x^2}$, then prove that:

$$\frac{dy}{dx} = \frac{\cos^{-1}x}{(1-x^2)^{3/2}}$$

Sol. Given, $\quad y = \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}} - \log\sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2}\left(1.\cos^{-1}x - \frac{x}{\sqrt{1-x^2}}\right) - \frac{x\cos^{-1}x(-2x)}{2\sqrt{1-x^2}}}{\left(\sqrt{1-x^2}\right)^2} + \frac{2x}{2(1-x^2)}$$

$$= \frac{\sqrt{1-x^2}\cos^{-1}x - x + \frac{x^2\cos^{-1}x}{\sqrt{1-x^2}}}{1-x^2} + \frac{x}{1-x^2}$$

$$= \frac{(1-x^2)\cos^{-1}x - x\sqrt{1-x^2} + x^2\cos^{-1}x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2}$$

$$= \frac{(1-x^2)\cos^{-1}x}{(1-x^2)^{3/2}} - \frac{x\sqrt{1-x^2}}{(1-x^2)^{3/2}} + \frac{x^2\cos^{-1}x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2}$$

$$= \frac{(1-x^2)\cos^{-1}x + x^2\cos^{-1}x}{(1-x^2)^{3/2}} = \frac{\cos^{-1}x}{(1-x^2)^{3/2}}$$

Hence Proved.

Q. 11. If $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2\le 1$, **then find** $\dfrac{dy}{dx}$.

Sol. Given, $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Put $x^2 = \cos\theta \Rightarrow \theta = \cos^{-1}x^2$

$$\therefore\quad y=\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}\right)$$

$$=\tan^{-1}\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)$$

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)$$

$$\therefore\quad y=\frac{\pi}{4}+\frac{\theta}{2}=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$$\therefore\quad \frac{dy}{dx}=-\frac{1}{2}\frac{1}{\sqrt{1-x^4}}2x=-\frac{x}{\sqrt{1-x^4}}$$

Q. 12. If $x = a(\cos t + t\sin t)$ **and** $y = a(\sin t - t\cos t)$, **then find the value of** $\dfrac{d^2y}{dx^2}$ **at** $t=\dfrac{\pi}{4}$.

Sol. $\dfrac{dy}{dt}=at\sin t;$ and $\dfrac{dx}{dt}=at\cos t$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\tan t$$

$$\Rightarrow\quad \frac{d^2y}{dx^2}=\sec^2 t\cdot\frac{dt}{dx}=\frac{\sec^3 t}{at}$$

$$\therefore\quad \left[\frac{d^2y}{dx^2}\right]_{t=\frac{\pi}{4}}=\frac{8\sqrt{2}}{a\pi}$$

Q. 13. If $y=\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)$ **and** $0 < x < 1$, **then find** $\dfrac{dy}{dx}$.

Sol. Put, $x=\sin\alpha, \sqrt{x}=\sin\beta$

$\therefore\quad \cos\alpha=\sqrt{1-x^2}$

and $\cos\beta=\sqrt{1-x}$

Now, $y = \sin^{-1}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$y = \sin^{-1}[\sin(\alpha - \beta)] = \alpha - \beta$

$$y=\sin^{-1}x-\sin^{-1}\sqrt{x}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x}\sqrt{1-x}}$$

Q. 14. If $(\tan^{-1}x)^y + y^{\cot x} = 1$, **then find** $\dfrac{dy}{dx}$.

Sol. Let $(\tan^{-1}x)^y = u$

and $y^{\cot x} = v$

$\therefore\quad \log u = y\log\tan^{-1}x$

and $\log v = \cot x\log y$

for u, $\dfrac{1}{u}\dfrac{du}{dx}=\dfrac{y}{\tan^{-1}x}\cdot\dfrac{1}{1+x^2}+\log\tan^{-1}x\dfrac{dy}{dx}$

$$\frac{du}{dx}=(\tan^{-1}x)^y\left[\frac{y}{\tan^{-1}x}\cdot\frac{1}{1+x^2}+\log\tan^{-1}x\frac{dy}{dx}\right]$$

for v, $\dfrac{1}{v}\dfrac{dv}{dx}=\dfrac{\cot x}{y}\cdot\dfrac{dy}{dx}-\operatorname{cosec}^2x\log y$

$$\frac{dv}{dx}=y^{\cot x}\left[\frac{\cot x}{y}\cdot\frac{dy}{dx}-\operatorname{cosec}^2x\log y\right]$$

Given, $u + v = 1$

$$\Rightarrow\quad \frac{du}{dx}+\frac{dv}{dx}=0$$

$$\frac{y\cdot(\tan^{-1}x)^{y-1}}{(1+x^2)}+(\tan^{-1}x)^y\cdot\log\tan^{-1}x\frac{dy}{dx}$$

$$+\cot x\cdot y^{\cot x-1}\frac{dy}{dx}-y^{\cot x}\operatorname{cosec}^2x\cdot\log y=0$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{y^{\cot x}\cdot\operatorname{cosec}^2x\log y+\dfrac{y(\tan^{-1}x)^{y-1}}{(1+x^2)}}{(\tan^{-1}x)^y\cdot\log\tan^{-1}x+\cot x.y^{\cot x-1}}$$

AI Q. 15. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$ **with respect to** $\cos^{-1}(2x\sqrt{1-x^2})$, **when** $x \ne 0$.

Sol. Let $u=\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$ and $v=\cos^{-1}(2x\sqrt{1-x^2})$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$$\therefore\quad u=\tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$$

$$u=\tan^{-1}(\tan\theta)$$

$$u = \theta = \cos^{-1}x$$

$$\frac{du}{dx}=\frac{-1}{\sqrt{1-x^2}}$$

and
$$v=\cos^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$$

$$= \cos^{-1}(\sin 2\theta)$$

$$=\cos^{-1}\left(\cos\left(\frac{\pi}{2}-2\theta\right)\right)$$

$$v=\frac{\pi}{2}-2\theta=\frac{\pi}{2}-2\cos^{-1}x$$

$$\therefore\quad \frac{dv}{dx}=\frac{2}{\sqrt{1-x^2}}$$

$$\therefore\quad \frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{-1}{\sqrt{1-x^2}}\times\frac{\sqrt{1-x^2}}{2}=\frac{-1}{2}$$

(In case, if $x = \sin\theta$, then answer is $\dfrac{1}{2}$)

Q. 16. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ **with respect to** $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, **when** $x \ne 0$.

Sol. Let $x = \tan\theta$, $\therefore$ $\theta = \tan^{-1}x$

$$u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$= \frac{1}{2}\tan^{-1}x$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{2(1+x^2)};$$

and $$v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1}x$$

$$\therefore \quad \frac{dv}{dx} = \frac{2}{1+x^2}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1}{2(1+x^2)} \times \frac{1+x^2}{2} = \frac{1}{4}$$

Note : (In case, if $x = \cot\theta$ then answer is $-\frac{1}{4}$)

Q. 17. Differentiate $\log(x^{\sin x} + \cot^2 x)$ w.r.t. x.

Sol. $$\frac{dy}{dx} = \frac{1}{x^{\sin x}+\cot^2 x}\frac{d}{dx}(x^{\sin x}+\cot^2 x) \quad \text{...(i)}$$

Let $u = x^{\sin x}$ and $v = \cot^2 x$

$\therefore$ $\log u = \sin x \cdot \log x$

$$\frac{1}{u}\frac{du}{dx} = \frac{\sin x}{x} + \log x \cdot \cos x$$

or $$\frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

and $$\frac{dv}{dx} = 2\cot x[-\text{cosec}^2 x]$$

From equ (i),

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x^{\sin x}+\cot^2 x}\left[x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right) - 2\cot x\,\text{cosec}^2 x\right]$$

Q. 18. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, then prove that: $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$.

Sol. Given, $x = 2\cos\theta - \cos 2\theta$

Differentiate w.r.t. θ,

$$\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$$

$$= 2(\sin 2\theta - \sin\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = 2\left[2\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right]$$

and $y = 2\sin\theta - \sin 2\theta$

$$\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$$

$$= 2\left[2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{4\sin\left(\frac{3\theta}{2}\right)\sin\frac{\theta}{2}}{4\cos\left(\frac{3\theta}{2}\right)\sin\frac{\theta}{2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$$

Q. 19. If $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, then find $\frac{dy}{dx}$.

Sol. Given, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$$\Rightarrow \quad y = e^{x\log(\sin x)} + \sin^{-1}\sqrt{x}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{x\log(\sin x)}\left[\log\sin x + x\frac{\cos x}{\sin x}\right] + \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^x[\log\sin x + x\cot x] + \frac{1}{2\sqrt{x-x^2}}$$

Q. 20. Find $\frac{dy}{dx}$, if $y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$.

Sol. Given, $y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

$$\Rightarrow \quad y = \sin^{-1}\left(\frac{2\cdot 2^x}{1+2^{2x}}\right)$$

Let $2^x = \tan\theta, \Rightarrow \theta = \tan^{-1}2^x$

$$y = \sin^{-1}(\sin 2\theta) \qquad \left[\because \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta\right]$$

$\Rightarrow$ $y = 2\theta$

$\Rightarrow$ $y = 2\tan^{-1}(2^x)$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = 2\cdot\frac{1}{1+(2^x)^2}\cdot\frac{d}{dx}(2^x)$$

$$= \frac{2}{1+4^x}\cdot 2^x \cdot \log_e 2$$

$$= \frac{2^{x+1}}{1+4^x}\log_e 2$$

AI Q. 21. If $y = x\log\left(\frac{x}{a+bx}\right)$, then prove that:

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$

Sol. Given, $y = x\log\left(\frac{x}{a+bx}\right)$

$$\Rightarrow \quad \frac{y}{x} = \log x - \log(a+bx)$$

Differentiating w.r.t. x,

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{b}{a+bx}$$

⇒ $$x\frac{dy}{dx} - y = x - \frac{bx^2}{a+bx} = \left(\frac{ax}{a+bx}\right) \quad \text{...(i)}$$

Differentiating again w.r.t. x

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = 1 - \frac{[(a+bx)2bx - bx^2(b)]}{(a+bx)^2}$$

⇒ $$x\frac{d^2y}{dx^2} = \frac{(a+bx)^2 - 2bx(a+bx) + b^2x^2}{(a+bx)^2}$$

⇒ $$x\frac{d^2y}{dx^2} = \frac{(a+bx)[a+bx-2bx] + b^2x^2}{(a+bx)^2}$$

⇒ $$x\frac{d^2y}{dx^2} = \frac{(a+bx)(a-bx) + b^2x^2}{(a+bx)^2}$$

⇒ $$x\frac{d^2y}{dx^2} = \frac{a^2 - b^2x^2 + b^2x^2}{(a+bx)^2}$$

⇒ $$x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2} = \left(\frac{a}{a+bx}\right)^2$$

⇒ $$x\frac{d^2y}{dx^2} = \left(\frac{a}{a+bx}\right)^2$$

$$x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 = \left(x.\frac{dy}{dx} - y\right)^2$$

using equ. (i)

So, $$x^3\frac{d^2y}{dx^2} = \left(x.\frac{dy}{dx} - y\right)^2$$ **Hence Proved.**

Q. 22. If $y = \log[x + \sqrt{x^2 + a^2}]$, then prove that:

$$(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Sol. Given, $y = \log[x + \sqrt{x^2 + a^2}]$

Differentiate with respect to x,

$$y' = \frac{1}{x + \sqrt{x^2 + a^2}}\left\{1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right\}$$

$$= \frac{1}{\sqrt{x^2 + a^2}}$$

⇒ $y' \times \sqrt{x^2 + a^2} = 1$

Squaring on both sides,

$$y'^2(x^2 + a^2) = 1$$

Differentiate again with respect to x

$$2y'y''(x^2 + a^2) + 2xy'^2 = 0$$

⇒ $$y'[(x^2 + a^2)y'' + xy'] = 0$$

⇒ $$(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$ **Hence Proved.**

AI Q. 23. If $y = \left(x + \sqrt{x^2 + 1}\right)^m$, then show that:

$$(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$$

Sol. Given, $y = \left\{x + \sqrt{x^2 + 1}\right\}^m$

Differentiate with respect to x

$$y' = m\left\{x + \sqrt{x^2 + 1}\right\}^{m-1}\left\{1 + \frac{2x}{2\sqrt{x^2 + 1}}\right\}$$

$$= m\left\{x + \sqrt{x^2 + 1}\right\}^{m-1}\left\{\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right\}$$

$$= \left\{\frac{my}{\sqrt{x^2 + 1}}\right\}$$

⇒ $$y'\sqrt{x^2 + 1} = my$$

Squaring on both sides,

$$y'^2(x^2 + 1) = (m^2y^2)$$

Differentiate again with respect to x

$$2y'y''(x^2 + 1) + 2xy'^2 = 2m^2yy'$$

⇒ $$y'[(x^2 + 1)y'' + xy'] = m^2yy'$$

⇒ $$(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$$ **Hence Proved.**

AI Q. 24. If $y^x = e^{y-x}$, prove that :

$$\frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$$

Sol. We have, $y^x = e^{y-x}$

On taking log on both sides,

$$x \log y = (y - x)\log e$$

⇒ $$x \log y = y - x \quad \text{...(i)}$$

$$x \log y + x = y$$

⇒ $$x(\log y + 1) = y$$

$$x = \frac{y}{1 + \log y}$$

Differentiating w.r.t. y on both sides,

$$\frac{dx}{dy} = \frac{(1 + \log y) \times 1 - y\left(0 + \frac{1}{y}\right)}{(1 + \log y)^2}$$

⇒ $$\frac{dx}{dy} = \frac{1 + \log y - 1}{(1 + \log y)^2} = \frac{\log y}{(1 + \log y)^2}$$

⇒ $$\frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$$

TOPIC-3

L'Hospital's Rule, Rolle's Theorem and Lagrange's Mean Value Theorem

Revision Notes

- **Indeterminate Forms :**

  If direct substitution of $x = a$ while evaluating $\lim\limits_{x\to a} f(x)$ leads to one of the following forms

  $$\frac{0}{0};\ \frac{\infty}{\infty};\ \infty-\infty;\ 0\times\infty;\ 1^{\infty},0^{0}\ \text{and}\ \infty^{0}$$

  Then, it is called indeterminate form. These limits be calculated by using L' Hospital's rule.

Scan to know more about this topic

L Hospital rule

- **L' Hospital's Rule :**

  If $f(x)$ and $g(x)$ are two functions of x such that

  (i) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$

  (ii) both are continuous at $x = a$

  (iii) both are differentiable at $x = a$

  (iv) $f'(x)$ and $g'(x)$ are defined at the point $x = a$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$.

  Above rule is also applicable, if $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$.

  If $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x)$, $g'(x)$ satisfy all the condition embedded in L' Hospital's rule.

  We can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = \lim\limits_{x\to a}\frac{f''(x)}{g''(x)}$

- **Rolle's Theorem :**

  Statement : Let f be real valued function defined on the closed interval $[a, b]$ such that,

  (i) It is continuous on the closed interval $[a, b]$.

  (ii) It is differentiable on the open interval (a, b).

  (iii) $f(a) = f(b)$.

  Then, there exists a real number $c \in (a, b)$ such that $f'(c) = 0$.

Scan to know more about this topic

Rolle's and Lagrange's mean value theorem

- **Lagrange's Mean Value Theorem**

  Statement : Let $f(x)$ be a function defined on $[a, b]$, such that

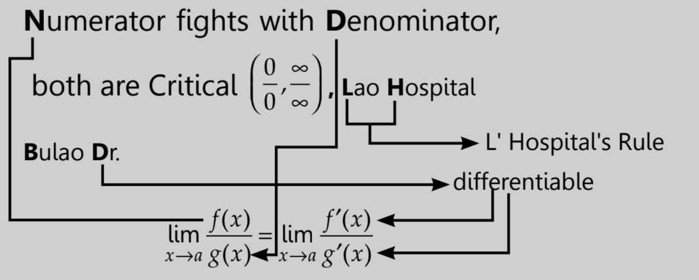
  (i) It is continuous on $[a, b]$.

  (ii) It is differentiable on (a, b).

  Then, there exists a real number $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Mnemonics

L' Hospital's Rule for Evaluating Limits


Numerator fights with **D**enominator,
both are Critical $\left(\frac{0}{0},\frac{\infty}{\infty}\right)$, **L**ao **H**ospital
Bulao **D**r.
L' Hospital's Rule
differentiable
$\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

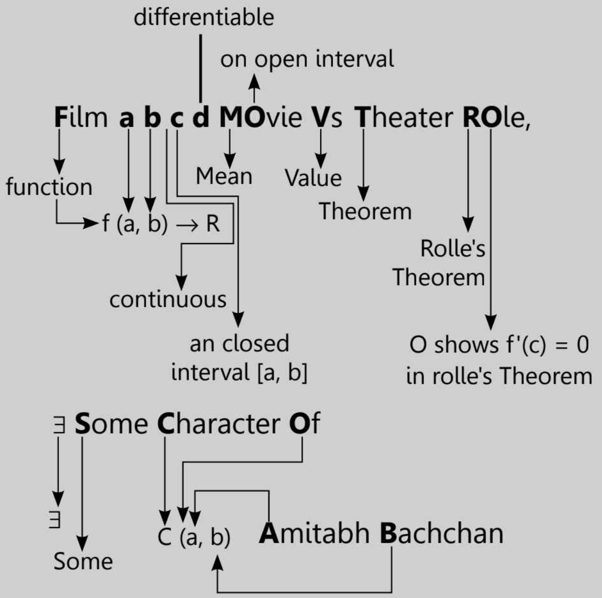
Interpretation :

if $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ takes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form

then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

where $f'(x)\frac{df(x)}{dx}$ and $g'(x)\frac{dg(x)}{dx}$

Mean Value Theorem and Rolle's Theorem


differentiable
on open interval
Film a b c d MOvie Vs Theater ROle,
function
f (a, b) → R
Mean
Value
Theorem
Rolle's Theorem
continuous
an closed interval [a, b]
O shows f'(c) = 0 in rolle's Theorem
∃ Some Character Of
∃
Some
C (a, b)
Amitabh Bachchan

Interpretation :

Mean Value Theorem-

If f:[*a*, *b*] → *R* Continuous on [*a*, *b*] and differential on (*a*, *b*), then ∃ some *c* in (*a*, *b* such that-

$$f'c = \frac{f(b)-f(a)}{b-a}$$

Interpretation :

If *f*:[*a*, *b*] → *R* Continuous on [*a*, *b*] and differentiable on [*a*, *b*], and *f*(*a*) = *f*(*b*) then ∃ some *c* in (*a*, *b*) s.t. *f*'(*c*) = 0

1 Mark Questions

Q 1. The value of c in Lagrange's mean value theorem for the function $f(x) = x^2 + x + 1, x \in [0, 4]$ is :

(a) 3 **(b)** 2 **(c)** 1 **(d)** $\frac{3}{2}$

[ISC Specimen Paper, 2021]

Sol. Correct Option : (b)

Explanation :

Given $f(x) = x^2 + x + 1$

$\Rightarrow$ $f'(x) = 2x + 1$

$\therefore$ $f'(c) = \frac{f(4)-f(0)}{4-0}$

$$2c + 1 = \frac{21-1}{4}$$

$\Rightarrow$ $2c = 4$

$\Rightarrow$ $c = 2$

Q. 2. Which of the following is not a necessary condition for Cauchy's Mean Value Theorem?

(a) The functions, $f(x)$ and $g(x)$ be continuous in $[a, b]$

(b) The derivation of $g'(x)$ be equal to 0

(c) The functions $f(x)$ and $g(x)$ be derivable in (a, b)

(d) There exists a value $c \in (a, b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

Sol. Correct Option : (b)

Explanation : Cauchy's Mean Value theorem is given by, $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$, where $f(x)$ and $g(x)$ be two functions which are derivable in $[a, b]$ and $g'(x) \neq 0$ for any value of x in $[a, b]$ and where $c \in (a, b)$.

AI **Q. 3. Find the value of c which satisfies the Mean Value Theorem for the given function, $f(x)= x^2 + 2x + 1$ on [1, 2].**

(a) $\frac{7}{2}$ **(b)** $-\frac{7}{2}$

(c) $\frac{13}{2}$ **(d)** $-\frac{13}{2}$

Sol. Correct Option : (b)

Explanation: Given function is, $f(x)= x^2 + 2x + 1$.

According to Mean Value Theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$f'(c)=2c+2$

$$2c+2 = \frac{(1+2+1)-(4+4+1)}{2-1} = \frac{4-9}{1} = -5$$

$2c = -7$

$c = -\frac{7}{2}$

Q. 4. What is the value of the given limit, $\lim_{x\to 0} \frac{2}{x}$?

(a) 2 **(b)** 0 **(c)** $\frac{1}{2}$ **(d)** $\frac{2}{3}$

Sol. Correct Option : (a)

Explanation: Given: $\lim_{x\to 0} \frac{2}{x}$

Using L'Hospital's Rule, by differentiating both the numerator and denominator with respect to x,

$$\lim_{x\to 0} \frac{2}{1} = 2$$

Q. 5. Find $\lim\limits_{x\to0}\dfrac{3e^x-2e^{2x}-e^{3x}}{e^x+e^{2x}-2e^{3x}}$

Sol. Form is $\dfrac{0}{0}$

Applying L hospitals rule we have

$$\lim_{x\to0}\frac{3e^x-4e^{2x}-3e^{3x}}{e^x+2e^{2x}-6e^{3x}}=\frac{3-4-3}{1+2-6}=\frac{-4}{-3}=\frac{4}{3}$$

AI Q. 6. Find relation between a and b such that the following limit is got after a single application of L hospitals Rule $\lim\limits_{x\to0}\dfrac{ae^x+be^{2x}}{be^x+ae^{2x}}$

Sol. Given differentiation is applied once we get

$ae^0+be^0=0=a+b$ (numerator → zero)

$be^0+ae^0=0=a+b$ (denominator → zero)

Thus the relation between (a, b) and is

$a+b=0$

or, $a=-b$.

2 Marks Questions

Q 1. Using L'Hospital's Rule, evaluate : $\lim\limits_{x\to0}\dfrac{8^x-4^x}{4x}$

[ISC Specimen Paper, 2021] [ISC, 2019]

Sol. $\lim\limits_{x\to0}\dfrac{8^x-4^x}{4x}$ $\left[\dfrac{0}{0}\text{form}\right]$

Applying L' Hospital's Rule,

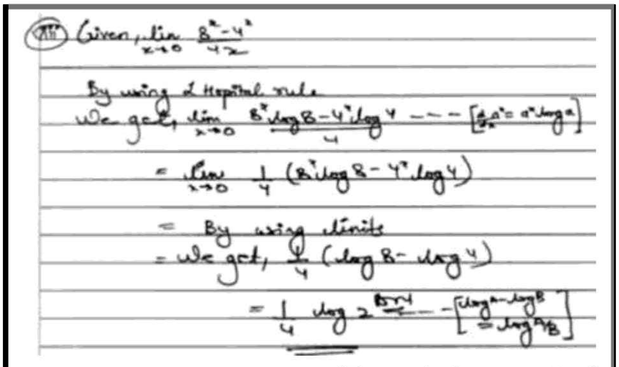
$$=\lim_{x\to0}\frac{8^x\ln8-4^x\ln4}{4}$$

$$=\frac{1}{4}(\ln8-\ln4)$$

$$=\frac{1}{4}\ln\frac{8}{4}=\frac{1}{4}\ln2$$

[ISC Marking Scheme, 2019]

OR



(xii) Given, $\lim\limits_{x\to0}\dfrac{8^x-4^x}{4x}$

By using L Hospital rule

We get, $\lim\limits_{x\to0}\dfrac{8^x\log8-4^x\log4}{4}$ --- $\left[\frac{d}{dx}a^x=a^x\log a\right]$

$=\lim\limits_{x\to0}\frac{1}{4}(8^x\log8-4^x\log4)$

= By using limits

= We get, $\frac{1}{4}(\log8-\log4)$

$=\frac{1}{4}\log2^{\,8\div4}$ --- $\left[\begin{array}{l}\log a-\log B\\=\log a/B\end{array}\right]$

[Topper's Answer, 2019]

AI Q. 2. Using L' Hospital's rule, evaluate :

$\lim\limits_{x\to\frac{\pi}{2}}\left(x\tan x-\dfrac{\pi}{2}\sec x\right)$ **[ISC, 2017]**

Sol. $\lim\limits_{x\to\frac{\pi}{2}}\left(x\tan x-\dfrac{\pi}{2}\sec x\right)$ [∞ – ∞ form]

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{x\sin x}{\cos x}-\frac{\pi/2}{\cos x}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{x\sin x-\pi/2}{\cos x}\right)\qquad\left[\frac{0}{0}\text{ form}\right]$$

Applying L' Hospital Rule, we get

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{x\cos x+\sin x}{-\sin x}\right)$$

$$=\frac{1}{-1}=-1$$ [applying limit]

[ISC Marking Scheme, 2017]

Examiner's Comment

- This point of the question was well attempted by most candidates.

Answering Tip

- Do adequate practice of calculating limits of Indeterminate form $\left(\frac{0}{0}\text{ or }\frac{\infty}{\infty}\right)$ by application of L' Hospital Rule.

Q. 3. Evaluate : $\lim\limits_{x\to\pi/2}\left[x\tan x-\dfrac{\pi}{2}\sec x\right]$ **[ISC, 2010]**

Sol. $\lim\limits_{x\to\frac{\pi}{2}}\left[x\tan x-\dfrac{\pi}{2}\sec x\right]$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{x\sin x-\frac{\pi}{2}}{\cos x}\right]\qquad\left(\frac{0}{0}\text{ form}\right)$$

By L' Hospital's Rule,

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{x\cos x+\sin x}{-\sin x}\right]=-1$$

Q. 4. Evaluate the following limit : $\lim\limits_{x\to\pi/4}\dfrac{1-\tan x}{\cos2x}$

Sol. $\lim\limits_{x\to\pi/4}\dfrac{1-\tan x}{\cos2x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

Using L' Hospital's rule,

$$\lim_{x\to\frac{\pi}{4}}\frac{-\sec^2x}{-2\sin2x}=\frac{-2}{-2\cdot1}=1$$

4 Marks Questions

Q 1. Verify Rolle's theorem for the function, $f(x) = -1 + \cos x$ in the interval $[0, 2\pi]$

[ISC, 2020]

Sol. $f(x) = -1 + \cos x\ [0, 2\pi]$

Since $\cos x$ is everywhere continuous and differential in the given interval

$\therefore$ $f(x) = -1 + \cos x$ is continuous in $[0, 2\pi]$

and $f(x) = -1 + \cos x$ is differentiable $(0, 2\pi)$

$$f(0) = -1 + \cos 0$$
$$= -1 + 1 = 0$$
$$f(2p) = -1 + \cos 2\pi$$
$$= -1 + 1 = 0$$
$$f(0) = f(2\pi)$$
$$f'(x) = -\sin x$$
$$f'(c) = -\sin c$$
$$0 = -\sin c$$
$$\sin c = 0$$

$c = n\pi$ i.e. π when $n = 1$

$\therefore$ $0 < c < 2\pi$

Hence Rolles theorem is verified.

Examiner's Comment

- While this question was attempted correctly by majority of the candidates, a few candidates could not write the correct intervals.

Answering Tip

- Understand the Mean Value Theorem in detail including difference between the closed and open intervals with their significance by sketching the graph of the same.

Q. 2 Using L' Hospital's rule, evaluate :

$$\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$$

[ISC, 2020]

Sol.
$$\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$$

$$\lim_{x\to 0}\frac{\frac{d}{dx}\left[xe^x - \log(1+x)\right]}{\frac{d}{dx}x^2}$$

$$\lim_{x\to 0}\frac{xe^x + e^x - \frac{1}{1+x}(1+0)}{2x}$$

$$\lim_{x\to 0}\frac{xe^x + e^x - \frac{1}{1+x}}{2x}$$

$$\lim_{x\to 0}\frac{\frac{d}{dx}\left(xe^x + e^x - \frac{1}{1+x}\right)}{\frac{d}{dx}(2x)}$$

$$\lim_{x\to 0}\frac{xe^x + e^x + e^x + \frac{1}{(1+x)^2}}{2}$$

$$\lim_{x\to 0}\frac{xe^x + 2e^x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{0\times e^0 + 2\times e0 + \frac{1}{(1+x)^2}}{2} = \frac{0+2+1}{2}$$

$$\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2} = \frac{3}{2}$$

Examiner's Comment

- Mistakes were made while converting in the $\frac{0}{0}$ form, also applied L' Hospital's Rule without observing the indefinite form $\frac{0}{0}$.

Answering Tip

- Application of L' Hospital's Rule for calculating Limits of Indeterminate $\left(\frac{0}{0}\text{ or }\frac{\infty}{\infty}\right)$ Forms should be learn.

AI **Q. 3. Verify the Lagrange's mean value theorem for the function : $f(x) = x + \frac{1}{x}$ in the interval [1, 3]**

[ISC, 2019]

Sol. $f(x) = x + \frac{1}{x}$ is continuous in the closed interval

$1 \le x \le 3$ *i.e* [1, 3]

$f'(x) = 1 - \frac{1}{x^2}$ is existing in the open interval

$1 < x < 3.$ *i.e* (1, 3)

The conditions of Lagrange's Mean Value Theorem are satisfied.

$$f(1) = 1 + 1 = 2, f(3) = 3 + \frac{1}{3} = \frac{10}{3}$$

To verify further, need to show that there exists a 'c' $\in$ (1, 3) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$1 - \frac{1}{x^2} = \frac{\frac{10}{2} - 2}{3 - 1} = \frac{4}{3}\times\frac{1}{2} = \frac{2}{3}$$

$$1 - \frac{1}{x^2} = \frac{2}{3}$$

$$\frac{1}{x^2} = 1 - \frac{2}{3} = \frac{1}{3}$$

$x = \pm\sqrt{3} = +1.73$ lies in (1, 3)

[ISC Marking Scheme, 2019]

Examiner's Comment

- Several candidates made errors while applying properties of mean value theorem due to lack of understanding of open and closed intervals. A few candidates made calculation errors.

Answering Tip

- Understand the concept of open and close intervals and theirs significance with the help of different examples.

Q. 4. Verify Rolle's Theorem for the following function :

$f(x) : e^{-x} \sin x$ on $[0, x]$ **[ISC, 2018]**

Sol. $f(x) = e^{-x} \sin$ is continuous in $[0, \pi]$

$f'(x) = e^{-x} (\cos x - \sin x)$ exists in $(0, \pi)$

$f(0) = f(\pi) = 0$

Rolle's theorem conditions are satisfied

$\therefore$ There exists at least one value of $x = c\ [c \in (0, \pi)]$

Such that $f'(c) = 0$

$\therefore \quad f'(c) = e^{-c} (\cos c - \sin c) = 0$

$\Rightarrow \quad \cos c - \sin c = 0$

$\Rightarrow \quad \tan c = 1$

$\Rightarrow \quad c = \dfrac{\pi}{4} \in (0, \pi)$

[ISC Marking Scheme, 2018]

AI Q. 5. Verify Rolle's Theorem for the following function :

$f(x) = e^x \sin x \ \forall\ x \in [0, \pi]$

[ISC Specimen Paper, 2018]

Sol. $f(x) = e^x \sin x \ \forall\ x \in [0, \pi]$

Here, e^x and $\sin x$ are continuous in $[0, \pi]$ and differentiable in $(0, \pi)$.

Also, $f(0) = e^0 \sin 0 = 0$

and $f(\pi) = e^{\pi} \sin \pi = 0$

$\Rightarrow \quad f(0) = f(\pi)$

All conditions of Rolle's theorem are satisfied.

$\therefore$ There exists $c \in (0, \pi)$ such that

$f'(c) = 0$

Now, $f'(x) = e^x \cos x + e^x \sin x$

$= e^x (\cos x + \sin x)$

$f'(c) = 0$

$\Rightarrow \quad e^c(\cos c + \sin c) = 0$

$\Rightarrow \quad \sin c + \cos c = 0 \qquad$ (as $e^c \in 0$)

$\Rightarrow \quad \tan c = -1$

$\Rightarrow \quad c = \dfrac{3\pi}{4} \in \left(0, \dfrac{\pi}{4}\right)$

Hence, Rolle's theorem is verified.

Q. 6. Verify Lagrange's mean value theorem for the function : $f(x) = x(1 - \log x)$ and find the value of 'c' in the interval [1, 2]. **[ISC, 2018]**

Sol. (i) $f(x) = x(1 - \log x)$ is continuous function in the closed interval [1, 2].

(ii) $f'(x) = x\left(\dfrac{-1}{x}\right) + (1 - \log x) = -\log x$

$\therefore$ $f'(x)$ existing in open interval (1, 2)

$\because$ All the conditions of Lagrange's mean value theorem are satisfied, there exists at least one value 'c' such that :

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Here, $f(a) = f(1) = 1(1 - \log 1) = 1$

and $f(b) = f(2) = 2(1 - \log 2)$

$\therefore \quad f'(c) = \dfrac{2(1 - \log 2) - 1}{2 - 1}$

$\Rightarrow \quad -\log c = 2 - 2\log 2 - 1$

$\Rightarrow \quad \log c = 2\log 2 - 1$

$\Rightarrow \quad \log c = 2\log 2 - \log e$

$\Rightarrow \quad \log c = \log\left(\dfrac{2^2}{e}\right)$

$\Rightarrow \quad \log c = \log\left(\dfrac{4}{e}\right)$

$\Rightarrow \quad c = \dfrac{4}{e} \in (1, 2)$

Examiner's Comment

- Many candidates got confused about closed interval and open interval while proving Lagrange's Mean value theorem, some candidates committed errors in solving logarithmic equations as they could not make distinguish between logarithmic function with base '10' and base 'e'.

Answering Tip

- Understand the concept of Lagrange's Mean value theorem and difference between closed and open intervals. Do adequate practice of problems based on Mean Value Theorem.

AI Q. 7. Using L' Hospital's rule, evaluate:

$$\lim_{x \to 0}\left(\frac{1}{x^2} - \frac{\cot x}{x}\right)$$ **[ISC, 2016]**

Sol. Given, $\displaystyle \lim_{x \to 0}\left(\frac{1}{x^2} - \frac{\cot x}{x}\right)$

$$= \lim_{x \to 0}\left(\frac{1}{x^2} - \frac{1}{x \tan x}\right)$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^2 \times \dfrac{\tan x}{x} \times x}$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^3} \qquad \left[\because \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

Differentiating numerator and denominator separately,

$$= \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$$

$$= \lim_{x\to 0} \frac{2\sec x \sec x . \tan x - 0}{6x}$$

$$= \lim_{x\to 0} \frac{4\sec^2 x + \tan^2 x + 2\sec^2 x}{6}$$

$$= \frac{1}{3}$$ **[ISC Marking Scheme, 2016]**

Detailed Answer :

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{\cot x}{x}\right)$$

$$= \lim_{x\to 0} \frac{\sin x - x\cos x}{x^2 \sin x}$$

$$= \lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} \cdot \lim_{x\to 0} \frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} \cdot 1$$

$$\left[\because \lim_{x\to 0} \frac{x}{\sin x} = 1\right]$$

$$= \lim_{x\to 0} \frac{\cos x - (\cos x - x\sin x)}{3x^2}$$

$$\left[\frac{0}{0}\text{form, applying L' Hospital's rule}\right]$$

$$= \lim_{x\to 0} \frac{x\sin x}{3x^2}$$

$$= \lim_{x\to 0} \frac{\sin x}{3x}$$

$$\left[\frac{0}{0}\text{form, applying L' Hospital's rule}\right]$$

$$= \lim_{x\to 0} \frac{\cos x}{3} = \frac{1}{3}$$

Q. 8. Using L' Hospital's rule, evaluate : $\lim_{x\to 0} \frac{x - \sin x}{x^2 \sin x}$

[ISC, 2015]

Sol. $\lim_{x\to 0} \frac{x - \sin x}{x^2 \sin x}$

$$= \lim_{x\to 0} \frac{x - \sin x}{x^3} \qquad \left[\because \lim_{x\to 0} \frac{x}{\sin x} = 1\right]$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2}$$

$$= x \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- Some candidates wrote incorrect differentiation of numerator and denominator. They wrote differentiation of '1 – cos *x*' as '1 – sin *x*' which was not correct. Some wrote the differentiation of sin x as (– cos *x*).

Answering Tip

- Indeterminate forms *i.e.* $\frac{0}{0}, \frac{\infty}{\infty}$ etc. should be learnt properly and revision of differentiation chapter must be done for practice. L' Hospital's rule must be learnt giving appropriate conditions to deal with different indeterminate forms.

Detailed Answer :

Given, $\lim_{x\to 0} \frac{x - \sin x}{x^2 \sin x}$ $\left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to 0} \frac{x - \sin x}{x^3} \cdot \frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{x - \sin x}{x^3} \cdot 1 \qquad \left(\because \lim_{x\to 0} \frac{x}{\sin x} = 1\right)$$

Applying L' Hospital's Rule, we have

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

Applying L' Hospital's Rule, we have

$$= \lim_{x\to 0} \frac{\sin x}{6x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \frac{1}{6} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{6}$$

Q. 9.Using L' Hospital's rule, evaluate:

$$\lim_{x\to 0}(1 + \sin x)^{\cot x}$$ **[ISC, 2014]**

Sol. Let $A = \lim_{x\to 0}(1 + \sin x)^{\cot x}$

$$\Rightarrow \quad \log A = \lim_{x\to 0} \cot x \log(1 + \sin x)$$

$$= \lim_{x\to 0} \frac{\log(1 + \sin x)}{\tan x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$\Rightarrow \log A \quad = \lim_{x\to 0}\left(\frac{\frac{\cos x}{1 + \sin x}}{\sec^2 x}\right) = 1$$

$\Rightarrow \quad A = e$ or $\log A = 1$ (Applying limit)

[ISC Marking Scheme, 2014]

Examiner's Comment

- Conversion into $\frac{0}{0}$ (indeterminate) form for the given limit directly or after taking logs on both sides was not done by some candidates. Some differentiated the fraction by $\frac{u}{v}$ rule instead of separately differentiating the numerator and denominator as is required in L' Hospital's Rule.

Answering Tip

- L′ Hospitals rule and when and how it can be applied needs to be demonstrated in a way that the students understand. Indeterminate forms $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$ etc. need to be explained.

AI **Q. 10. Using L′ Hospital's rule, evaluate:**

$$\lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2x}{x - \sin x}\right) \qquad \text{[ISC, 2013]}$$

Sol. $$\lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2x}{x - \sin x}\right) \qquad \left(\frac{0}{0}\text{ form}\right)$$

Using L′ Hospital's Rule,

$$\Rightarrow \lim_{x\to 0}\frac{e^x + e^{-x} - 2}{1 - \cos x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

Again using L′ Hospital's Rule,

$$\Rightarrow \lim_{x\to 0}\frac{e^x - e^{-x}}{\sin x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

Once again using L′ Hospital's Rule,

$$= \lim_{x\to 0}\frac{e^x + e^{-x}}{\cos x} = \frac{1+1}{1} = 2$$

Q. 11. Evaluate : $\lim\limits_{x\to\pi/2}(\cos x\cdot\log\tan x)$ **[ISC, 2012]**

Sol. $$\lim_{x\to\pi/2}(\cos x\cdot\log\tan x) \qquad \textbf{(0·∞ form)}$$

$$= \lim_{x\to\pi/2}\frac{\log\tan x}{\sec x} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

Using L′ Hospital's Rule,

$$= \lim_{x\to\pi/2}\frac{\frac{1}{\tan x}\times\sec^2 x}{\sec x\cdot\tan x}$$

$$= \lim_{x\to\pi/2}\frac{\sec^2 x}{\tan^2 x\cdot\sec x}$$

$$= \lim_{x\to\pi/2}\frac{\sec x}{\tan^2 x}$$

$$= \lim_{x\to\pi/2}\frac{1}{\cos x}\times\frac{\cos^2 x}{\sin^2 x}$$

$$= \lim_{x\to\pi/2}\frac{\cos x}{\sin^2 x}$$

$$= \frac{\cos\pi/2}{\sin^2\pi/2} = \frac{0}{1} = 0$$

Q. 12. Evaluate : $\lim\limits_{y\to 0}\dfrac{y - \tan^{-1}y}{y - \sin y}$ **[ISC, 2011]**

Sol. Given: $$\lim_{y\to 0}\frac{y - \tan^{-1}y}{y - \sin y} \qquad \left[\frac{0}{0}\text{ form}\right]$$

On applying L′ Hospital's rule, we get

$$= \lim_{y\to 0}\frac{1 - \frac{1}{1+y^2}}{1 - \cos y} \qquad \left[\frac{0}{0}\text{ form}\right]$$

Again, applying L′ Hospital's rule, we get

$$= \lim_{y\to 0}\frac{0 - (-1)(1+y^2)^{-2}\cdot 2y}{\sin y}$$

$$= \lim_{y\to 0}\frac{2y}{(1+y^2)^2\cdot\sin y}$$

$$= \lim_{y\to 0}\frac{2}{(1+y^2)^2}\cdot\left(\frac{y}{\sin y}\right)$$

$$= \frac{2}{(1+0)^2}\cdot 1 = 2 \qquad \left(\because \lim_{y\to 0}\frac{\sin y}{y} = 1\right)$$

Q. 13. Examine the validity and condition of Lagrange's mean value theorem for the function $f(x) = x(x-1)(x-2)$ for every $x \in \left[0, \frac{1}{2}\right]$ **[ISC, 2000]**

Sol. Given function is

$$f(x) = x(x-1)(x-2) = (x^2 - x)(x-2)$$
$$= x^3 - 2x^2 - x^2 + 2x$$
$$= x^3 - 3x^2 + 2x \qquad \text{...(i)}$$

So, $f(x)$ is continuous for all $x \in \left[0, \frac{1}{2}\right]$.

Now, $f'(x) = 3x^2 - 6x + 2$ [differentiate eq. (i)]

Since, $f(x)$ is a polynomial, so $f(x)$ is differentiable for all $x \in \left(0, \frac{1}{2}\right)$.

As $f(x)$ satisfies both the conditions of Lagrange's mean value theorem, so there exist $c \in \left(0, \frac{1}{2}\right)$, such that

$$f'(c) = \frac{f(1/2) - f(0)}{1/2 - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{(1/2)^3 - 3(1/2)^2 + 2(1/2)}{1/2}$$

$$\Rightarrow 3c^2 - 6c + 2 = 2\left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{1}{4} - \frac{3}{2} + 2$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{1 - 6 + 8}{4}$$

$$\Rightarrow 12c^2 - 24c + 8 = 1 - 6 + 8$$

$$\Rightarrow 12c^2 - 24c + 8 - 3 = 0$$

$$\Rightarrow 12c^2 - 24c + 5 = 0$$

$$\therefore \quad c = \frac{24 \pm \sqrt{(-24)^2 - 4\times 12\times 5}}{2\times 12}$$

$$\left[\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$= \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{24 \pm \sqrt{336}}{24}$$

$$= \frac{24 \pm 4\sqrt{21}}{24} = \frac{6 \pm \sqrt{21}}{6}$$

$$= \frac{6 + 4.6}{6} \text{ or } \frac{6 - 4.6}{6}$$

$$= \frac{10.6}{6} \text{ or } \frac{1.4}{6} = 1.7 \text{ or } 0.23$$

$$\because \quad 1.7 \notin \left[0, \frac{1}{2}\right] \text{ and } 0.23 \in \left[0, \frac{1}{2}\right]$$

Hence, 1.7 is not possible but 0.23 is acceptable.

6 Marks Questions

AI Q. 1. Verify the conditions of Rolle's theorem for the following function:

$f(x) = \log(x^2 + 2) - \log 3$ on $[-1, 1]$

Find a point in the interval, where the tangent to the curve is parallel to x-axis. [ISC, 2016]

Sol. $f(x) = \log(x^2 + 2) - \log 3$ is continuous $[-1, 1]$

$$f'(x) = \frac{2x}{(x^2+2)}$$

$f'(x)$ exists in $(-1, 1)$, $f(-1) = f(1) = 0$

All the conditions of Rolle's theorem are satisfied then there exists 'c' in $(-1, 1)$ such that $f'(c) = 0$

$$\frac{2c}{(1+c^2)} = 0$$

$c = 0$ lies between -1 and 1. Hence, Rolle's theorem is verified. The point where the tangent is parallel to x axis is $\left(0, \log\frac{2}{3}\right)$.

[ISC Marking Scheme, 2016]

Examiner's Comment

- Several candidates made errors because of confusion between the closed and open interval for continuity and differentiability of the function. Most of the candidates did not find the point of contact of the tangent and the curve.

Answering Tip

- The different criteria for Rolle's and Lagrange's Mean Value theorems need to be understood and differentiated. Difference between 'closed' and 'open' intervals and its significance should be practiced by sketching the curve, etc.

Q. 2. Verify Lagrange's Mean Value Theorem for the following function:

$f(x) = 2 \sin x + \sin 2x$ on $[0, \pi]$. [ISC, 2015]

Sol. $f(x) = (2 \sin x + \sin 2x)$ is continuous in $[0, \pi]$

$f'(x)$ exists in $(0, \pi)$

$f'(x) = 2 \cos x + 2 \cos 2x$

$f(0) = 0, f(\pi) = 0$

All the conditions of Lagrange's Mean Value theorem are satisfied, there exist 'c' in $(0, \pi)$

$$\text{such that } f'(c) = \frac{f(b) - f(a)}{b - a}$$

$2\cos c + 2 \cos 2c = 0$

$2 \cos^2 c + \cos c - 1 = 0$ $[\because \cos 2x = 2 \cos^2 x - 1]$

$\cos c = -1, 1/2 \Rightarrow \cos c = \cos \pi$

$\Rightarrow$ $c = \pi$ (not possible)

or $\cos c = \cos\frac{\pi}{3}$

$\Rightarrow$ $c = \frac{\pi}{3} \in (0, \pi)$

$c = \pi/3$ which lies between 0 to π, hence, lagrange's mean value theorem is verified.

[ISC Marking Scheme, 2015]

Examiner's Comment

- Many candidates failed to state all the criteria for application of Lagrange's theorem correctly. The concept of closed or open was not clear to many candidates. A few candidates got confused with Rolle's theorem' condition $f(a) = f(h)$.

Q. 3. Verify Rolle's Theorem for the function

$f(x) = e^x(\sin x - \cos x)$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. [ISC, 2014]

Sol. $f(x) = e^x(\sin x - \cos x)$ is continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

$f(x)$ is differentiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$$f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x)$$
$$= 2e^x \sin x$$

$f'(x)$ exists in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$f(\pi/4) = f(5\pi/4) = 0$

There exist 'c' in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f'(c) = 0$

$2e^c \sin c = 0, c = 0, \pi$

$\Rightarrow c = \pi$ lies between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. Hence, Rolle's Theorem is verified. **[ISC Marking Scheme, 2014]**

Examiner's Comment

- Several candidates failed to state all the criteria for application of Rolle's Mean Value Theorem correctly. Some failed to differentiate the function correctly. The concept of closed and open interval was not clear to many.

Answering Tip

- Enumerate the criteria for Mean Value Theorems correctly. Firstly, functions have to be continuous in the closed interval; secondly, the derivative needs to exist in the open interval and thirdly, $f(a) = f(b)$ needs to be established. Copious practice is a must to understand derivative operations and that while solving $f'(x) = 0$ the value of x (say c) needs to be found in the open interval.

Q. 4. Verify Lagrange's Mean Value Theorem for the function $f(x) = \sqrt{x^2 - x}$ in the interval [1, 4].

[ISC, 2013]

Sol. Given, $f(x)=\sqrt{x^2-x}, x\in[1,4]$

Since, x^2-x is exists in [1, 4]

and $\sqrt{x^2-x}$ is exists in [1, 4]

⇒ $f(x)$ is continuous in [1, 4]

Differentiating the given function w.r.t. x

$$f'(x)=\frac{1}{2}(x^2-x)^{-1/2}\cdot(2x-1)$$

$$=\frac{2x-1}{2\sqrt{x^2-x}}$$

which exists $\forall\ x\in R$

∴ $f(x)$ is differentiable in (1, 4).

Thus, both the conditions of Lagrange's mean value theorem is satisfied therefore, ∃ c in (1, 4)

Such that $$f'(c)=\frac{f(4)-f(1)}{4-1}$$

$$\frac{2c-1}{2\sqrt{c^2-c}}=\frac{\sqrt{12}}{3}$$

$$3(2c-1)=2\sqrt{c^2-c}\cdot\sqrt{12}$$

Squaring both the sides,

$$9(4c^2-4c+1)=48(c^2-c)$$

$$3(4c^2-4c+1)=16(c^2-c)$$

$$\Rightarrow\quad 12c^2-12c+3=16c^2-16c$$

$$\Rightarrow\quad 4c^2-4c-3=0$$

$$\Rightarrow\quad (2c-3)(2c+1)=0$$

$$c=\frac{3}{2},\frac{-1}{2}$$

Thus, there exist $c=\frac{3}{2}\in(1,4)$

such that $$f'\left(\frac{3}{2}\right)=\frac{f(4)-f(1)}{4-1}$$

Hence, Lagrange's Mean Value theorem is verified and $c=\frac{3}{2}$ **Hence Proved.**

AI Q. 5. Verify Rolle's theorem for the function :

$$f(x)=\log\left\{\frac{x^2+ab}{(a+b)x}\right\}$$

in the interval $[a, b]$ where $0\notin[a, b]$. [ISC, 2012]

Sol. Given, $$f(x)=\log\left\{\frac{x^2+ab}{(a+b)x}\right\}$$

$$=\log(x^2+ab)-\log(a+b)-\log x \quad \ldots\text{(i)}$$

Since, $a>0$ and $\log x$ is continuous for all $x>0$, therefore, $f(x)$ is continuous in $[a, b]$.

$f(x)$ is differentiable in (a, b), and

$$f(a)=\log\left(\frac{a^2+ab}{(a+b)a}\right)=\log 1=0$$

$$f(b)=\log\left(\frac{b^2+ab}{(a+b)b}\right)=\log 1=0$$

$$\Rightarrow\quad f(a)=f(b)$$

Thus, all the three conditions of Rolle's theorem are satisfied. Therefore, there exists at least one real number c in (a, b) such that $f'(c)=0$.

Differentiating (i) w.r.t. x, we get

$$f'(x)=\frac{1}{x^2+ab}\cdot 2x-0-\frac{1}{x}$$

$$=\frac{2x}{x^2+ab}-\frac{1}{x}$$

$$=\frac{x^2-ab}{x(x^2+ab)}$$

Now, $f'(c)=0$

$$\Rightarrow\quad \frac{c^2-ab}{c(c^2+ab)}=0$$

$$\Rightarrow\quad c^2-ab=0\Rightarrow c^2=ab$$

$$\Rightarrow\quad c=\pm\sqrt{ab}$$

$$\because\quad 0\notin[a,b]\ \Rightarrow\ -ab\notin[a,b]$$

$$\therefore\quad c=\sqrt{ab}$$

So, there exists a real number

$c=\sqrt{ab}$ in (a, b) such that

$f'(c)=0$

Hence, Rolle's theorem is verified and $c=\sqrt{ab}$.

Q. 6. Verify Lagrange's mean value theorem for the function $f(x)=\sin x-\sin 2x$ in the interval $[0, \pi]$. [ISC, 2011]

Sol. The function $f(x)=\sin x-\sin 2x$ is differentiable for all values of x, and hence differentiable in $[0, \pi]$

Also, $f'(x)=\cos x-2\cos 2x$

⇒ By Lagrange's mean value theorem, we have

$$f'(c)=\frac{f(\pi)-f(0)}{\pi-0}$$

$$\Rightarrow\ \cos c-2\cos 2c=\frac{0-0}{\pi}=0$$

$$\Rightarrow\quad \cos c-2(2\cos^2 c-1)=0$$

$$\Rightarrow\quad 4\cos^2 c-\cos c-2=0\ [\because \cos 2x=2\cos^2 x-1]$$

$$\Rightarrow\quad \cos c=\frac{1\pm\sqrt{1+32}}{8}=\frac{1\pm\sqrt{33}}{8}$$

$$\text{or}\quad \cos c=\frac{1\pm 5.7}{8}$$

$$\Rightarrow\quad \cos c=\frac{1+5.7}{8}\ \text{or}\ \cos c=\frac{1-5.7}{8}$$

$$\Rightarrow\quad \cos c=0.84 \text{ or } \cos c=-0.59$$

$$\Rightarrow\quad c=33^\circ \text{ or } c=126.2^\circ \text{ (approx.)}$$

Both values of c lie in the interval $[0, \pi]$

Hence, the Mean value theorem is verified.

AI Q. 7. Using Rolle's theorem, find a point on the curve $y=\sin x+\cos x-1,\ x\in\left[0,\frac{\pi}{2}\right]$ where the tangent is parallel to x-axis. [ISC, 2010]

Sol. Given, $y = f(x) = \sin x + \cos x - 1$

Since, cosine and sine function is continuous for all values, so $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$ and differentiable in $\left(0, \frac{\pi}{2}\right)$.

Now, $f'(x) = \cos x - \sin x$

$\therefore$ $f'(x)$ also exists in the open interval $\left(0, \frac{\pi}{2}\right)$

$$f(0) = \sin 0 + \cos 0 - 1 = 0$$

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} - 1 = 0$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

Thus, all the three conditions of Rolle's theorem are satisfied.

$$\therefore \quad f'(x) = 0, f'(x) = \cos x - \sin x$$

$$\Rightarrow \cos x - \sin x = 0$$

$$\Rightarrow \quad \tan x = 1 = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad x = \frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right]$$

When $x = \frac{\pi}{4}$, then

$$y = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - 1$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

Hence, the required point $\left(\frac{\pi}{4}, \sqrt{2} - 1\right)$.

Q. 8. It is given that Rolle's theorem holds good for the function:

$$f(x) = x^3 + ax^2 + bx, x \in [1, 2] \text{ at the point } x = \frac{4}{3}.$$

Find the values of *a* and *b*.

Sol. Given, $f(x) = x^3 + ax^2 + bx, x \in [1, 2], f$ is continuous for $x \in [1, 2]$

$f'(x) = 3x^2 + 2ax + b$ exists in $x \in [1, 2]$

$$f(1) = a + b + 1$$

$$f(2) = 8 + 4a + 2b$$

Now, $f(1) = f(2)$

$$a + b + 1 = 8 + 4a + 2b$$

$$\Rightarrow \quad 3a + b + 7 = 0$$

$$\Rightarrow \quad 3a + b = -7 \qquad \ldots\text{(i)}$$

$$f'\left(\frac{4}{3}\right) = 3 \times \frac{16}{9} + 2a \times \frac{4}{3} + b = 0$$

$$\Rightarrow 8a + 3b = -16 \qquad \ldots\text{(ii)}$$

Solving (i) and (ii), we get,

$$\therefore \quad a = -5, b = 8$$

AI **Q. 9. Use Lagrange's mean value theorem to determine a point *P* on the curve $y = \sqrt{x - 2}$ defined in the interval [2, 3], where the tangent is parallel to the chord joining the end points on the curve.** **[ISC, 2008]**

Sol. Let $f(x) = y = \sqrt{x - 2}$ exists $\forall\ x \in [2, 3]$

$\therefore$ y is continuous in [2, 3]

$$f'(x) = \frac{dy}{dx} = \frac{1}{2\sqrt{x - 2}}$$

which exists for all values of $x \in (2, 3)$

Thus, the function is differentiable in (2, 3) and therefore, the condition of Lagrange's mean value theorem is satisfied.

$$\therefore \quad f'(c) = \frac{f(3) - f(2)}{3 - 2}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{c - 2}} = \frac{1 - 0}{1}$$

$$\Rightarrow \quad \sqrt{c - 2} = \frac{1}{2}$$

$$\Rightarrow \quad c - 2 = \frac{1}{4}$$

$$\Rightarrow \quad c = \frac{1}{4} + 2 = \frac{9}{4} \in (2, 3)$$

When $x = \frac{9}{4}, y = \frac{1}{2}$,

Also, from $y = \sqrt{x - 2}$, we have $y = 0$ when $x = 2$ and $y = 1$ when $x = 3$.

Hence, by theorem the tangent at $P\left(\frac{9}{4}, \frac{1}{2}\right)$ to the curve $y = \sqrt{x - 2}$ is parallel to the chord joining the points (2, 0) and (3, 1).

Q. 10. Verify Rolle's theorem for the function $f(x) = e^{2x}(\sin 2x - \cos 2x)$ defined in the interval $\left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$.

Sol. Given, $f(x) = e^{2x}(\sin 2x - \cos 2x)$

Since, e^{2x}, $\sin 2x$ and $\cos 2x$ are continuous in $\left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$ and differentiable in $\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$.

So, $f(x)$ is continuous in $\left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$ and differentiable in $\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$.

Now,
$$f\left(\frac{\pi}{8}\right) = e^{2 \times \frac{\pi}{8}}\left[\sin\frac{2\pi}{8} - \cos\frac{2\pi}{8}\right]$$

$$= e^{\frac{\pi}{4}}\left[\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right]$$

$$= e^{\frac{\pi}{4}}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] = 0$$

and
$$f\left(\frac{5\pi}{8}\right) = e^{2 \times \frac{5\pi}{8}}\left[\sin 2 \times \frac{5\pi}{8} - \cos 2 \times \frac{5\pi}{8}\right]$$

$$= e^{\frac{5\pi}{4}}\left[\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right]$$

$$= 0$$

$$\left[\because \sin\frac{5\pi}{4} = \cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}\right]$$

$\therefore \qquad f\left(\frac{\pi}{8}\right) = f\left(\frac{5\pi}{4}\right)$

As all the three conditions of Rolle's theorem are satisfied.

So, there exist a point $c \in \left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$ such that,

$f'(c) = 0$

Now, $\qquad f'(x) = e^{2x}(2\cos 2x + 2\sin 2x) + 2e^{2x}(\sin 2x - \cos 2x)$

Consider, $f'(x) = 0 \ \therefore \ \sin 2c = 0$

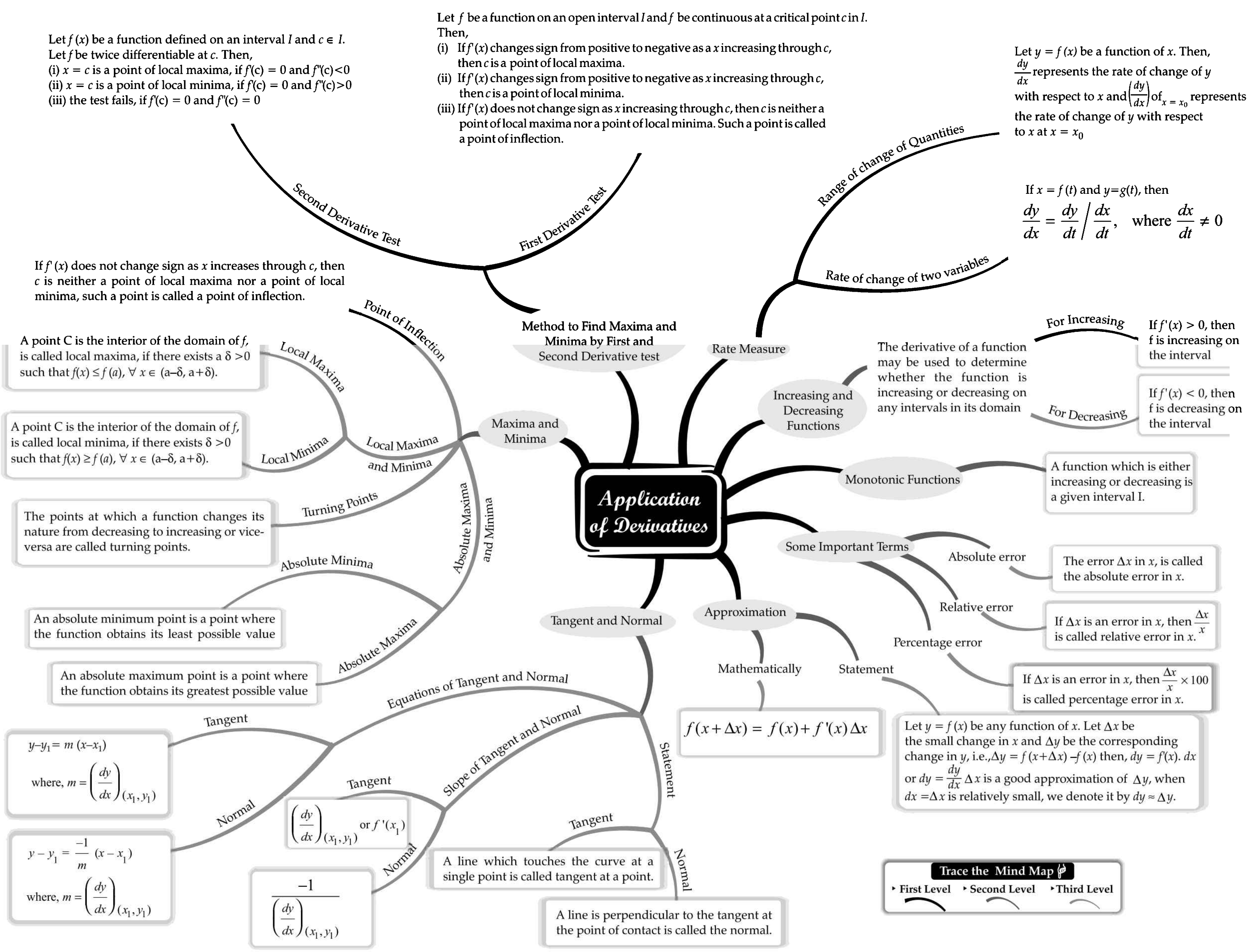
[since $e^{2x} \neq 0$ for any finite value of x]

$\Rightarrow \qquad 2c = \pi \qquad [\because \sin \pi = 0]$

$\therefore \qquad c = \frac{\pi}{2} \in \left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$

Hence, Rolle's theorem is verified.

◆



Application of Derivatives

Rate Measure

Range of change of Quantities

Let $y = f(x)$ be a function of x. Then, $\frac{dy}{dx}$ represents the rate of change of y with respect to x and $\left(\frac{dy}{dx}\right)$ of$_{x = x_0}$ represents the rate of change of y with respect to x at $x = x_0$

Rate of change of two variables

If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \quad \text{where } \frac{dx}{dt} \neq 0$$

Increasing and Decreasing Functions

The derivative of a function may be used to determine whether the function is increasing or decreasing on any intervals in its domain

For Increasing

If $f'(x) > 0$, then f is increasing on the interval

For Decreasing

If $f'(x) < 0$, then f is decreasing on the interval

Monotonic Functions

A function which is either increasing or decreasing is a given interval I.

Some Important Terms

Absolute error

The error Δx in x, is called the absolute error in x.

Relative error

If Δx is an error in x, then $\frac{\Delta x}{x}$ is called relative error in x.

Percentage error

If Δx is an error in x, then $\frac{\Delta x}{x} \times 100$ is called percentage error in x.

Approximation

Mathematically

$$f(x + \Delta x) = f(x) + f'(x)\Delta x$$

Statement

Let $y = f(x)$ be any function of x. Let Δx be the small change in x and Δy be the corresponding change in y, i.e.,$\Delta y = f(x + \Delta x) - f(x)$ then, $dy = f'(x).dx$ or $dy = \frac{dy}{dx}\Delta x$ is a good approximation of Δy, when $dx = \Delta x$ is relatively small, we denote it by $dy \approx \Delta y$.

Tangent and Normal

Statement

Tangent

A line which touches the curve at a single point is called tangent at a point.

Normal

A line is perpendicular to the tangent at the point of contact is called the normal.

Slope of Tangent and Normal

Tangent

$\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ or $f'(x_1)$

Normal

$$\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

Equations of Tangent and Normal

Tangent

$y - y_1 = m(x - x_1)$

where, $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

Normal

$y - y_1 = \frac{-1}{m}(x - x_1)$

where, $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

Maxima and Minima

Local Maxima and Minima

Local Maxima

A point C is the interior of the domain of f, is called local maxima, if there exists a $\delta > 0$ such that $f(x) \leq f(a)$, $\forall\, x \in (a - \delta, a + \delta)$.

Local Minima

A point C is the interior of the domain of f, is called local minima, if there exists $\delta > 0$ such that $f(x) \geq f(a)$, $\forall\, x \in (a - \delta, a + \delta)$.

Turning Points

The points at which a function changes its nature from decreasing to increasing or vice-versa are called turning points.

Absolute Maxima and Minima

Absolute Minima

An absolute minimum point is a point where the function obtains its least possible value

Absolute Maxima

An absolute maximum point is a point where the function obtains its greatest possible value

Point of Inflection

If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima, such a point is called a point of inflection.

Method to Find Maxima and Minima by First and Second Derivative test

First Derivative Test

Let f be a function on an open interval I and f be continuous at a critical point c in I. Then,

(i) If $f'(x)$ changes sign from positive to negative as a x increasing through c, then c is a point of local maxima.

(ii) If $f'(x)$ changes sign from positive to negative as x increasing through c, then c is a point of local minima.

(iii) If $f'(x)$ does not change sign as x increasing through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called a point of inflection.

Second Derivative Test

Let $f(x)$ be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then,

(i) $x = c$ is a point of local maxima, if $f'(c) = 0$ and $f''(c) < 0$

(ii) $x = c$ is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$

(iii) the test fails, if $f'(c) = 0$ and $f''(c) = 0$

Trace the Mind Map

- First Level
- Second Level
- Third Level

CHAPTER 6

APPLICATIONS OF DERIVATIVES

SYLLABUS

Applications of derivatives : rate of change of bodies, increasing / decreasing functions, tangents and normals, use of derivatives in approximation, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations.

- ***Equation of Tangent and Normal***
- ***Approximation***
- ***Rate Measure***
- ***Increasing and decreasing functions***
- ***Maxima and minima***
  - ***Stationary/turning points***
  - ***Absolute maxima/minima***
  - ***Local maxima/minima***
  - ***First derivatives test and second derivatives test***
  - ***Point of inflexion***
  - ***Application problems based on maxima and minima***

Trend Analysis

List of Concepts	2016	2017	2018	2019	2020
Rate of change of Bodies			1 Q (4 marks)	1 Q (4 marks)	1 Q (2 marks)
Tangents and Normals			1 Q (4 marks)		1 Q (4 marks)
Approximate Values, Differentials and Errors			1 Q (2 marks)		
Increasing / Decreasing Functions				1 Q (2 marks)	
Maxima and Minima	1 Q (6 marks)	1 Q (6 marks)	1 Q (6 marks)	1 Q (6 marks)	1 Q (6 marks)

TOPIC - 1
Rate of Change of Bodies .... P. 148
TOPIC - 2
Tangents and Normals .... P. 152
TOPIC - 3
Approximate Values, Differentials and Errors .... P. 160
TOPIC - 4
Increasing / Decreasing Functions .... P. 162
TOPIC - 5
Maxima and Minima .... P. 165

TOPIC-1
Rate of Change of Bodies

Revision Notes

- **Interpretation of $\frac{dy}{dx}$ as a rate measure :**

  If two variables x and y are varying with respect to another variables say t, *i.e.*, if $x = f(t)$ and $y = g(t)$, then by the Chain Rule, we have

  $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},\ \frac{dx}{dt} \neq 0$$

  Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x both with respect to t.

Scan to know more about this topic

Rate of change of bodies

Also, if y is a function of x and they are related as $y = f(x)$ then, $f'(\alpha)$, *i.e.*, represents the rate of change of y with respect to x at the instant when $x = \alpha$.

1 Mark Questions

Q. 1. The rate of change of area of circle with respect to its radius r at $r = 3$ cm is :

(a) 6π (b) 8π (c) 12π (d) 3π

[ISC Specimen Paper, 2021]

Sol. Correct option : (a)

Explanation :

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\therefore \left[\frac{dA}{dr}\right]_{r=3} = 2 \times \pi \times 3 = 6\pi$$

AI Q. 2. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is :

(a) 10 cm^2/s (b) $\sqrt{3}$ cm^2/s

(c) $10\sqrt{3}$ cm^2/s (d) $\frac{10}{3}$ cm^2/s

Sol. Correct option : (c)

Explanation : Let the side of an equilateral triangle be x cm.

$\therefore$ Area of equilateral triangle, $A = \frac{\sqrt{3}}{4}x^2$ ...(i)

Also, $\frac{dx}{dt} = 2$ cm/s

On differentiating equation (i) with respect to t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2x \times \frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{4} \times 2 \times 10 \times 2$$

$$\left[\because x = 10 \text{ and } \frac{dx}{dt} = 2\right]$$

$$= 10\sqrt{3} \text{ cm}^2/\text{s}$$

Q. 3. If a circular plate is heated uniformly, its area expands $3c$ times as fast as its radius, then the value of c when the radius is 6 units, is

(a) 4π (b) 2π (c) 6π (d) 8π

Sol. Correct option : (a)

Explanation : $A = \pi r^2$

$$\frac{dA}{dt} = \pi \times 2r\frac{dr}{dt}$$

$$3c \times \frac{dr}{dt} = \pi \times 2r\frac{dr}{dt}$$

$$3c = \pi \times 2 \times 6$$

$$c = 4\pi$$

AI Q. 4. The total cost associated with provision of free mid-day meals to x students of a school in primary classes is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$. If the marginal cost is given by rate of change $\frac{dC}{dx}$ of total cost, write the marginal cost of food for 300 students.

Sol. We have, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$

$$\Rightarrow \frac{dC}{dx} = C'(x) = 0.015x^2 - 0.04x + 30 + 0$$

At $x = 300$,

$$C'(300) = 0.015(300)^2 - 0.04(300) + 30$$

$$= 1368$$

So, the marginal cost of food for 300 students is 1,368.

Q. 5. The total expenditure (in ₹) required for providing the cheap edition of a book for poor and deserving students is given by $R(x) = 3x^2 + 36x$, where x is the number of set of books. If the marginal expenditure is defined as $\frac{dR}{dx}$, write the marginal expenditure required for 1200 such sets.

Sol. Here, $R(x) = 3x^2 + 36x$, where x is the number of sets of book

$$\therefore \frac{dR}{dx} = 6x + 36$$

$$\left(\frac{dR}{dx}\right)_{x=1200} = 6(1200) + 36 = 7{,}236$$

So, the marginal cost of books for 1200 such sets = ₹ 7,236

Q. 6. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $p(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added.

Sol. Here, pollution content is given by $p(x) = 0.005x^3 + 0.02x^2 + 30x$, where x is the number of diesel vehicles

$$\therefore \frac{dp}{dx} = 0.015x^2 + 0.04x + 30$$

$$\left(\frac{dp}{dx}\right)_{x=3} = 0.015(3)^2 + 0.04(3) + 30$$

$$= 30.255$$

AI Q. 7. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in ₹) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$.

Sol. Total revenue is given by $R(x) = 3x^2 + 36x + 5$

Marginal revenue $= \frac{dR}{dx} = 6x + 36$

At $x = 5$, $\left(\frac{dR}{dx}\right)_{x=5} = 6 \times 5 + 36 = 66$

Q. 8. The contentment obtained after eating x-units of a new dish at a trial function is given by the function $f(x) = x^3 + 6x^2 + 5x + 3$. If the marginal contentment is defined as the rate of change of $f(x)$ with respect to the number of units consumed at an instant, then find the marginal contentment when three units of dish are consumed.

Sol. $f(x) = x^3 + 6x^2 + 5x + 3$

$$\frac{df(x)}{dx} = 3x^2 + 12x + 5$$

Marginal contentment

$$\left[\frac{df(x)}{dx}\right]_{x=3} = 3 \times (3)^2 + 12 \times 3 + 5$$

$$= 27 + 36 + 5 = 68 \text{ units}$$

2 Marks Questions

Q. 1. The edge of a variable cube is increasing at the rate of 10 cm/sec. How fast is the volume of the cube increasing when the edge is 5 cm long ?

[ISC, 2020]

Sol. Let the edge of cube be x cm

$$\therefore \quad \frac{dx}{dt} = 10 \text{ cm/sec (given)}$$

Volume of cube $(V) = x^3$

$$\frac{dV}{dt} = \frac{d}{dt}x^3 = 3x^2\frac{dx}{dt}$$

$$\left(\frac{dV}{dt}\right)_{x=5} = 3(5)^2(10)$$

$$= 750 \text{ cm}^3\text{/sec.}$$

$\therefore$ Volume of cube increasing by 750 cm^3/sec when edge of cube is 5 cm.

[ISC Marking Scheme, 2020]

Examiner's Comment

- Few students are not able to form this mathematically.

Answering Tip

- Practice adequate problems from this topic as it is important from exam point of view.

AI **Q. 2. The length x of a rectangle is decreasing at the rate of 5 cm/min and the width y increasing at the rate of 4 cm/min, find the rate of change its area when $x = 5$ cm and $y = 8$ cm.**

Sol. Let A denote the area of rectangle at instant t.

$$\therefore \quad A = xy \quad \text{(area of rectangle),}$$

$$\Rightarrow \quad \left.\begin{aligned}\frac{dx}{dt} &= -5 \text{ cm/min}\\ \frac{dy}{dt} &= 4 \text{ cm/min}\end{aligned}\right\} \text{(given)}$$

$$\therefore \quad \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{x=5,\,y=8} = 5 \times 4 + 8 \times (-5) \text{ cm}^2\text{/min}$$

$$\Rightarrow \quad \frac{dA}{dt} = (20 - 40) \text{ cm}^2\text{/min}$$

$$\Rightarrow \quad \frac{dA}{dt} = -20 \text{ cm}^2\text{/min}$$

Here, (–) ve sign shows that area is decreasing at the rate of 20 cm^2/min.

Q. 3. The volume of a cube is increasing at the rate of 9 cm^3/sec. How fast is the surface area increasing when the length of an edge is 10 cm.

Sol. Let x denote the edge of cube, V denote the volume and s denotes the surface area of cube at instant t.

$$\frac{dV}{dt} = 9 \text{ cm}^3\text{/sec}$$

$$\because \quad V = x^3$$

$$\therefore \quad \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{9}{3x^2} = \frac{dx}{dt}$$

Now, $$s = 6x^2$$

$$\frac{ds}{dt} = 12x \times \frac{dx}{dt}$$

$$\Rightarrow \quad \left(\frac{ds}{dt}\right)_{x=10} = 12 \times 10 \times \frac{9}{3 \times 10 \times 10}$$

$$= \frac{12 \times 9}{3 \times 10} = \frac{36}{10}$$

$$= 3.6 \text{ cm}^2\text{/sec}$$

AI **Q. 4. Find the point on the curve $y^2 = 8x + 3$ for which the y-coordinate changes 8 times more than coordinate of x.**

Sol. $$y^2 = 8x + 3 \quad \text{...(i) (given)}$$

$$\therefore \quad 2y\frac{dy}{dt} = 8\frac{dx}{dt}$$

$$\frac{dy}{dt} = 8\frac{dx}{dt} \quad \text{...(ii) (given)}$$

$$\therefore \quad 2y \cdot 8\frac{dx}{dt} = 8\frac{dx}{dt} \quad \text{(using (i) \& (ii))}$$

$$\Rightarrow \quad y = \frac{8}{16} = \frac{1}{2}$$

For $$y = \frac{1}{2}$$

From (i), $$\left(\frac{1}{2}\right)^2 = 8x + 3$$

$$\frac{1}{4} - 3 = 8x$$

$$\Rightarrow \quad x = \frac{1-12}{4} \times \frac{1}{8} = -\frac{11}{32}$$

Hence, required point is $\left(-\frac{11}{32}, \frac{1}{2}\right)$.

Q. 5. The radius r of a right circular cone is decreasing at the rate of 3 cm/minute and the height h is increasing at the rate of 2 cm/minute. When r = 9 cm and h = 6 cm, find the rate of change of its volume.

Sol.

$$\frac{dr}{dt} = -3 \text{ cm/min},$$

$$\frac{dh}{dt} = 2 \text{ cm/min}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + 2hr\frac{dr}{dt}\right]$$

$$\left(\frac{dV}{dt}\right)_{\text{at } r=9,\, h=6} = \frac{\pi}{3}[81 \times 2 - 2 \times 6 \times 9 \times 3]$$

$$\left(\frac{dV}{dt}\right)_{\text{at } r=9,\, h=6} = -54\,\pi \text{ cm}^3\text{/min}$$

or Volume is decreasing at the rate 54π cm^3/min.

Q. 6. The volume of a sphere is increasing at the rate of 8 cm^3/sec. Find the rate at which its surface area is increasing when the radius of the sphere is 12 cm.

Sol. $\frac{dV}{dt}$ = 8 cm^3/s, where V is the volume of sphere

i.e., $V = \frac{4}{3}\pi r^3$

or $$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

or $$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times \frac{dV}{dt}$$

Since, $$S = 4\pi r^2$$

or $$\frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi r \times \frac{1}{4\pi r^2} \times 8$$

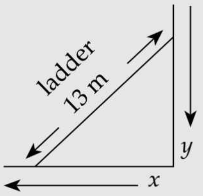
or $$= \frac{2 \times 8}{12} = \frac{4}{3} \text{ cm}^2\text{/s}$$ ½

4 Marks Questions

AI Q. 1. A 13 m long ladder is leaning against a wall, touching the wall at a certain height from the ground level. The bottom of the ladder is pulled away from the wall, along the ground, at the rate of 2 m/s. How fast is the height of the wall decreasing when the foot of the ladder is 5 m away from the wall?

[ISC, 2019]

Sol.



$$x^2 + y^2 = 169$$

$$\text{at } x = 5$$

$$y^2 = 169 - 25 = 144$$

$$y = 12$$

$$\frac{dx}{dt} = 2 \text{ m/s}$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

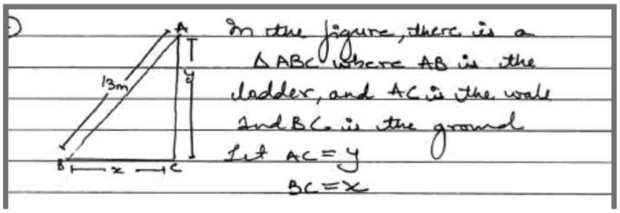
$$x = 5,\ y = 12 \text{ and } \frac{dx}{dt} = 2\text{m/s}$$

$$5 \times 2 + 12.\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{5}{6} \text{ m/s}$$

[ISC Marking Scheme, 2019]

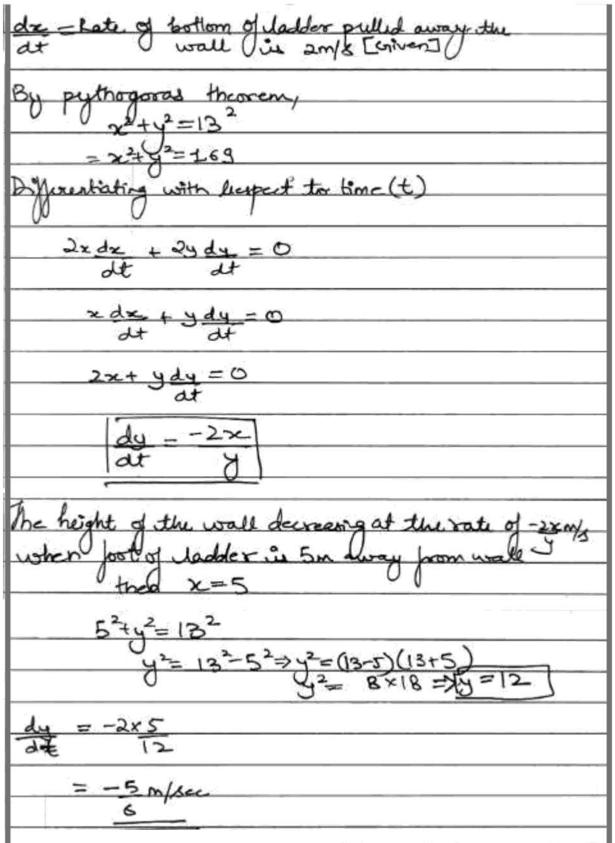
OR



In the figure, there is a $\triangle ABC$ where AB is the ladder, and AC is the wall and BC is the ground

Let AC = y

BC = x



$\frac{dx}{dt}$ = Rate of bottom of ladder pulled away the wall is 2m/s [Given]

By pythogoras theorem,

$x^2+y^2=13^2$

$= x^2+y^2=169$

Differentiating with respect to time (t)

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$2x + y\frac{dy}{dt} = 0$

$\boxed{\frac{dy}{dt} = \frac{-2x}{y}}$

The height of the wall decreasing at the rate of $-\frac{2x}{y}$ m/s when foot of ladder is 5m away from wall then $x=5$

$5^2+y^2=13^2$

$y^2 = 13^2-5^2 \Rightarrow y^2=(13-5)(13+5)$

$y^2 = 8\times18 \Rightarrow \boxed{y=12}$

$\frac{dy}{dt} = \frac{-2\times5}{12}$

$= \frac{-5}{6}$ m/sec

[Topper's Answer, 2019]

Examiner's Comment

- Several candidates could not form the function. Many candidates made errors while differentiating the function with reference to time and ended up differentiating with respect to '*x*' instead of time '*t*'.

Answering Tip

- Understand the concept of application of derivatives under different topics.

Q. 2. Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of 2 cm^2/sec in the surface, through a tiny hole at the vertex of bottom. When the slant height of water level is 4 cm, find the rate of decrease of the slant height of the water. [ISC, 2018]

Sol. Let r be the radius and l be the slant height of water surface

$$\sin\frac{\pi}{4} = \frac{r}{l}$$

or, $$r = l.\frac{1}{\sqrt{2}}$$

Curved surface, $$s = \pi rl$$

$\therefore$ $$s = \pi.l.\frac{1}{\sqrt{2}}.l$$

$$= \frac{1}{\sqrt{2}}\pi l^2$$

How, $$\frac{ds}{dt} = \frac{1}{\sqrt{2}}\pi.2l\frac{dl}{dt}$$

Given that $$\frac{ds}{dt} = -2\ cm^2/s$$

$\therefore$ $$-2 = \frac{\pi}{\sqrt{2}}.2l\frac{dl}{dt}$$

$\Rightarrow$ $$\frac{dl}{dt} = \frac{-\sqrt{2}}{\pi l}$$

At $l = 4$ cm

$$\left(\frac{dl}{dt}\right)_{l=4} = \frac{-\sqrt{2}}{\pi.4} = \frac{-\sqrt{2}}{4.\pi}\ \text{cm/s}$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Many candidates considered volume as function instead of surface area. Majority of candidates did not apply proper sign through it was specified in the question that find the rate of decrease of slant height of the water.

Answering Tip

- The difference between rate of increase and rate of decrease need to be understood carefully. Practice a number of problems based on application of derivatives.

Q. 3. A circular disc of radius 3 cm. is heated. Due to expansion its radius increases at the rate of 0.05 cm/s. Find the rate of which its area increases when the radius is 3.2 cm. [ISC Specimen Paper, 2018]

Sol. Let $r = 3$ cm be the radius of the disc

Given, $$\frac{dr}{dt} = 0.05\ \text{cm/s}$$

Let area, $$A = \pi r^2$$

$\therefore$ $$\frac{dA}{dt} = \frac{d}{dr}(\pi r^2)\frac{dr}{dt}$$

$$= 2\pi r.(0.05)$$

$$= \frac{22}{7} \times r \times (0.1)$$

Now, $$\left(\frac{dA}{dt}\right)_{r=3.2\text{ cm}} = \frac{22}{7} \times 3.2 \times 0.1$$

$$= \frac{22}{7} \times 0.32$$

$$= 1.005\ cm^2/s$$

$\therefore$ Area increases at the rate of 1.005 cm^2/s

TOPIC-2

Tangents and Normals

Revision Notes

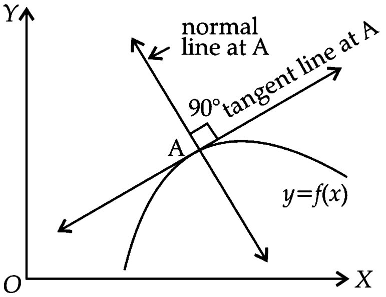
➤ **Slope or gradient of a line :**

If a line makes an angle θ with the positive direction of X-axis in anti-clockwise direction, then $\tan\theta$ is called the slope or gradient of the line. [Note that θ is taken as positive or negative accordingly as it is measured in anti-clockwise (*i.e.*, from positive direction of X-axis to the positive direction of Y-axis) or clockwise direction respectively.]

Scan to know more about this topic

Tangent and normal

- **Pictorial representation of tangent and normal :**



- **Facts about the slope of a line :**

  (a) If a line is parallel to x-axis (or perpendicular to y-axis), then its slope is 0 (zero) or $\frac{dy}{dx}=0$.

  (b) If a line is parallel to y-axis (or perpendicular to x-axis), then its slope is $\frac{1}{0}$, *i.e.*, not defined or $\frac{dx}{dy}=0$.

  (c) If two lines are perpendicular then product of their slopes equals – 1, *i.e.*, $m_1 \times m_2 = -1$. Whereas, for two parallel lines, their slopes are equal, *i.e.*, $m_1 = m_2$. (Here in both the cases m_1 and m_2 represent the slopes of respective lines).

- **Equation of tangent at (x_1, y_1) :**

  $(y - y_1) = m_T(x - x_1)$, where m_T is the slope of tangent such that $m_T = \left[\frac{dy}{dx}\right]_{at(x_1,y_1)}$

- **Equation of Normal at (x_1, y_1) :**

  $(y - y_1) = m_N(x - x_1)$, where m_N is the slope of such that $m_N = \dfrac{-1}{\left[\frac{dy}{dx}\right]_{at(x_1,y_1)}}$

  Note that $m_T \times m_N = -1$, which is obvious because tangent and normal are perpendicular to each other. In other words, the tangent and normal lines are inclined at right angle on each other.

- **Acute angle between the two curves whose slopes m_1 and m_2 are known :**

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 \cdot m_2}\right| \quad \Rightarrow \quad \theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1 \cdot m_2}\right|$$

  It is absolutely sufficient to find one angle (generally the acute angle) between the two curves. Other angle between the curves is given by $\pi - \theta$.

  Note that if the curves cut orthogonally (*i.e.*, they cut each other at right angles), then it means $m_1 \times m_2 = -1$, where m_1 and m_2 represent the slopes of the tangent of curves at the intersection point.

- **Finding the slope of a line $ax + by + c = 0$:**

  Step 1 : Express the given line in the standard slope-intercept form $y = mx + c$, *i.e.*, $y = \left(-\frac{a}{b}\right)x - \frac{c}{b}$.

  Step 2 : By comparing to the standard form $y = mx + c$, we can conclude $-\frac{a}{b}$ is the slope of given line $ax + by + c = 0$.

1 Mark Questions

Q.1 Identify from the given options the slope of the normal to the curve $x^2 + 3y + y^2 = 5$ at (1, 1) :

(a) $-\frac{2}{5}$ **(b)** $\frac{5}{2}$ **(c)** $\frac{2}{5}$ **(d)** $-\frac{5}{2}$

Sol. Correct option : (b)

Explanation :

Given equations of curve :

$$x^2 + 3y + y^2 = 5$$

$$2x + (3 + 2y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{3+2y}$$

$$\text{Slope of the normal} = \frac{-1}{\left(\frac{-2x}{3+2y}\right)}$$

$$\therefore \text{Slope of the normal at (1, 1)} = \left[\frac{3+2y}{2x}\right]_{(1,1)}$$

$$= \frac{5}{2}$$

AI Q. 2. The curve $y = x^{1/5}$ has at (0, 0)

(a) a vertical tangent (parallel to y-axis)
(b) a horizontal tangent (parallel to x-axis)
(c) an oblique tangent
(d) no tangent

Sol. Correct option : (a)

Explanation : Given that, $y = x^{1/5}$

On differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5}x^{-4/5}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(0,0)} = \frac{1}{5}\times(0)^{-4/5} = \infty$$

So, the curve $y = x^{1/5}$ has a vertical tangent at (0, 0), which is parallel to y-axis.

Q. 3. If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of a is :

(a) 1 **(b) 0** **(c) –6** **(d) 6**

Sol. Correct option : (d)

Explanation : Given that, $ay + x^2 = 7$ and $x^3 = y$

On differentiating with respect to x in both equations, we get

$$a.\frac{dy}{dx} + 2x = 0 \text{ and } 3x^2 = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a} \text{ and } \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-2}{a} = m_1 \text{ and } \left(\frac{dy}{dx}\right)_{(1,1)} = 3.1 = 3 = m_2$$

Since, the curve cuts orthogonally at (1, 1).

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \left(\frac{-2}{a}\right).3 = -1$$

$$\therefore \quad a = 6$$

Q. 4. Write the equation of tangent drawn to the curve $y = \sin x$ at the point (0, 0).

Sol. $\dfrac{dy}{dx} = \cos x$

Slope of tangent at (0, 0) = $\left[\dfrac{dy}{dx}\right]_{(0,\,0)} = \cos 0° = 1$

Equation of tangent is $y = x$

AI **Q. 5. Find the slope of tangent to the curve $y = 3x^2 - 6$ at the point on it whose x-coordinate is 2.**

Sol. Given, $y = 3x^2 - 6$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 6x$$

Now, slope of tangent

$$= \left(\frac{dy}{dx}\right)_{x=2} = 6(2) = 12$$

Hence, required slope is 12.

2 Marks Questions

Q. 1. Find the slope of tangent and normal to the curve $x^2 + 2y + y^2 = 0$ at (– 1, 2).

Sol. Given, $x^2 + 2y + y^2 = 0$

$$2x + 2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2+2y) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2(1+y)} = -\frac{x}{1+y}$$

Slope of tangent at (– 1, 2)

$$\left[\frac{dy}{dx}\right]_{(-1,\,2)} = \frac{-(-1)}{1+2} = \frac{1}{3}$$

Slope of normal at (– 1, 2)

$$\frac{-1}{\left[\frac{dy}{dx}\right]_{(-1,\,2)}} = -\frac{3}{1} = -3.$$

Q. 2. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

Sol. We have, $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$

Since, both the curves touch each other at (1, 2), *i.e.*, curves are passing through (1, 2).

$$\therefore \quad 2y\cdot\frac{dy}{dx} = 4$$

$$\text{and } 2x + 2y\frac{dy}{dx} = 6$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{2y}$$

$$\text{and} \quad \frac{dy}{dx} = \frac{6-2x}{2y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{4}{4} = 1$$

$$\text{and} \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{6-2\cdot1}{2\cdot2} = \frac{4}{4} = 1$$

$$\Rightarrow \quad m_1 = 1 \text{ and } m_2 = 1$$

Thus, we see that slope of both the curves are equal to each other, *i.e.*, $m_1 = m_2 = 1$ at the point (1, 2). Hence, both the curves touch each other.

Q. 3. Show that the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ touches the curve $y = b.e^{\frac{-x}{a}}$ at the point where the curve intersects the axis of y.

Sol. We have the equation of line given by $\dfrac{x}{a} + \dfrac{y}{b} = 1$, which touches the curve $y = b.e^{\frac{-x}{a}}$ at the point, where the curve intersects the axis of y, *i.e.*, $x = 0$.

$$\therefore \quad y = b\cdot e^{-0/a} = b \qquad [\because e^0 = 1]$$

So, the point of intersection of the curve with y-axis is (0, b).

Now, slope of the given line at (0, b) is given by

$$\frac{1}{a}\cdot 1 + \frac{1}{b}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{a}\cdot b = \frac{-b}{a} = m_1 \qquad \text{[say]}$$

Also, the slope of the curve at (0, b) is

$$\frac{dy}{dx} = b\cdot e^{-x/a}\cdot\frac{-1}{a}$$

$$\frac{dy}{dx} = \frac{-b}{a}e^{-x/a}$$

$$\left(\frac{dy}{dx}\right)_{(0,b)} = \frac{-b}{a}e^{-0} = \frac{-b}{a} = m_2 \qquad \text{[say]}$$

Since, $\qquad m_1 = m_2 = \dfrac{-b}{a}$

Hence, the line touches the curve at the point, where the curve intersects the axis of y.

4 Marks Questions

Q. 1. The equation of tangent at (2, 3) on the curve $y^2 = px^3 + q$ is $y = 4x - 7$. Find the values of 'p' and 'q'. [ISC, 2020]

Sol. $\qquad y^2 = px^3 + q$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}px^3 + \frac{d}{dx}q$$

$$2y\frac{dy}{dx} = 3px^2 + 0$$

$$\left(\frac{dy}{dx}\right) = \frac{3px^2}{2y}$$

Slope of the line, $\left(\dfrac{dy}{dx}\right)_{(2,3)} = \dfrac{3p(2)^2}{2\times 3} = 2p \ = 2p$

Since the slope of given tangent line is 4.

Hence, $\quad 2p = 4$

or $\qquad p = 2$

Since, the equation of the curve $y^2 = px^3 + q$ is also passing through (2, 3)

$\therefore \qquad (3)^2 = \text{p}(2)^3 + q$

$\qquad 9 = p(8) + q$

$\qquad 9 = 16 + q$

$\therefore \qquad q = -7$

Hence, $\qquad p = 2, q = -7.$

Examiner's Comment

- Most of the students solved this question correctly but few of them made mistake while finding the slope of line at given point.

Answering Tip

- Understand the concept of tangent and normal with respect to application of derivatives and practice as much questions as you can.

Q. 2. Find the points on the curve $y = 4x^3 - 3x + 5$ at which the equation of the tangent is parallel to the x-axis. [ISC, 2018]

Sol. Given curve $y = 4x^3 - 3x + 5 \qquad$ ...(i)

Differentiating eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = 12x^2 - 3$$

$\because$ Tangent is parallel to x-axis

$$\therefore \qquad \frac{dy}{dx} = 0$$

$$\Rightarrow \qquad 12x^2 - 3 = 0 \Rightarrow x = \pm\frac{1}{2}$$

When $x = \dfrac{1}{2}$, $\qquad y = 4.\left(\dfrac{1}{2}\right)^3 - 3\left(\dfrac{1}{2}\right) + 5$

$$= 4\times\frac{1}{8} - \frac{3}{2} + 5$$

$$= \frac{1}{2} - \frac{3}{2} + 5$$

$$= \frac{1-3+10}{2} = 4$$

When $x = \dfrac{-1}{2}$, $\qquad y = 4.\left(\dfrac{-1}{2}\right)^3 - 3\left(\dfrac{-1}{2}\right) + 5$

$$= 4\left(\frac{-1}{8}\right) + \frac{3}{2} + 5$$

$$= \frac{-1}{2} + \frac{3}{2} + 5$$

$$= \frac{-1+3+10}{2} = 6$$

Hence, required points are $\left(\dfrac{1}{2}, 4\right)$ and $\left(\dfrac{-1}{2}, 6\right)$.

[ISC Marking Scheme, 2018]

Examiner's Comment

- This question was solved correctly by most of the candidates. A few candidates made mistakes while finding the point of contact at which the equation of tangent to the curve where tangent is parallel to the x-axis.

Answering Tip

- Understand geometrical interpretation of differentiation with ample examples.

Q. 3. Find the equations of the normals to curve $y = x^3 + 2x + 6$, which are parallel to the line $x + 14y + 4 = 0$. [ISC Specimen Paper, 2018]

Sol. Given curve, $y = x^3 + 2x + 6$

Slope of tangent to the curve $= y' = 3x^2 + 2$

Slope of normal to the curve $= -\dfrac{1}{y'} = \dfrac{-1}{3x^2+2}$

$[\therefore m_T \times m_N = -1]$

Given, that normal is parallel to the line

$$x + 14y + 4 = 0$$

$$\Rightarrow \quad 14y = -x - 4$$

$$\Rightarrow \quad y = \frac{-1}{14}x - \frac{4}{14}$$

$\therefore$ Slope of normal $= \dfrac{-1}{14}$

At point (h, k), slope of normal is $\dfrac{-1}{3h^2+2}$

$$\Rightarrow \quad \frac{-1}{3h^2+2} = \frac{-1}{14}$$

$$\Rightarrow \quad 3h^2 + 2 = 14$$

$$\Rightarrow \quad 3h^2 = 12$$

$$\Rightarrow \quad h^2 = 4 \Rightarrow h = \pm 2$$

When $h = 2, k = 8 + 4 + 6 = 18$

and $h = -2, k = -8 - 4 + 6 = -6$

$\therefore$ Required normal is at point (2, 18) and (– 2, – 6)

$\therefore$ The equations of required normals are

$$y - 18 = \frac{-1}{14}(x-2)$$

or, $\quad x + 14y - 254 = 0$

and $\quad y + 6 = \dfrac{-1}{14}(x+2)$

or, $\quad x + 14y + 86 = 0$

AI **Q. 4. Show that the equation of normal at any point t on the curve $x = 3\cos t - \cos^3 t$ and $y = 3\sin t - \sin^3 t$ is $4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$.**

Sol. $x = 3\cos t - \cos^3 t$

$$\frac{dx}{dt} = -3\sin t + 3\cos^2 t \cdot \sin t$$

$$= -3\sin t(1 - \cos^2 t) = -3\sin^3 t$$

Again, $y = 3\sin t - \sin^3 t$

$$\frac{dy}{dt} = 3\cos t - 3\sin^2 t \cdot \cos t$$

$$\frac{dy}{dt} = 3\cos t(1 - \sin^2 t) = 3\cos^3 t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{\cos^3 t}{\sin^3 t}$$

$\therefore$ Slope of normal $= -\dfrac{dx}{dy} = \dfrac{\sin^3 t}{\cos^3 t}$

$\therefore$ Equation of normal is

$$y - (3\sin t - \sin^3 t) = \frac{\sin^3 t}{\cos^3 t}\{x - (3\cos t - \cos^3 t)\}$$

$$\Rightarrow y\cos^3 t - 3\sin t\cos^3 t + \sin^3 t\cos^3 t$$
$$= x\sin^3 t - 3\cos t\sin^3 t + \sin^3 t\cos^3 t$$

$$\Rightarrow y\cos^3 t - x\sin^3 t = 3\sin t\cos^3 t - 3\cos t\sin^3 t$$

$$\Rightarrow y\cos^3 t - x\sin^3 t = 3\sin t\cos t(\cos^2 t - \sin^2 t)$$

$$\Rightarrow y\cos^3 t - x\sin^3 t = 3\sin t\cos t\cos 2t$$

Multiply both sides by 4, we get

$$4(y\cos^3 t - x\sin^3 t) = 3 \times 4\sin t\cos t\cos 2t$$

$$= 3 \times 2\sin 2t\cos 2t$$

$$= 3\sin 4t$$ **Hence Proved.**

Q. 5. The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of a and b.

Sol. Given, $y^2 = ax^3 + b$

Differentiate with respect to x, we get

$$2y\frac{dy}{dx} = 3ax^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3ax^2}{2y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3ax^2}{\pm 2\sqrt{ax^3+b}} \qquad [\because y^2 = ax^3 + b]$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{(2,3)} = \frac{3a(2)^2}{\pm 2\sqrt{a(2)^3+b}}$$

$$= \frac{12a}{\pm 2\sqrt{8a+b}}$$

$$= \frac{6a}{\pm\sqrt{8a+b}}$$

Since, (2, 3) lies on the curve

$$y^2 = ax^3 + b$$

$$\Rightarrow \quad 9 = 8a + b \qquad \text{...(i)}$$

Also, from equation of tangent

$$y = 4x - 5 \qquad \text{[Given]}$$

Slope of the tangent $= \dfrac{dy}{dx} = 4$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{(2,3)} = \frac{6a}{\pm\sqrt{8a+b}} \text{ becomes}$$

$$4 = \frac{6a}{\pm\sqrt{9}} \qquad \text{\{from (i)\}}$$

$$\therefore \quad 4 = \frac{6a}{\pm 3}$$

$$\therefore \quad 4 = \frac{6a}{3} \text{ or } 4 = \frac{6a}{-3}$$

$$\Rightarrow \quad a = 2 \text{ or } a = -2$$

For $\quad a = 2 \quad 9 = 8(2) + b$

$\Rightarrow \quad b = -7$

and for $a = -2 \quad 9 = 8(-2) + b$

$\Rightarrow \quad b = 25$

$\therefore \quad a = 2$ and $b = -7$

or $\quad a = -2$ and $b = 25$

Q. 6. Find the equations of the tangent and normal to the curve $\dfrac{x^2}{a} - \dfrac{y^2}{b} = 1$ at the point $(\sqrt{2}a, b)$.

Sol. Given, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

On differentiating w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

Slope of tangent at $(\sqrt{2}a, b) = \dfrac{b^2\sqrt{2}a}{a^2b} = \dfrac{\sqrt{2}b}{a}$

Slope of normal at $(\sqrt{2}a, b) = -\dfrac{a}{\sqrt{2}b} \quad [\because m_T . m_N = -1]$

Equation of tangent at $(\sqrt{2}a, b)$ is

$$y - b = \frac{\sqrt{2}b}{a}(x - \sqrt{2}a),$$

i.e., $\sqrt{2}bx - ay = ab$

and equation of normal at $(\sqrt{2}a, b)$ is

$$y - b = -\frac{a}{\sqrt{2}b}(x - \sqrt{2}a)$$

i.e., $ax + \sqrt{2}by = \sqrt{2}(a^2 + b^2)$

Q. 7. For the curve $y = 4x^3 - 2x^5$, find all those points at which the tangent passes through the origin.

Sol. Given, $y = 4x^3 - 2x^5$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 12x^2 - 10x^4$$

Tangent at (x_1, y_1) is given by:

$$y - y_1 = (12x_1^2 - 10x_1^4)(x - x_1)$$

[As, $(y - y_0) = m(x - x_0)$]

Since, it passes through the origin,

$$0 - y_1 = (12x_1^2 - 10x_1^4)(0 - x_1)$$

$$\Rightarrow \quad y_1 = 12x_1^3 - 10x_1^5 \quad \text{...(i)}$$

We known that (x_1, y_1) lies on the curve

$$\Rightarrow \quad y_1 = 4x_1^3 - 2x_1^5 \quad \text{...(ii)}$$

From eq. (i) and eq. (ii),

$$8x_1^3 - 8x_1^5 = 0$$

$$\Rightarrow \quad x_1^3(1 - x_1^2) = 0$$

If $x_1^3 = 0$, then $x_1 = 0$

If $1 - x_1^2 = 0$, then $x_1 = 1$ or $x_1 = -1$

Substituting these values of x_1 in eq. (i), we get

$y_1 = 0$ (when $x_1 = 0$)

$y_1 = 12 - 10 = 2$ (when $x_1 = 1$)

$y_1 = [12(-1) - 10(-1)]$ (when $x_1 = -1$)

$= -2$

So, required points: (0, 0), (1, 2) and (–1, –2).

AI **Q. 8. Find the equations of the tangent and normal to the curve $x = a\sin^3\theta, y = b\cos^3\theta$ at $\theta = \frac{\pi}{4}$.**

Sol. Given, $x = a \sin^3\theta$ and $y = b \cos^3\theta$

$$\therefore \quad \frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\text{and} \quad \frac{dy}{d\theta} = -3b\cos^2\theta\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{b}{a}\cot\theta$$

At $\theta = \frac{\pi}{4}$,

$$\text{Slope : } \left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\frac{b}{a}\cot\frac{\pi}{4} = -\frac{b}{a}$$

When $\theta = \frac{\pi}{4}, x = \frac{a}{2\sqrt{2}}, y = \frac{b}{2\sqrt{2}}$

[as $x = a \sin^3\theta$ and $y = b \cos^3\theta$]

Equation of tangent at $\theta = \frac{\pi}{4}$ is

$$y - \frac{b}{2\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{2\sqrt{2}}\right)$$

$$\Rightarrow \quad \sqrt{2}(bx + ay) = ab$$

$$\text{Slope of normal} = -\left(\frac{1}{-\frac{b}{a}}\right)$$

[$\because m_T \times m_N = -1$]

Equation of normal at $\theta = \frac{\pi}{4}$ is

$$y - \frac{b}{2\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{2\sqrt{2}}\right)$$

$$\Rightarrow \quad 2\sqrt{2}y - b = \frac{a}{b}(2\sqrt{2}x - a)$$

$$\Rightarrow \quad 2\sqrt{2}(by - ax) = b^2 - a^2$$

Q. 9. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Sol. Given, $ay^2 = x^3$

Differentiating w.r.t. 'x', we get

$$2ay\frac{dy}{dx} = 3x^2$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(am^2,\, am^3)} = \frac{3(am^2)^2}{2a(am^3)}$$

$$= \frac{3a^2m^4}{2a^2m^3} = \frac{3}{2}m$$

$$\text{Slope of normal} = -\frac{2}{3m} \quad [\because m_T \times m_N = -1]$$

Eq. of normal at (x_1, y_1) is

$$y - y_1 = -\frac{2}{3m}(x - x_1)$$

$\therefore$ Eq. of normal at (am^2, am^3)

$$y - am^3 = -\frac{2}{3m}(x - am^2)$$

$$\Rightarrow \quad 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow \quad 2x + 3my = 3am^4 + 2am^2$$

$$\Rightarrow \quad 2x + 3my = am^2(3m^2 + 2)$$

AI **Q. 10. Show that the equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$.**

Sol. Given, $y^2 = 4ax$

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

This gives the slope of the line tangent to the parabola at (x_1, y_1)

$\therefore$ Slope at (x_1, y_1), *i.e.*, $m = \frac{2a}{y_1}$

$\therefore$ Required tangent line is $y - y_1 = m(x - x_1)$

$\Rightarrow \qquad y - y_1 = \frac{2a}{y_1}(x - x_1)$

$\Rightarrow \qquad yy_1 - y_1^2 = 2ax - 2ax_1$

But (x_1, y_1) lies on $y^2 = 4ax$

$\therefore \qquad y_1^2 = 4ax_1$

Substituting this value of y_1^2, we get

$yy_1 - 4ax_1 = 2ax - 2ax_1$

$\Rightarrow \qquad yy_1 = 2a(x + x_1)$

This is the required equation.

Q. 11. Find the point on the curve $y = x^3 - 11x + 5$ at which the equation of tangent is $y = x - 11$.

Sol. Slope of the tangent from $y = x - 11$ is 1.

Given, $\qquad y = x^3 - 11x + 5$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 11$$

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 11$$

$$3x_1^2 - 11 = 1$$

$\Rightarrow \qquad x_1 = \pm 2$

When $x_1 = 2, y_1 = 8 - 22 + 5 = -9$

When $x_1 = -2, y_1 = -8 + 22 + 5 = 19$

Since, $(-2, 19)$ does not lie on the tangent $y = x - 11$

$\therefore$ Required point is $(2, -9)$.

AI **Q. 12. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$, which is parallel to the line $4x - 2y + 5 = 0$.**

Sol. Given, $\qquad y = \sqrt{3x - 2}$

Differentiating w.r.t 'x', we get

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x - 2}}$$

$\because 4x - 2y + 5 = 0$ has slope

$\therefore \qquad \frac{dy}{dx} = -\frac{4}{(2)} = 2$

Since, the tangent is parallel to the line

$\therefore \qquad \frac{3}{2\sqrt{3x - 2}} = 2 \qquad \left(\text{As, } \frac{dy}{dx} = 2\right)$

$\Rightarrow \qquad 3 = 4\sqrt{3x - 2}$

$\Rightarrow \qquad \sqrt{3x - 2} = \frac{3}{4}$

On squaring both sides, we get

$\Rightarrow \qquad 3x - 2 = \frac{9}{16}$

$\Rightarrow \qquad x = \frac{41}{48}$

When $\qquad x = \frac{41}{48}, y = \frac{3}{4}$

$\therefore$ The point is $\left(\frac{41}{48}, \frac{3}{4}\right)$

The eqn. of tangent is :

$\Rightarrow \qquad y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$

$\Rightarrow \qquad \frac{4y - 3}{4} = \frac{2(48x - 41)}{48}$

$\Rightarrow \quad 24y - 18 = 48x - 41$

$\Rightarrow \quad 48x - 24y = 23$

$\Rightarrow \quad 48x - 24y - 23 = 0$

6 Marks Questions

AI **Q. 1. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is**

(i) parallel to the line $2x - y + 9 = 0$.

(ii) perpendicular to the line $5y - 15x = 13$.

Sol. Given curve, $y = x^2 - 2x + 7 \qquad$ ...(i)

Slope of tangent to given curve $= \frac{dy}{dx} = 2x - 2$

(i) Tangent parallel to the line $2x - y + 9 = 0$

Let slope of line $2x - y + 9 = 0$ be m_1

$\therefore \qquad m_1 = 2$

Since, tangent to the curve is parallel to line

$$2x - y + 9 = 0$$

$\therefore \qquad \frac{dy}{dx} = m_1$

$\Rightarrow \qquad 2x - 2 = 2 \Rightarrow x = 2$

when $x = 2, y = 7 \qquad$ [from (i)]

Hence, equation through $(2, 7)$ and parallel to line $2x - y + 9 = 0$ is given by

$$y - 7 = 2(x - 2)$$

or $\qquad y = 2x + 3$

(ii) Tangent perpendicular to the line $5y - 15x = 13$

Let slope of line $5y - 15x = 13$ be m_2

$\therefore \qquad m_2 = 3$

Since, tangent to the curve is perpendicular to line $5y - 15x = 13$

$\therefore \qquad \frac{dy}{dx} \times m_2 = -1$

$\Rightarrow \qquad (2x - 2) \times 3 = -1$

$\Rightarrow \qquad 6x - 6 = -1$

$\Rightarrow \qquad x = \frac{5}{6}$

when $x = \frac{5}{6}, y = \frac{217}{36} \qquad$ [from (i)]

Hence, equation of tangent through $\left(\frac{5}{6}, \frac{217}{36}\right)$ and perpendicular to the line $5y - 15x = 13$ is

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right)$$

or $\qquad 36y + 12x - 227 = 0$

Q. 2. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Sol. Given, curves, $x = y^2$ ...(i)

and $xy = k$ ...(ii)

From (i) & (ii), $y^2y = k$

$\Rightarrow$ $y^3 = \text{k}$

$\Rightarrow$ $y = k^{\frac{1}{3}}$

$\therefore$ $x = k^{\frac{2}{3}}$ [from (i)]

The point of intersection is $(k^{\frac{2}{3}}, k^{\frac{1}{3}})$.

Differentiating (i) & (ii), we get

From (i) $\Rightarrow$ $1 = 2y\dfrac{dy}{dx}$ ...(iii)

From (ii) $\Rightarrow$ $x\dfrac{dy}{dx} + y = 0$ ...(iv)

From (iii), we have

$$\left(\frac{dy}{dx}\right)_{(k^{\frac{2}{3}}, k^{\frac{1}{3}})} = \frac{1}{2k^{\frac{1}{3}}} = m_1$$

From (iv), we have

$$\left(\frac{dy}{dx}\right)_{(k^{\frac{2}{3}}, k^{\frac{1}{3}})} = -\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = m_2$$

$$\therefore \quad m_1m_2 = \frac{1}{2k^{\frac{1}{3}}} \times \frac{-k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = -\frac{k^{-\frac{2}{3}}}{2}$$

Since $8k^2 = 1$

$\Rightarrow$ $k^2 = \dfrac{1}{8}$

$$\therefore \quad m_1m_2 = -\frac{(8)^{\frac{1}{3}}}{2} = -\frac{2}{2} = -1$$

$\therefore$ (i) & (ii) cut at right angles.

Q. 3. Find the equation of the normal at a point on the curve $x^2=4y$, which passes through the point (1, 2). Also find the equation of the corresponding tangent.

Sol. Let the point at which the normal on the curve $x^2 = 4y$ passes through the point (1, 2) be $P(x_1, y_1)$

Now, $x^2 = 4y$ ...(i)

Since, $P(x_1, y_1)$ lies on (i), so

$$x_1^2 = 4y_1 \quad \text{...(ii)}$$

Differentiate (i) w.r.t. x both sides,

$$2x = 4\frac{dy}{dx}$$

or, $\left(\dfrac{dy}{dx}\right) = \dfrac{x}{2}$

Now, Slope of normal at $\text{P}(x_1, y_1)$

$$m_N = -\left(\frac{dx}{dy}\right)_{x_1, y_1} = -\frac{2}{x_1}$$

Now, equation of normal:

$$y - y_1 = m_N(x - x_1)$$

Since, the normal passes through (1, 2).

So, $2 - y_1 = \left(-\dfrac{2}{x_1}\right)(1 - x_1)$

Using (ii),

$$\left(2 - \frac{x_1^2}{4}\right) = \left(2 - \frac{2}{x_1}\right) \Rightarrow x_1 = 2$$

When $x_1 = 2, y_1 = 1$

So, the point is $P(2, 1)$.

Hence, equation of normal is:

$$y - 1 = \left(-\frac{2}{2}\right)(x - 2)$$

$\Rightarrow$ $y - 1 = -x + 2$

$\Rightarrow$ $x + y = 3$

and the equation of tangent is :

$$y - 1 = \frac{2}{2}(x - 2)$$

$\Rightarrow$ $y - 1 = x - 2$

$\Rightarrow$ $x - y = 1.$

AI Q. 4. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which passes through the point $\left(\frac{4}{3}, 0\right)$.

Sol. Given curve, $3x^2 - y^2 = 8$ ...(i)

Differentiating w.r.t. 'x' we get

$$6x - 2y\frac{dy}{dx} = 0$$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{3x}{y}$

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{y_1}$$

$\therefore$ The eq. of tangent at (x_1, y_1)

$$y - y_1 = \frac{3x_1}{y_1}(x - x_1)$$

Tangent passes through the point $\left(\frac{4}{3}, 0\right)$

$\therefore$ $0 - y_1 = \dfrac{3x_1}{y_1}\left(\dfrac{4}{3} - x_1\right)$

$\Rightarrow$ $-y_1^2 = 3x_1\left(\dfrac{4}{3} - x_1\right)$

$\Rightarrow$ $-y_1^2 = 4x_1 - 3x_1^2$

Using (i),

$\Rightarrow$ $8 - 3x_1^2 = 4x_1 - 3x_1^2$

$x_1 = 2$

$\therefore$ From (i), $3(2)^2 - y_1^2 = 8$

[$\because$ point (x_1, y_1) also lie on the curve]

$\Rightarrow$ $12 - y_1^2 = 8$

$\Rightarrow$ $y_1^2 = 4$

$y_1 = \pm 2$

$\therefore$ The points are (2, 2) and (2, –2).

∴ The eq. of tangent at (2, 2) is :

$$\Rightarrow \quad y - 2 = 3(x - 2) \qquad \left[\because \left.\frac{dy}{dx}\right|_{(2,\,2)} = 3\right]$$

$$y = 3x - 4$$

The eq. of tangent at (2, –2) is :

$$\Rightarrow \quad y + 2 = -3(x - 2)$$

$$y = -3x + 4 \qquad \left[\because \left.\frac{dy}{dx}\right|_{(2,\,-2)} = -3\right]$$

TOPIC-3

Approximate Values, Differentials and Errors

Revision Notes

Scan to know more about this topic

Approximations

- **Approximate change in the value of functions $y = f(x)$:**

Let $\qquad y = f(x)$

From the definition of derivatives, $\qquad \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$

∴ By definition of limit as $\delta x \to 0$, $\dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}$

i.e., when δx is very near to zero, then we have $\dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$ (approximately).

Therefore, $\delta y = \dfrac{dy}{dx} \cdot \delta x$, where δy represents the approximate change in y.

In case δx is relatively small when compared with x, dy is approximation of δy and we denote it by $dy \approx \delta y$.

Also, δx and δy are known as differentials. δx is known as absolute error in x, $\dfrac{\delta x}{x}$ is known as relative error in x.

- **Approximate value :**

By the definition of derivatives (first principle),

$$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

∴ By the definition of limit when $h \to 0$

We have $\qquad \dfrac{f(x+h) - f(x)}{h} \to f'(x)$

∴ If h is very near to zero, then we have

$$\frac{f(x+h) - f(x)}{h} = f'(x) \text{ (approximately)}$$

or $\qquad f(x + h) = f(x) + hf'(x)$ approximately when $h \to 0$

2 Marks Questions

AI **Q. 1. Find the approximate change in the volume 'V' of a cube of side x meter caused by decreasing the side by 1%. [ISC, 2018]**

Sol. Volume of cube of side x is given by

$$V = x^3$$

$$\therefore \quad \frac{dV}{dx} = 3x^2$$

Given, $\qquad \delta x = 1\%$ of side (x)

i.e., $\qquad \delta x = \dfrac{1}{100}x$

$$\therefore \quad \delta V = \frac{dV}{dx}.\delta x = 3x^2.\frac{1}{100}x$$

$$= 0.03x^3 \text{ m}^3$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Many candidates did not have idea of approximation concept as application of differentiation and few candidates made mistakes in writing the final answer in the correct form.

Answering Tip

- Approximation is one the applications of derivatives. Understand the concept of approximation carefully and do enough practice of problems related to it.

Q. 2. A circular metal plate expands under heating so that its radius increase by 2%. Find the approximate increase in the area of plate if the radius of the plate before heating is 10 cm.

Sol. Let r be the radius and A be the area of the plate.

$\therefore \quad A = \pi r^2$

Also, let Δr be the change in the radius and ΔA be the corresponding change in the area of the plate.

Then, given $\frac{\Delta r}{r} \times 100 = 2$

$\Rightarrow \quad \frac{\Delta r}{10} \times 100 = 2$

When $r = 10$,

$\Rightarrow \quad \Delta r = \frac{2}{10} \Rightarrow dr = \frac{2}{10} \quad [\because \Delta x \cong dx]$

Now, $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$\therefore \quad \left(\frac{dA}{dr}\right)_{r=10} = 20\pi$

$\therefore \quad dA = \frac{dA}{dr} dr$

$\Rightarrow \quad dA = 20\pi \times \frac{2}{10} = 4\pi \Rightarrow \Delta A = 4\pi \text{ cm}^2$

Q. 3. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Sol. Let r be the radius of the sphere and Δr be the error in measuring the radius. Then, $r = 9$ cm and $\Delta r = 0.03$ cm.

Let V be the volume of the sphere. Then,

$V = \frac{4}{3}\pi r^3$

$\therefore \quad \frac{dV}{dr} = 4\pi r^2$

$\Rightarrow \quad \left(\frac{dV}{dr}\right)_{r=9} = 4\pi \times (9)^2 = 324\,\pi$

Let ΔV be the error in V due to error Δr in r.

Then, $\quad \Delta V = \frac{dV}{dr} \times \Delta r$

$\Rightarrow \quad \Delta V = 324\,\pi \times 0.03 = 9.72\,\pi \text{ cm}^3.$

4 Marks Questions

Q. 1. Using differentials, find the approximate value of $(3.968)^{3/2}$.

Sol. Let $\quad y = f(x) = x^{3/2}, x = 4$

Let $x + \Delta x = 3.968$

$\therefore \quad \Delta x = 3.968 - 4 = -0.032$

$\Delta y = \left[\frac{dy}{dx}\right]_{x=4} \Delta x$

$\Rightarrow \quad \Delta y = \left[\frac{3}{2}x^{1/2}\right]_{x=4} \Delta x$

$\Rightarrow \quad \Delta y = \frac{3}{2} \times 2(-0.032) = -0.096$

$\therefore \quad \Delta y = (x + \Delta x)^{3/2} - x^{3/2}$

$\Rightarrow \quad -0.096 = (3.968)^{3/2} - (4)^{3/2}$

$\Rightarrow (3.968)^{3/2} = 8 - 0.096$

$= 7.904$

AI **Q. 2. Find the approximate value of $f(3.02)$, up to 2 places of decimal, where $f(x) = 3x^2 + 5x + 3$.**

Sol. Given, $f(x) = 3x^2 + 5x + 3$

$f'(x) = 6x + 5$

Let $x = 3$ and $\Delta x = 0.02$

$\because \quad f'(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$\therefore f(x + \Delta x) = (\Delta x)f'(x) + f(x)$

$\therefore \quad f(3.02) = (0.02)f'(3) + f(3)$

$= (0.02)(23) + 45$

$= 45.46$

Q. 3. Using differentials, find the approximate value of $\sqrt{49.5}$.

Sol. Let $y = \sqrt{x}$ and $x = 49, \Delta x = 0.5$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\therefore \quad y + \Delta y = \sqrt{x + \Delta x}$

$\Rightarrow \quad y + \frac{dy}{dx} \cdot \Delta x = \sqrt{x + \Delta x}$

$\Rightarrow \sqrt{x} + \frac{1}{2\sqrt{x}} \cdot \Delta x = \sqrt{x + \Delta x}$

Putting $x = 49$ and $\Delta x = 0.5$, we get

$\sqrt{49} + \frac{1}{2\sqrt{49}} \cdot (0.5) = \sqrt{49.5}$

$\Rightarrow \quad \sqrt{49.5} = 7 + \frac{1}{28} = 7.0357$

Q. 4. Use differentials, find the approximate value of $(127)^{1/3}$.

Sol. Let $\quad y = x^{1/3} \quad$ ...(i)

Let $\quad x = 125$ and $x + \Delta x = 127$

$\Rightarrow \quad \Delta x = 2$

$y = (125)^{1/3} = 5$

From (i),

$$\frac{dy}{dx} = \frac{1}{3}(x)^{-2/3}$$

$$\Rightarrow \quad \Delta y = \frac{dy}{dx}\times\Delta x = \frac{x^{-2/3}}{3}\times\Delta x$$

$$= \frac{2}{3(125)^{\frac{2}{3}}} = \frac{2}{75} = 0{\cdot}0266$$

$$\therefore \quad y + \Delta y = 5 + 0{\cdot}0266 = 5{\cdot}0266$$

$$\Rightarrow \quad \sqrt[3]{127} = 5{\cdot}0266$$

AI Q. 5. Use differentials find the approximate value of $(0{\cdot}037)^{1/2}$.

Sol. Let $x = 0{\cdot}04$

$x + \Delta x = 0{\cdot}037,$

then $\Delta x = 0{\cdot}037 - 0{\cdot}040$

$\Rightarrow \quad \Delta x = -\,0{\cdot}003$

Let $y = x^{1/2}$ ...(i)

$\Rightarrow \quad y = (0{\cdot}04)^{1/2} = 0{\cdot}2$

Differentiating (i) w.r.t. x, we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\therefore \quad \Delta y = \frac{\Delta x}{2\sqrt{x}} \qquad \left[\because \Delta y = \frac{dy}{dx}.\Delta x\right]$$

$$\Rightarrow \quad \Delta y = \frac{-0{\cdot}003}{2\sqrt{0{\cdot}04}} = \frac{-0{\cdot}003}{2\times0{\cdot}2}$$

$$= \frac{-3}{4\times100} = \frac{-0{\cdot}75}{100}$$

$$\Rightarrow \quad \Delta y = -\,0{\cdot}0075$$

$$y + \Delta y = 0{\cdot}2 - 0{\cdot}0075$$

$$= 0{\cdot}1925$$

$$\therefore \quad \sqrt{0{\cdot}037} = 0{\cdot}1925$$

TOPIC-4
Increasing / Decreasing Functions

Scan to know more about this topic

Increasing and Decreasing Functions

Revision Notes

- A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases, *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta, f(\alpha) > f(\beta)$.

  If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.

- A function $f(x)$ is said to be a decreasing function in $[a, b]$, if as x increase, $f(x)$ decreases, *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.

  If $f'(x) \leq 0$ lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.

  - A function $f(x)$ is a constant function in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$.
  - By monotonic function $f(x)$ in interval I, we mean that f is either only increasing in I or only decreasing in I.

- Finding the intervals of increasing and/or decreasing of a function:

  Algorithm

  Step 1 : Consider the function $y = f(x)$.

  Step 2 : Find $f'(x)$.

  Step 3 : Put $f'(x) = 0$ and solve to get the critical point(s).

  Step 4 : The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

Mnemonics

Increasing and Decreasing Function

Dead	**Z**ombies	**C**onsider	**G**reen	**L**emons
↓	↓	↓	↓	↓
Find **Derivative** of function $f(x)$ i.e., $f'(x)$	Put Derivative equal to **zero** i.e., $f'(x) = 0$	Get **Critical** points *i.e.*, values of x	If $f'(x)$ is **Greater** than zero *i.e.*, $f'(x) > 0$, then $f(x)$ is increasing	If $f'(x)$ is **Less** than zero i.e., $f'(x) < 0$, then $f(x)$ is decreasing

1 Mark Questions

Q. 1. Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$.

(a) sin 2*x* **(b) tan x**

(c) cos *x* **(d) cos 3*x***

Sol. Correct option : (c)

Explanation : In the given interval $\left(0, \frac{\pi}{2}\right)$

$f(x) = \cos x$

On differentiating with respect to *x*, we get

$f'(x) = -\sin x$

which gives $f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

AI Q. 2. The function $f(x) = \tan x - x$

(a) always increases

(b) always decreases

(c) never increases

(d) sometimes increases and sometimes decreases

Sol. Correct option : (a)

Explanation : We have,

$f(x) = \tan x - x$

On differentiating with respect to *x*, we get

$f'(x) = \sec x - 1$

$\Rightarrow f'(x) > 0, \forall x \in F$

So, $f(x)$ always increases.

Q. 3. Show that the function given by $f(x) = \sin x$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

Sol. Consider, $f(x) = \sin x$

$f'(x) = \cos x$ ....(i)

$\cos x < 0$ for each $x \in \left(\frac{\pi}{2}, \pi\right)$ $= \frac{1}{x} > 0$

$\therefore$ $f(x) < 0$ [from (i)]

Hence, function is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

AI Q. 4. Show that function $y = 4x - 9$ is increasing for all $x \in R$.

Sol. Given, $y = 4x - 9$

$$\frac{dy}{dx} = 4 > 0 \text{ for all } x \in R.$$

Hence, function is increasing for all $x \in R$.

Q. 5. Find the interval for which the function $f(x) = \cot^{-1} x + x$ increases.

Sol. Given, $f(x) = \cot^{-1}x + x$

$$f'(x) = \frac{-1}{1+x^2} + 1 = \frac{-1+1+x^2}{1+x^2}$$

$$= \frac{x^2}{1+x^2} \geq 0 \text{ for all } x \in R.$$

$\therefore f$ increases for $(-\infty, \infty)$.

2 Marks Questions

AI Q. 1. Prove that the function $f(x) = x^3 - 6x^2 + 12x + 5$ is increasing on *R*. **[ISC, 2019]**

Sol. $f(x) = x^3 - 6x^2 + 12x + 5$

$f'(x) = 3x^2 - 12x + 12$

$= 3(x^2 - 4x + 4)$

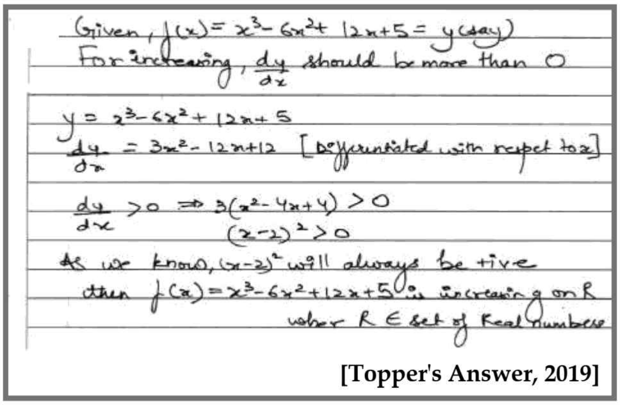
$= 3(x-2)^2 \geq 0$ for all $x \in R$

$\therefore f'(x) \geq 0 \ \forall x \in R$

$\therefore f(x)$ increasing function on *R*

[ISC Marking Scheme, 2019]

OR



Given, $f(x) = x^3 - 6x^2 + 12x + 5 = y$ (say)

For increasing, $\frac{dy}{dx}$ should be more than 0

$y = x^3 - 6x^2 + 12x + 5$

$\frac{dy}{dx} = 3x^2 - 12x + 12$ [Differentiated with respect to x]

$\frac{dy}{dx} > 0 \Rightarrow 3(x^2 - 4x + 4) > 0$

$(x-2)^2 > 0$

As we know, $(x-2)^2$ will always be +ive

then $f(x) = x^3 - 6x^2 + 12x + 5$ is increasing on R

where R ∈ set of Real numbers

[Topper's Answer, 2019]

Q. 2. Find the intervals in which the function $f(x)$ is strictly increasing where, $f(x) = 10 - 6x - 2x^2$. **[ISC Specimen Paper, 2018]**

Sol. Given, $f(x) = 10 - 6x - 2x^2$

$\therefore$ $f'(x) = -6 - 4x$

$f'(x) = -6 - 4x = 0$

$\Rightarrow$ $x = \frac{-6}{4} = \frac{-3}{2}$

$\therefore$ The intervals are $\left(-\infty, \frac{-3}{2}\right)$ and $\left(\frac{-3}{2}, \infty\right)$

$f'(x) > 0$ in $\left(-\infty, \frac{-3}{2}\right)$

$\therefore f(x)$ is strictly increasing in $\left(-\infty, \frac{-3}{2}\right)$

Q. 3. Prove that the function $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing on *R*.

Sol. We have, $f(x) = x^3 - 3x^2 + 3x - 100$

$\therefore$ $f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$

Now, $x \in R \Rightarrow (x-1)^2 \geq 0 \Rightarrow f'(x) \geq 0$

Thus, $f'(x) \geq 0$ for all $x \in R$.

Hence, $f(x)$ is increasing on *R*.

AI **Q. 4. Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing on $(-1, 1)$.**

Sol. We have,

$$f(x) = x^2 - x + 1$$

$$\therefore \quad f'(x) = 2x - 1 = 2\left(x - \frac{1}{2}\right)$$

Now, $-1 < x < \frac{1}{2} \Rightarrow \left(x - \frac{1}{2}\right) < 0 \Rightarrow 2\left(x - \frac{1}{2}\right) < 0$

$$\Rightarrow \quad f'(x) < 0$$

and $\frac{1}{2} < x < 1 \Rightarrow x - \frac{1}{2} > 0 \Rightarrow 2\left(x - \frac{1}{2}\right) > 0$

$$\Rightarrow \quad f'(x) > 0$$

Thus, $f'(x)$ does not have the same sign throughout the interval $(-1, 1)$.

Hence, $f(x)$ is neither increasing nor decreasing on $(-1, 1)$.

4 Marks Questions

AI **Q. 1. Find the intervals in which $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is (a) strictly increasing (b) strictly decreasing.**

Sol. Given, $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$

$$\therefore \quad f'(x) = \frac{12}{10}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}$$

$$= \frac{6}{5}(x-1)(x+2)(x-3)$$

Put, $\quad f'(x) = 0 \Rightarrow \frac{6}{5}(x-1)(x+2)(x-3) = 0$

$\Rightarrow \quad x = 1, -2, 3$

$\therefore$ Intervals are $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$

$\because$ $f'(x) > 0$ for $(-2, 1) \cup (3, \infty)$

Hence, $f(x)$ is strictly increasing in $(-2, 1)$ and $(3, \infty)$.

$\because$ $f'(x) < 0$ for $(-\infty, -2)$ and $(1, 3)$

Hence, $f(x)$ is strictly decreasing in $(-\infty, -2)$ and $(1, 3)$.

Q. 2. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is (a) strictly increasing (b) strictly decreasing.

Sol. Try yourself, similar to Q. 1 in 4 marks Question.

Q. 3. Find the intervals in which the function $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$ is :

(i) strictly increasing

(ii) strictly decreasing.

Sol. Try yourself, similar to Q. 1 in 4 marks Question.

Q. 4. Separate the intervals $\left[0, \frac{\pi}{2}\right]$ into sub intervals in which $f(x) = \sin^4 x + \cos^4 x$ is increasing or decreasing.

Sol. Given $f(x) = \sin^4 x + \cos^4 x$

$$\therefore \quad f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$$

$$= -4\sin x \cos x[-\sin^2 x + \cos^2 x]$$

$$= -2\sin 2x \cos 2x = -\sin 4x$$

On equating

$$f'(x) = 0 \Rightarrow -\sin 4x = 0$$

$$\Rightarrow \quad 4x = 0, \pi, 2\pi, \ldots$$

$$\Rightarrow \quad x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \ldots$$

Sub-intervals are $\left[0, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

Now, $f'(x) < 0$ in $\left[0, \frac{\pi}{4}\right]$

$\Rightarrow$ $f(x)$ is decreasing in $\left[0, \frac{\pi}{4}\right]$

and, $f'(x) > 0$ in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

$\therefore$ $f'(x)$ is increasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Q. 5. Find the intervals in which the function given by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is

(a) increasing

(b) decreasing

Sol. Given, $f(x) = \sin x + \cos x$

$$\therefore \quad f'(x) = \cos x - \sin x$$

Now, $f'(x) = 0$ gives $\sin x = \cos x$ which gives

$$x = \frac{\pi}{4}, \frac{5\pi}{4} \text{ as } 0 \le x \le 2\pi$$

The points $x = \frac{\pi}{4}, \frac{5\pi}{4}$ divides the interval $[0, 2\pi]$ into 3 disjoint intervals, $\left[0, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$.

Here, $f'(x) > 0$ in $x \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]$ or $f(x)$ is strictly increasing in intervals $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$.

Also, $f'(x) <$ when if $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ or $f(x)$ is strictly decreasing in this intervals.

Interval	Sign of $f'(x)$	Nature of function
$\left[0, \frac{\pi}{4}\right]$	+ ve	f is strictly increasing
$\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$	– ve	f is strictly decreasing
$\left[\frac{5\pi}{4}, 2\pi\right]$	+ ve	f is strictly increasing

AI Q. 6. Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$, is an increasing function of x throughout its domain.

Sol. Given, $y = \log(1+x) - \frac{2x}{2+x}, x > -1$

Differentiating, w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{1+x} - \frac{[(2+x)(2) - 2x]}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{[4+2x-2x]}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(2+x)^2(1+x)}$$

$$= \frac{4+x^2+4x-4-4x}{(2+x)^2(1+x)}$$

$$= \frac{x^2}{(2+x)^2(1+x)}$$

For increasing function,

$$\frac{dy}{dx} \geq 0$$

$$\frac{x^2}{(2+x)^2(x+1)} \geq 0$$

$$\Rightarrow \left(\frac{x}{2+x}\right)^2 \cdot \left(\frac{1}{x+1}\right) > 0$$

$$\Rightarrow \quad \left(\frac{1}{x+1}\right) > 0$$

This is possible only when $1 + x > 0$ *i.e.*, $x > -1$

So, $\frac{dy}{dx} > 0$ for $x > -1$

When $x > -1$

$\frac{dy}{dx}$ is always greater than zero.

$\therefore y = \log(1+x) - \frac{2x}{2+x}$ is always increasing through out its domains.

Q. 7. Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing.

Sol. Try yourself, similar to Q. 5 in 4 marks Question.

Q. 8. Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence, find the intervals in which $f(x)$ is :

(a) strictly increasing

(b) strictly decreasing

Sol. $\therefore \quad f'(x) = 2x - 1$

Now, $f'(x) > 0, \forall x \in \left(\frac{1}{2}, 1\right)$

and $f'(x) < 0, \forall x \in \left(-1, \frac{1}{2}\right)$

$\therefore f(x)$ is neither increasing nor decreasing in $(-1, 1)$.

Therefore, $f(x)$ is strictly increasing on $\left(\frac{1}{2}, 1\right)$ and

$f(x)$ is strictly decreasing on $\left(-1, \frac{1}{2}\right)$

TOPIC-5

Maxima and Minima

Revision Notes

- **Understanding maxima and minima :**

  Consider $y = f(x)$ be a well defined function on an interval I, then

  (a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$

  The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the **point of maximum value of f in I.**

  (b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in I$.

  The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of f in I.**

  (c) f is said to have an extreme value in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

  The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point**.

Scan to know more about this topic

Maxima and Minima

- Let f be a real valued function and also take a point c from its domain. Then

  (i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local maximum value of f.**

  (ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local minimum value** of f.

- **Critical points :**

  It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes, *i.e.*, $f'(c) = 0$ or f is not differentiable.

➢ **First Derivative Test :**

Consider $y = f(x)$ be a well defined function on an open interval I. Now proceed as have been mentioned in the following algorithm :

Step 1 : Find $\frac{dy}{dx}$.

Step 2 : Find the critical point(s) by putting $\frac{dy}{dx} = 0$. Suppose $c \in I$ (where I is the interval) be any critical point and f be continuous at this point c. Then we may have following situations :

- $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.
- $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.
- $\frac{dy}{dx}$ **does not change sign** as x increases through c, then $x = c$ is **neither** a point of **local maximum nor** a point of **local minimum**. Rather in this case, the point $x = c$ is called the **point of inflection**.

➢ **Second Derivative Test :**

Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that :

- $x = c$ is a point of local maxima if $f''(c) = 0$ and $f''(c) < 0$
  The value $f(c)$ is called the local maximum value of f.
- $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
  The value $f(c)$ is called the local minimum value of f.

This test fails if $f'(c) = 0$ and $f'(c) = 0$. In such a case, we use **first derivative test** as discussed in the above.

➢ **Absolute maxima and absolute minima :**

If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

Algorithm

Step 1 : Find all the critical points of f in the given interval, *i.e.*, find all the points x where either $f'(x) = 0$ or f is not differentiable.

Step 2 : Take the end points of the given interval.

Step 3 : At all these points (*i.e.*, the points found in Step 1 and Step 2) calculate the values of f.

Step 4 : Identify the maximum and minimum value of f out of the values calculated in Step 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

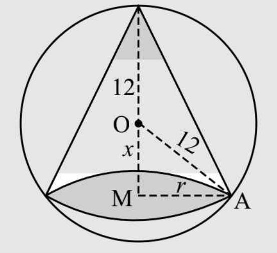
Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly, absolute minimum value is called as **global minimum value** or the **least value**.

6 Marks Questions

AI **Q. 1. A cone is inscribed in a sphere of radius 12 cm. If the volume of the cone is maximum, find its height. [ISC Specimen Paper, 2021] [ISC, 2018]**

Sol. Let r be the radius of cone and h be the height of cone, let x be the distance between centre of sphere and centre of base of cone, Given, radius of sphere $R = 12$ cm

$$\therefore \quad h = 12 + x \quad \text{...(i)}$$



$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(12^2 - x^2)(12 + x)$$

$$[\because \text{ In } \Delta OMA,\ 12^2 = x^2 + r^2\ ]$$

$$= \frac{1}{3}\pi(144 - x^2)(12 + x)$$

$$\therefore \quad \frac{dv}{dx} = \frac{1}{3}\pi\left[(144 - x^2).1 + (12 + x)(-2x)\right]$$

$$= \frac{1}{3}\pi\left[144 - x^2 - 24x - 2x^2\right]$$

$$= \frac{1}{3}\pi\left[144 - 24x - 3x^2\right]$$

For maximum or minimum volume, put $\frac{dv}{dx} = 0$

i.e., $\frac{1}{3}\pi[144 - 24x - 3x^2] = 0$

$\Rightarrow \quad 144 - 24x - 3x^2 = 0$

$\Rightarrow \quad x^2 + 8x - 48 = 0$

$\Rightarrow \quad (x + 12)(x - 4) = 0$

$\Rightarrow \quad x = -12 \text{ and } x = 4$

Now,

$$\frac{d^2v}{dx^2} = \frac{1}{3}\pi[-24 - 6x]$$

$$\therefore \quad \left(\frac{d^2v}{dx^2}\right)_{x=4} = \frac{1}{3}\pi[-24 - 24] < 0$$

Hence, at $x = 4$ volume of cone is maximum and height of cone (h) = 12 + x = 12 + 4 = 16 cm.

[ISC Marking Scheme, 2018]

Q. 2. Show that the radius of a closed right circular cylinder of given surface area and maximum volume is equal to half of its height.

[ISC 2020]

Sol. Given : Total surface area of cylinder

$$= 2\pi r^2 + 2\pi rh$$

$$A = 2\pi r^2 + 2\pi rh$$

$$\therefore \quad h = \frac{A - 2\pi r^2}{2\pi r} \quad \text{...(i)}$$

[Where r is the radius and h is the height of the cylinder]

Volume of cylinder (V) $= \pi r^2 h$

$$V = \pi r^2 \frac{(A - 2\pi r^2)}{2\pi r} \quad \text{from (i)}$$

$$V = \frac{r}{2}(A - 2\pi r^2)$$

$$V = \frac{Ar}{2} - \pi r^3$$

Now,

$$\frac{dV}{dr} = \frac{d}{dr}\frac{Ar}{2} - \frac{d}{dr}(\pi r^3)$$

$$\frac{dV}{dr} = \frac{A}{2} - 3\pi r^2$$

For maximum/minimum;

$$\frac{dV}{dr} = 0$$

$$\therefore \quad \frac{A}{2} - 3\pi r^2 = 0$$

$$\frac{A}{2} = 3\pi r^2$$

$$A = 6\pi r^2$$

So, $2\pi r^2 + 2\pi rh = 6\pi r^2$

$$2\pi rh = 4\pi r^2$$

$$h = 2r$$

$$r = \frac{h}{2} \quad \text{...(ii)}$$

Again,

$$\frac{dV}{dr} = \frac{A}{2} - 3\pi r^2$$

and,

$$\frac{d^2V}{dr^2} = 0 - 6\pi r$$

for any value of r i.e. $r > 0$

$$\frac{d^2V}{dr^2} < 0$$

Hence, volume is maximum when $r = h^2$

Hence Proved.

[ISC Marking Scheme, 2020]

Examiner's Comment

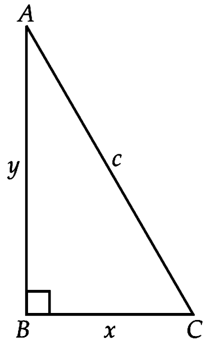
- Most of the candidates attempted this question incorrectly. Some candidates could not express volume of the cone as function in mathematical form. A number of calculation mistakes were also made while differentiating.

Answering Tip

- The concept of maxima and minima need to be understood carefully. Do adequate practice of problems based on maxima and minima. Also, practice problem based on mensuration related concept.

Q. 3. Prove that the area of right-angled triangle of given hypotenuse is maximum when the triangle is isosceles. **[ISC, 2020]**

Sol. Given ΔABC is right angle triangle and hypotenuse is c unit.



To Prove : Area is maximum when triangle is an isosceles triangle.

Proof : In ΔABC, ∠B = 90°

$$\therefore \quad c^2 = x^2 + y^2 \quad \text{...(i)}$$

$$\text{Area of } \Delta ABC = \frac{1}{2} \text{ base} \times \text{height}$$

$$A = \frac{1}{2}xy$$

$$A^2 = \frac{1}{4}x^2y^2 \text{ (Let } A^2 = Z)$$

$$Z = \frac{1}{4}x^2(c^2 - x^2)$$

$$Z = \frac{c^2x^2}{4} - \frac{x^4}{4} \quad \text{...(ii)}$$

$$\frac{dZ}{dx} = \frac{d}{dx}\frac{c^2x^2}{4} - \frac{d}{dx}\frac{x^4}{4}$$

$$\frac{dZ}{dx} = \frac{2c^2x}{4} - \frac{4x^3}{4}$$

$$\frac{dZ}{dx} = \frac{c^2x}{2} - x^3$$

For maximum/minimum,

$$\frac{dZ}{dx} = 0$$

$$\frac{c^2x}{2} - x^3 = 0$$

$$x\left(\frac{c^2}{2} - x^2\right) = 0$$

either $x = 0$ (Not possible $x > 0$)

So, $$\frac{c^2}{2} - x^2 = 0$$

$$\frac{c^2}{2} = x^2$$

$$c^2 = 2x^2$$

Now, from equation (i) $x^2 + y^2 = 2x^2$

$$y^2 = x^2$$

$$y = x$$

Again, $$\frac{dZ}{dx} = \frac{c^2x}{2} - x^3$$

Again, $$\frac{d^2Z}{dx^2} = \frac{c^2}{2} - 3x^2$$

$$\frac{d^2Z}{dx^2} = \frac{x^2+y^2}{2} - 3x^2$$

$$\frac{d^2Z}{dx^2} = \frac{2x^2}{2} - 3x^2 = -2x^2$$

$$\frac{d^2Z}{dx^2} < 0$$

∴ Area is maximum when $x = y$.

∴ ΔABC is an isosceles right angle triangle.

Hence Proved.

[ISC Marking Scheme, 2020]

Q. 4. The volume of a closed rectangular metal box with a square base is 4096 cm^3. The cost of polishing the outer surface of the box is ₹ 4 per cm^2. Find the dimensions of the box for the minimum cost of polishing it. [ISC, 2019]

Sol. Let the side of a square base $= x$

Height $= h$

$$x^2h = 4096$$

or $$h = \frac{4096}{x^2}$$

T.S. $A(S) = 2x^2 + 4xh$

$$S = 2x^2 + 4x \times \frac{4096}{x^2}$$

$$S = 2x^2 + \frac{16384}{x}$$

$$\text{Cost of polishing (c)} = \left(2x^2 + \frac{16384}{x}\right)$$

$$\frac{dc}{dx} = 4\left[4x + \frac{16384}{x^2}\right]$$

$$\frac{dc}{dx} = 0 \Rightarrow 4x = \frac{16384}{x^2}$$

$$x^3 = 4096$$

$$x = 16$$

$$\frac{d^2c}{dx^2} = 4\left[4 + \frac{32768}{x^3}\right]$$

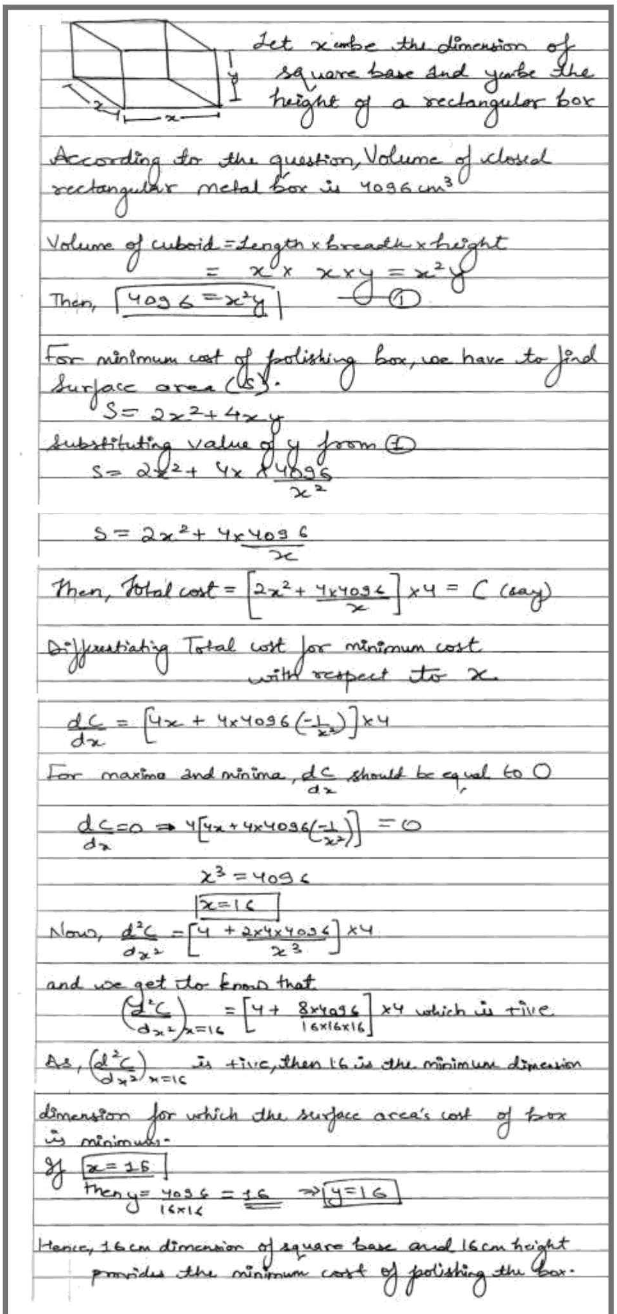
at $x = 16$, $\frac{d^2c}{dx^2}$ = positive

therefore the cost is minimum at $x = 16$

hence, $h = \frac{4096}{16^2} = 16$

[ISC Marking Scheme, 2019]

OR



Let x be the dimension of square base and y be the height of a rectangular box

According to the question, Volume of closed rectangular metal box is 4096 cm^3

Volume of cuboid = Length × breadth × height

$= x \times x \times y = x^2y$

Then, $\boxed{4096 = x^2y}$ —①

For minimum cost of polishing box, we have to find Surface area (S).

$S = 2x^2 + 4xy$

Substituting value of y from ①

$S = 2x^2 + 4x \times \frac{4096}{x^2}$

$S = 2x^2 + \frac{4 \times 4096}{x}$

Then, Total cost $= \left[2x^2 + \frac{4 \times 4096}{x}\right] \times 4 = C$ (say)

Differentiating Total cost for minimum cost with respect to x.

$\frac{dC}{dx} = \left[4x + 4 \times 4096\left(-\frac{1}{x^2}\right)\right] \times 4$

For maxima and minima, $\frac{dC}{dx}$ should be equal to 0

$\frac{dC}{dx} = 0 \Rightarrow 4\left[4x + 4 \times 4096\left(-\frac{1}{x^2}\right)\right] = 0$

$x^3 = 4096$

$\boxed{x = 16}$

Now, $\frac{d^2C}{dx^2} = \left[4 + \frac{2 \times 4 \times 4096}{x^3}\right] \times 4$

and we get to know that

$\left(\frac{d^2C}{dx^2}\right)_{x=16} = \left[4 + \frac{8 \times 4096}{16 \times 16 \times 16}\right] \times 4$ which is +ive

As, $\left(\frac{d^2C}{dx^2}\right)_{x=16}$ is +ive, then 16 is the minimum dimension dimension for which the surface area's cost of box is minimum.

If $\boxed{x = 16}$

then $y = \frac{4096}{16 \times 16} = 16 \Rightarrow \boxed{y = 16}$

Hence, 16 cm dimension of square base and 16 cm height provides the minimum cost of polishing the box.

[Topper's Answer, 2019]

Q. 5. Find the point on the straight line $2x + 3y = 6$, which is closest to the origin.

[ISC, 2019]

Sol. Straight line : $2x + 3y = 6$

Let the point be $(x, y) \Rightarrow y = \frac{6-2x}{3}$

The shortest distance is the perpendicular distance

$$d = \sqrt{x^2+y^2} \text{ or } D = x^2 + y^2$$

$$D = x^2 + y^2$$

$$D = x^2 + \left(\frac{6-2x}{3}\right)^2$$

$$\frac{dD}{dx} = 2x + 2\left(\frac{6-2x}{3}\right)\frac{-2}{3} = \frac{26x-24}{9}$$

with $\frac{dD}{dx} = 0$

$$x = \frac{12}{13}$$

$$\frac{d^2D}{dx} = \frac{26}{9} = [> 0]$$

Therefore, the distance is minimum at $x = \frac{12}{13}$

so $y = \frac{18}{13}$

$\left(\frac{12}{13}, \frac{18}{13}\right)$ is the required point on the line.

[ISC Marking Scheme, 2019]

Examiner's Comments

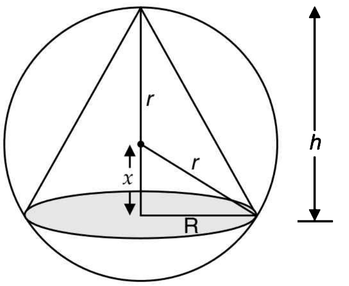
- Several candidates failed to apply the correct formula for the surface area of a closed box with square base and express the same in terms of one variable. Some could not complete the 2nd order derivative test for maximization/minimization. A few candidates found the value of the variable but could not find the dimensions of the box.
- A number of candidates identified distance function in terms of one variable in the square root form where differentiation of it involves long calculations. Some candidates could not complete the second order derivative test.

Answering Tips

- Do proper revision of menstruation formulae of 2-dimensional and 3-dimensional figures/objects.
- Rewrite the objective function in terms of one variable.

AI **Q. 6. Show that the altitude of a right circular cone of maximum that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.** **[ISC Specimen Paper, 2018]**

Sol. Let R be the radius of base and h be the height of the cone, let x be the distance between the centre of sphere and centre of base of cone.



$\therefore \quad h = r + x$

and $\quad R^2 = r^2 - x^2$

Volume of cone $V(x) = \frac{1}{3}\pi R^2 h$

$$= \frac{1}{3}\pi (r^2 - x^2)(r + x)$$

$$\therefore \quad V'(x) = \frac{1}{3}\pi \{-2x(r + x) + (r^2 - x^2)\}$$

$$= \frac{1}{3}\pi (r + x)(-2x + r - x)$$

$$= \frac{1}{3}\pi (r + x)(r - 3x)$$

For max. and min. volume

$$V'(x) = 0$$

$\Rightarrow \quad (r + x)(r - 3x) = 0$

$\Rightarrow \quad r = -x$ (not possible) or $r = 3x$

$\Rightarrow \quad x = \frac{r}{3}$

Now, $\quad V''(x) = \frac{1}{3}\pi\{(r - 3x) - 3(r + x)\}$

at $x = \frac{r}{3}$, $V''\left(\frac{r}{3}\right) < 0$

$\therefore$ Volume is maximum when $x = \frac{r}{3}$

$\therefore \quad h = r + x = r + \frac{r}{3} = \frac{4r}{3}$

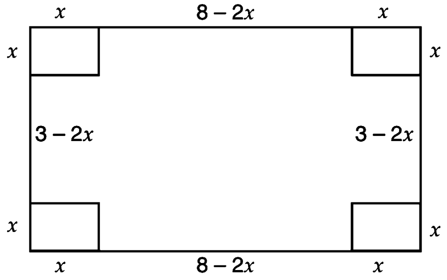
i.e., height of cone is $\frac{4r}{3}$

Q. 7. An open topped box is to be made by removing equal squares from each corner of a 3 m by 8m rectangle sheet of aluminium and by folding up the side. Find the volume of the largest such box.

[ISC Specimen Paper, 2018]

Sol. Given rectangle is of breadth = 3 m

and length = 8 m



Let x be the length of each square removed from the corners

Volume of box $= V(x) = (8-2x)(3-2x)\,x$

$$= x(24-16x-6x+4x^2)$$

$$= 24x-22x^2+4x^3$$

$$V'(x) = 24-44x+12x^2$$

For max. and min. volume

$$V'(x) = 0$$

$$\Rightarrow \quad 12x^2-44x+24 = 0$$

$$\Rightarrow \quad 3x^2-11x+6 = 0$$

$$\Rightarrow 3x^2-9x-2x+6 = 0$$

$$\Rightarrow \quad (x-3)(3x-2) = 0$$

$\Rightarrow \quad x = 3$ (not possible) or $3x = 2$

$$\Rightarrow \quad x = \frac{2}{3}$$

Now, $\quad V''(x) = -44+24x$

at $x = \frac{2}{3}$ $\quad V''\left(\frac{2}{3}\right) = -44+24\left(\frac{2}{3}\right) < 0$

$\therefore$ Volume is maximum when $x = \frac{2}{3}$

$$\text{Volume }(V) = \left(8-2\left(\frac{2}{3}\right)\right)\left(3-2\left(\frac{2}{3}\right)\right)\left(\frac{2}{3}\right)$$

$$= \left(8-\frac{4}{3}\right)\left(3-\frac{4}{3}\right)\left(\frac{2}{3}\right)$$

$$= \frac{20}{3}\times\frac{5}{3}\times\frac{2}{3}$$

$$= \frac{200}{27}\text{ m}^3$$

AI **Q. 8. Show that the surface area of closed cuboid with square base and given volume is minimum when it is a cube. [ISC, 2017]**

Sol. Let the length and breadth be x units (square base), height of cuboid be h unit.

Volume of cuboid $(v) = x^2h$

$$\therefore \quad h = \frac{v}{x^2}.$$

Surface area of cuboid $(s) = 2(x^2+xh+xh)$

$$\therefore \quad s = 2x^2+4xh$$

$$\Rightarrow \quad s = 2x^2+4x\times\frac{v}{x^2}$$

$$\Rightarrow \quad s = 2x^2+4v\frac{1}{x}$$

$$\therefore \quad \frac{ds}{dx} = 4x-4v\frac{1}{x^2}$$

For maximum and minimum surface, $\frac{ds}{dx} = 0$

$$\therefore \quad 4x-4v.\frac{1}{x^2} = 0$$

$$\Rightarrow \quad 4x = \frac{4v}{x^2}$$

$$\Rightarrow \quad x^3 = v$$

$$\Rightarrow \quad x^3 = x^2h$$

$$\Rightarrow \quad x = h$$

Now, $\quad \frac{d^2s}{dx^2} = 4+\frac{8v}{x^3}$

At $\quad x = h,\ \frac{d^2s}{dx^2} = 4+\frac{8v}{h^3} > 0$

$\therefore$ Surface area is minimum, when $x = h$.

i.e., when it is a cube, surface area will be minimum. **[ISC Marking Scheme, 2017]**

Examiner's Comment

- Many candidates did not express volume and surface area of cube with square base by correct formula. Most of the candidates did not show the working of second order derivatives and failed to prove $\frac{d^2s}{dx^2} > 0$.

Answering Tips

- Learn the formulas for area, perimeter, surface and volume of 2-dimensional and 3-dimensional geometrical figures.
- Express the function to be optimized in terms of a single variable by using given data.

Q. 9. A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle to get maximum area. Also, find the maximum area. [ISC, 2016]

Sol.

$$OQ = r\cos\theta$$

$$QR = r\sin\theta$$

$$A(\theta) = \text{Area of the rectangle}$$

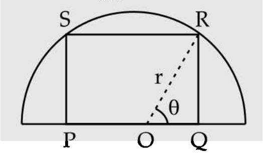
$$= 2OQ\times QR$$

$$= 2r\cos\theta.r\sin\theta$$

$$= r^2\sin 2\theta$$

For maximum and minimum area

$$\frac{dA}{d\theta} = 2r^2\cos 2\theta = 0$$



$$\cos 2\theta = 0$$

$$\Rightarrow \quad 2\theta = \frac{\pi}{2}$$

$\Rightarrow \quad \theta = \frac{\pi}{4}$ is the critical point

$$\frac{d^2A}{d\theta^2} = -4r^2\sin 2\theta$$

$$\left(\frac{d^2A}{d\theta^2}\right)_{\theta=\frac{\pi}{4}} = -4r^2 < 0$$

$\Rightarrow$ Area is maximum at $\theta = \frac{\pi}{4}$

So, sides of rectangle are

$$QR = r\sin\theta = r.\sin\frac{\pi}{4} = \frac{r}{\sqrt{2}} = \frac{\sqrt{2}r}{2}$$

and $PQ = 2OQ = 2r\cos\theta = 2r\cos\left(\frac{\pi}{4}\right)$

$$= 2r\frac{1}{\sqrt{2}}$$

$$= \sqrt{2}r$$

Maximum area $= PQ \times RQ$

$$= \sqrt{2}r \times \frac{\sqrt{2}r}{2}$$

$$= r^2$$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Many candidates were not able to write the expression in mathematical form. They were not able to express the equation in one variable. A number of candidates made calculation mistakes while differentiating.

Q. 10. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius 10 cm is a square of side $10\sqrt{2}$ cm. [ISC, 2015]

Sol. $AB = 2x$; $BC = 2y$

In ΔABC,

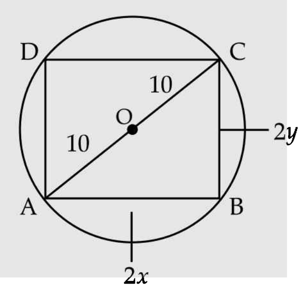
$$(2x)^2 + (2y)^2 = (20)^2$$

$$\Rightarrow \quad 4x^2 + 4y^2 = 400 \Rightarrow x^2 + y^2 = 100 \quad ...(i)$$

$$P = 4x + 4y = 4x + 4\cdot\sqrt{100 - x^2}$$

$$\therefore \quad \frac{dP}{dx} = 4 - \frac{4x}{\sqrt{100 - x^2}} = 0 \quad \text{[from (i)]}$$

$$\Rightarrow \quad x = 5\sqrt{2} \text{ cm}$$



$$\frac{d^2P}{dx^2} = -4\left\{\frac{\sqrt{100 - x^2} - \dfrac{x(-x)}{\sqrt{100 - x^2}}}{100 - x^2}\right\}$$

$$= \frac{-4 \times 100}{(100 - x^2)^{\frac{3}{2}}} < 0$$

Hence, perimeter is maximum, when $x = 5\sqrt{2}$.

$$\therefore \quad y = 5\sqrt{2} \text{ [from (i)]} \Rightarrow x = y$$

$\therefore$ $ABCD$ is square of side $10\sqrt{2}$ cm

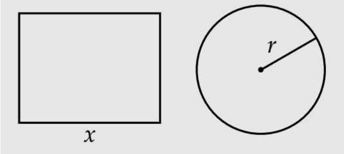
[ISC Marking Scheme, 2015]

AI Q. 11. A wire of length 50 m is cut into two pieces. One piece of the wire is bent in the shape of a square and the other in the shape of a circle. What should be the length of each piece so that the combined area of the two is minimum? [ISC, 2014]

Sol. Let the side of the square be x and radius of the circle r.

$$4x + 2\pi r = 50$$

$$\Rightarrow \quad 2\pi r = 50 - 4x$$



$$\Rightarrow \quad r = \frac{50 - 4x}{2\pi}$$

$$A = x^2 + \pi r^2$$

$$= x^2 + \pi\left(\frac{50 - 4x}{2\pi}\right)^2$$

$$= x^2 + \frac{1}{4\pi}(50 - 4x)^2$$

For maximum and minimum value of A,

$$\frac{dA}{dx} = 0$$

$$\frac{dA}{dx} = 2x + \frac{1}{4\pi}2(50 - 4x)(-4) = 0$$

$$\Rightarrow \quad 2\pi x - 100 + 8x = 0$$

$$\Rightarrow \quad x = \frac{50}{\pi + 4}$$

$$\frac{d^2A}{dx^2} = +\text{ve}, A \text{ is minimum at } x = \frac{50}{\pi + 4}$$

$$\therefore \text{ Length of square } = \frac{200}{\pi + 4} \text{ m}$$

$$\text{Length of circle } = 50 - \frac{200}{\pi + 4} = \frac{50\pi}{\pi + 4} \text{ m}$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- The required sum of areas of the two parts had to be expressed correctly and in terms of a single variable. Some candidates failed in this. Differentiation and simplification errors were also made by several candidates.

Answering Tip

- Differentiation rules for different functions and forms need continuous revision and practice. Obtaining the result in the required form needs simplification, cross multiplication, substitution, etc. Hence enough revision and practices is required.

- If $4x + 2\pi r = 50$ then $r = \dfrac{50-4x}{2\pi}$. Hence $A = x^2 + \pi\left(\dfrac{50-4x}{2\pi}\right)^2$. In this manner 'A' can be expressed in terms of a single variable '*x*'. Differentiation and solving follows. Also, second order derivative needs to be shown as negative for maximum, and positive for minimum. So, practice such these problems regularly.

Detailed Answer :

Length of wire = 50 m (Given)

Let length of one piece for shape of square = x m

$\therefore$ Length of other piece for shape of circle

$$= (50 - x) \text{ m}$$

Now perimeter of square = $4a = x$

$$\Rightarrow \quad a = \frac{x}{4}$$

and circumference of circle = $2\pi r = 50 - x$

$$\Rightarrow \quad r = \frac{50-x}{2\pi}$$

Combined Area = $a^2 + \pi r^2$

$$= \frac{x^2}{16} + \pi \cdot \left(\frac{50-x}{2\pi}\right)^2$$

$$= \frac{x^2}{16} + \pi \cdot \frac{(50-x)^2}{4\pi^2}$$

$$A = \frac{x^2}{16} + \frac{(50-x)^2}{4\pi}$$

Differentiating w.r. t. x, we get

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{2(50-x)(-1)}{4\pi}$$

$$= \frac{x}{8} + \frac{(x-50)}{2\pi}$$

$$= \frac{\pi x + 4x - 200}{8\pi}$$

$$= \frac{x(4+\pi) - 200}{8\pi}$$

For maximum and minimum value of A, $\dfrac{dA}{dx} = 0$

$\therefore \quad x(4 + \pi) - 200 = 0$

$$x = \frac{200}{4+\pi}$$

At $x = \dfrac{200}{4+\pi}$, $\dfrac{d^2A}{dx^2} > 0$

A is minimum at $x = \dfrac{200}{4+\pi}$

$\therefore$ Length of square wire, $x = \dfrac{200}{4+\pi}$ m

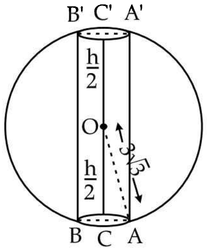
and length of circle wire = $50 - x$

$$= 50 - \frac{200}{4+\pi}$$

$$= \frac{50\pi}{4+\pi} \text{ m}$$

Q. 12. Find the maximum volume of the cylinder which can be inscribed in a sphere of radius $3\sqrt{3}$ cm. (Leave the answer in terms of π). [ISC, 2013]

Sol. Let h units be the height of the cylinder and R unit be the radius of the cylinder. Given, $3\sqrt{3}$ cm is the radius of the sphere.



If V is the volume of the cylinder, then

$$V = \pi R^2 h \qquad \text{...(i)}$$

Let O be the centre of the sphere and $C'OC \perp B'A'$ as well as BA.

From right angled ΔOCA

$$(3\sqrt{3})^2 = \left(\frac{h}{2}\right)^2 + R^2$$

$$\Rightarrow \quad R^2 = 27 - \frac{h^2}{4} \qquad \text{...(ii)}$$

$$\therefore \quad V = \pi\left(27 - \frac{h^2}{4}\right) \cdot h$$

$$V = 27\pi h - \pi\frac{h^3}{4}$$

$$\therefore \quad \frac{dV}{dh} = 27\pi - \frac{3\pi h^2}{4}$$

$$\text{and} \quad \frac{d^2V}{dh^2} = -\frac{6\pi h}{4}$$

For Maxima/Minima, $\dfrac{dV}{dh} = 0$

$$\therefore \quad 27\pi = \frac{3\pi h^2}{4}$$

$$h^2 = \frac{4 \times 27}{3}$$

$$h^2 = 36$$

$$h = 6, -6$$

$$\text{and} \quad \left(\frac{d^2V}{dh^2}\right)_{h=6} = \frac{-6\pi \times 6}{4} < 0$$

$\therefore$ V is maximum when $h = 6$, putting $h = 6$ in equation (ii),

$$R^2 = 27 - \frac{36}{4}$$

$$R^2 = 18$$

$\therefore$ From (i), $V = \pi R^2 h$

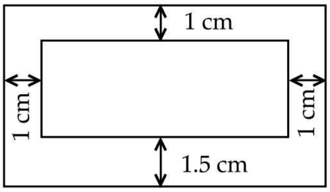
$$= \pi \times 18 \times 6$$

$$= 108\pi \text{ cm}^3$$

Q. 13. A printed page is to have total area of 80 sq. cm with a margin of 1 cm at the top and on each side and a margin of 1.5 cm at the bottom. What should be the dimensions of the page so that the printed area will be maximum? [ISC, 2012]

Sol. Let x cm and y cm be the dimensions of the printed page.

$$xy = 80 \qquad \text{...(i)}$$



Let A (x sq. cm) be the printed area then,

$$A = (x-2)(y-5/2)$$

$$\Rightarrow \quad A = xy - \frac{5x}{2} - 2y + 5$$

$$\Rightarrow \quad A = 80 - \frac{5x}{2} - \frac{2 \times 80}{x} + 5 \qquad \text{[using (i)]}$$

$$\Rightarrow \quad A = 85 - \frac{5x}{2} - \frac{160}{x}$$

Differentiating w.r.t. x, we get

$$\frac{dA}{dx} = -\frac{5}{2} + \frac{160}{x^2}$$

Differentiating once again w.r.t. x

$$\frac{d^2A}{dx^2} = -\frac{320}{x^3}$$

For maximum and minimum Area,

$$\therefore \quad \frac{dA}{dx} = 0$$

$$-\frac{5}{2} + \frac{160}{x^2} = 0$$

$$\Rightarrow \quad \frac{160}{x^2} = \frac{5}{2}$$

$$\Rightarrow \quad x^2 = 64$$

$$\Rightarrow \quad x = 8$$

Also, $\left(\dfrac{d^2A}{dx^2}\right)_{x=8} = -\dfrac{320}{512} < 0$

$\Rightarrow A$ is maximum at $x = 8$

From (i), $\quad xy = 80$

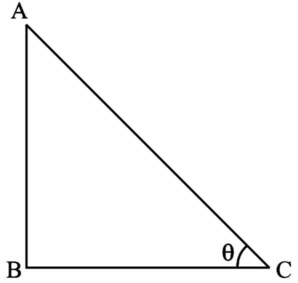
$$\Rightarrow \quad 8 \times y = 80$$

$$\Rightarrow \quad y = 10$$

Therefore, the dimensions of the printed page are 10 cm and 8 cm.

Q. 14. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$. [ISC, 2011]

Sol. Consider the following triangle ABC



Let $AC = y$ and $BC = x$ and $x + y = k$ (k is a constant).

Let A be the area of triangle, then

$$A = \frac{1}{2} BC \times AB$$

$$\Rightarrow \quad A = \frac{1}{2} x \times \sqrt{y^2 - x^2}$$

$$\Rightarrow \quad A^2 = \frac{1}{4} x^2 (y^2 - x^2)$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}\left\{(k-x)^2 - x^2\right\}$$

$$\Rightarrow \quad A^2 = \frac{k^2x^2 - 2kx^3}{4} \qquad \text{...(i)}$$

Differentiating eq(i) w.r.t. x, we get

$$\frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A} \qquad \text{...(ii)}$$

for maximum or minimum value, put $\dfrac{dA}{dx} = 0$

i.e., $\quad \dfrac{k^2x - 3kx^2}{4A} = 0$

$$x = \frac{k}{3}$$

Again differentiating eq.(ii), we get

$$2\left(\frac{dA}{dx}\right)^2 + 2A\frac{d^2A}{dx^2} = \frac{2k^2 - 12kx}{4} \qquad \text{...(iii)}$$

Now, put $\dfrac{dA}{dx} = 0$ and $x = \dfrac{k}{3}$ in eq.(iii), we get

$$\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0$$

Thus, A is maximum when $x = \dfrac{k}{3}$

Now, when $x = \dfrac{k}{3}$, $y = k - \dfrac{k}{3} = \dfrac{2k}{3}$

Therefore, $\quad \cos\theta = \dfrac{x}{y}$

$$\Rightarrow \quad \cos\theta = \frac{\frac{k}{3}}{\frac{2k}{3}} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Q. 15. A closed right circular cylinder has volume $\frac{539}{2}$ cubic units. Find the radius and the height of the cylinder so that the total surface area is minimum. [ISC, 2010]

Sol. Let r be the radius and h be the height of the cylinder.

$$V = \pi r^2 h = \frac{539}{2}$$

$$\Rightarrow \quad h = \frac{539}{2\pi r^2}$$

Let S be the total surface area, then

$$S = 2\pi rh + 2\pi r^2$$

$$= 2\pi r\left(\frac{539}{2\pi r^2}\right) + 2\pi r^2 = \frac{539}{r} + 2\pi r^2$$

$$\frac{dS}{dr} = -\frac{539}{r^2} + 4\pi r$$

For maxima or minima, $\frac{dS}{dr} = 0$

$$\Rightarrow \quad -\frac{539}{r^2} + 4\pi r = 0$$

$$\Rightarrow \quad -539 + 4\pi r^3 = 0$$

$$\Rightarrow \quad r^3 = \frac{539}{4\pi} = \frac{539 \times 7}{4 \times 22}$$

$$\Rightarrow \quad r^3 = \left(\frac{7}{2}\right)^3$$

$$\Rightarrow \quad r = \frac{7}{2} \text{ units}$$

$$\therefore \quad \frac{d^2S}{dr^2} = \frac{1078}{r^3} + 4\pi,$$

$$\left(\frac{d^2s}{dr^2}\right)_{r=\frac{7}{2}} = \frac{1078}{\left(\frac{7}{2}\right)^3} + 4\pi > 0$$

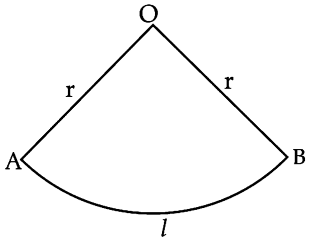
$\therefore$ S is minimum, at $r = \frac{7}{2}$ units

and $$h = \frac{539 \times 7 \times 2 \times 2}{2 \times 22 \times 7 \times 7} = 7$$

$\therefore$ Radius of the cylinder be $\frac{7}{2}$ units and height be 7 units.

Q. 16. A wire of length 20 m is available to fence off a flower bed in the form of a sector of a circle. What must be the radius of the circle, if we wish to have a flower bed with the greatest possible area? [ISC, 2009]

Sol.



Given,

$$2r + l = 20 \text{ m} \qquad \text{...(i)}$$

$$\text{Area} = \frac{1}{2}rl$$

$$A = \frac{1}{2}r(20 - 2r) = 10r - r^2 \qquad \text{[from (i)]}$$

$$\frac{dA}{dr} = 10 - 2r = 0$$

For maximum or minimum area, $\frac{dA}{dr} = 0$

$$\Rightarrow \quad 10 - 2r = 0 \Rightarrow r = 5$$

$$\frac{d^2A}{dr^2} = -2 \text{ (negative) at } r = 5$$

$\therefore$ A is maximum at $r = 5$ m

Q. 17. An open box with a square base is to be made out of a given quantity of cardboard whose area is c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units. [ISC, 2008]

Sol. Given, surface area of open box = c^2

$$\therefore \quad 2bh + 2hl + lb = c^2$$

But base is square. So, $l = b = a$

$$\therefore \quad 2ah + 2ah + a^2 = c^2$$

$$\therefore \quad a^2 + 4ah = c^2$$

$$\therefore \quad h = \frac{c^2 - a^2}{4a} \qquad \text{...(i)}$$

Now, Volume (V) = a^2h

or, $$V = a^2\left(\frac{c^2 - a^2}{4a}\right) = \frac{1}{4a}(ac^2 - a^3)$$

$$\therefore \quad \frac{dV}{da} = \frac{1}{4}(c^2 - 3a^2) = 0$$

For maximum or minimum, $\frac{dV}{da} = 0$

$$\Rightarrow \quad a = \frac{c}{\sqrt{3}}$$

$$\frac{d^2V}{da^2} = \frac{1}{4}(-6a) = -\text{ve}$$

$\therefore$ V is maximum at $a = \frac{c}{\sqrt{3}}$

$$\therefore \quad \text{Maximum volume} = a^2h = \frac{a^2(c^2 - a^2)}{4a}$$

$$= \frac{1}{4}(ac^2 - a^3)$$

$$= \frac{1}{4}\left(\frac{c^3}{\sqrt{3}} - \frac{c^3}{3\sqrt{3}}\right)$$

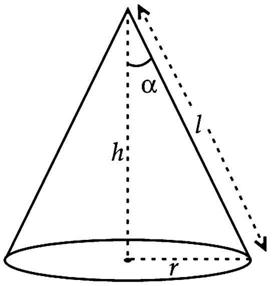
$$= \frac{c^3}{4}\left[\frac{2}{3\sqrt{3}}\right] = \frac{c^3}{6\sqrt{3}} \text{ cubic units}$$

Hence Proved.

AI **Q. 18. Show that the semi-vertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}(\sqrt{2})$. [ISC, 2007]**

Sol. Let r, h and l be the radius, height and slant height of the cone respectively.

As shown in the figure,



$$\sin\alpha = \frac{r}{l} \quad \Rightarrow \quad r = l\sin\alpha$$

and $$\cos\alpha = \frac{h}{l} \quad \Rightarrow \quad h = l\cos\alpha$$

∴ $$\text{Volume } = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi(l\sin\alpha)^2 l\cos\alpha$$

$$= \frac{1}{3}\pi l^3\sin^2\alpha\cos\alpha$$

∴ $$\frac{dV}{d\alpha} = \frac{1}{3}\pi l^3[-\sin^3\alpha + 2\cos\alpha\cdot\sin\alpha\cos\alpha]$$

For maximum or minimum, $\frac{dV}{d\alpha} = 0$

$$\Rightarrow \quad \sin\alpha[-\sin^2\alpha + 2\cos^2\alpha] = 0$$

$$\Rightarrow \quad -\sin^2\alpha + 2 - 2\sin^2\alpha = 0 \qquad [\sin\alpha \neq 0]$$

$$\Rightarrow \quad \sin\alpha = \sqrt{\frac{2}{3}} \quad \Rightarrow \quad \tan\alpha = \sqrt{2}$$

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

Now, $$\frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi r l^3[-3\sin^2\alpha\cdot\cos\alpha + 2\cos\alpha\cos 2\alpha + \sin 2\alpha(-\sin\alpha)]$$

$$= \frac{1}{3}\pi r l^3\,[-3\sin^2\alpha\,.\cos\alpha + 2\cos\alpha\,(2\cos^2\alpha - 1) + 2\sin\alpha.\cos\alpha(-\sin\alpha)]$$

$$= \frac{1}{3}\pi r l^3\left[-3\times\frac{2}{3}\times\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\left(2\times\frac{1}{3} - 1\right) + 2\times\sqrt{\frac{2}{3}}\times\frac{1}{\sqrt{3}}\left(-\sqrt{\frac{2}{3}}\right)\right]$$

$$= \frac{1}{3}\pi l^3\left[-\frac{2}{\sqrt{3}} - \frac{2}{3\sqrt{3}} - \frac{4}{3\sqrt{3}}\right] < 0$$

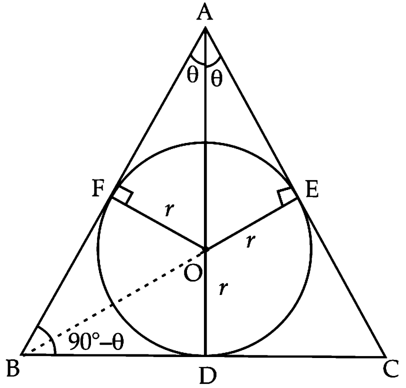
∴ Volume is maximum for $\tan\alpha = \sqrt{2}$.

i.e., $\alpha = \tan^{-1}(\sqrt{2})$

Hence Proved.

Q. 19. Prove that the least perimeter of an isosceles triangle in which a circle of radius *r* can be inscribed is $6\sqrt{3}r$.

Sol. Let Δ *ABC* be an isosceles triangle with *AB* = *AC*



Let ∠*BAC* = 2θ

AO bisects ∠*BAC*.

Join *OE, OF, OD*.

Where *O* is the centre of the circle and

$$OE = OF = OD = r$$

Also, $OE \perp AC, OD \perp BC$ and $OF \perp AB$.

In Δ *AOE*, $\tan\theta = \frac{r}{AE}$

$$\Rightarrow \quad AE = r\cot\theta$$

Similarly, $AF = r\cot\theta$

In ∠ *ABD*, $AD \perp BC$ (Δ *ABC* is isosceles)

∴ $$\angle ABD = 90° - \theta = \frac{\pi}{2} - \theta$$

OB bisects ∠*ABD*, ∴ $\angle OBF = \angle OBD = \frac{\pi}{4} - \frac{\theta}{2}$

In Δ *OFB*,

$$\Rightarrow \quad BF = r\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = r\cot\left(\frac{\pi - 2\theta}{4}\right)$$

Similarly, $BD = DC = CE = r\cot\left(\frac{\pi - 2\theta}{4}\right)$

We have, perimeter of Δ *ABC*

$$P = AB + BC + CA$$

$$= AF + BF + BD + DC + AE + EC$$

$$= 2r\cot\theta + 4r\cot\left(\frac{\pi - 2\theta}{4}\right)$$

Differentiate with respect to θ

∴ $$\frac{dP}{d\theta} = 2r(-\operatorname{cosec}^2\theta) + 4r\left(-\operatorname{cosec}^2\left(\frac{\pi - 2\theta}{4}\right)\times -\frac{1}{2}\right)$$

$$= -2r\operatorname{cosec}^2\theta + 2r\operatorname{cosec}^2\left(\frac{\pi - 2\theta}{4}\right)$$

⇒ For minimum and maximum perimeter,

$$\frac{dP}{d\theta} = 0$$

$$\Rightarrow \quad 2r\operatorname{cosec}^2\left(\frac{\pi - 2\theta}{4}\right) = 2r\operatorname{cosec}^2\theta$$

$$\Rightarrow \quad \sin^2\left(\frac{\pi - 2\theta}{4}\right) = \sin^2\theta$$

$$\Rightarrow \quad \sin\left(\frac{\pi - 2\theta}{4}\right) = \sin\theta$$

or $\sin\left(\frac{\pi-2\theta}{4}\right) = -\sin\theta$

But $0 \le \theta \le \frac{\pi}{2}$

⇒ $\sin 2\theta \ge \sin\theta$

∴ $\sin\left(\frac{\pi-2\theta}{4}\right) \ne -\sin\theta$

Hence, $\sin\left(\frac{\pi-2\theta}{4}\right) = \sin\theta$

⇒ $\frac{\pi-2\theta}{4} = \theta$

⇒ $\pi - 2\theta = 4\theta$ ⇒ $\pi = 6\theta$

⇒ $\theta = \frac{\pi}{6}$ ⇒ $2\theta = \frac{\pi}{3}$

⇒ *ABC* is an equilateral triangle.

By second derivative test,

$$\frac{d^2P}{d\theta^2} = -2r\{2\,\text{cosec}\,\theta(-\text{cosec}\,\theta\cot\theta)\} + 2r\left[2\,\text{cosec}\left(\frac{\pi-2\theta}{4}\right)\left\{-\text{cosec}\left(\frac{\pi-2\theta}{4}\right)\cot\left(\frac{\pi-2\theta}{4}\right)\right\}\right]\times\left(-\frac{1}{2}\right)$$

$$= 4r\,\text{cosec}^2\theta\cot\theta + \frac{4r}{2}\text{cosec}^2\left(\frac{\pi-2\theta}{4}\right)\cot\left(\frac{\pi-2\theta}{4}\right)$$

∴ $\frac{d^2P}{d\theta^2} \ge 0$ at $\theta = \frac{\pi}{6}$

∴ Perimeter is minimum, when $\theta = \frac{\pi}{6}$

Hence, $P = 2r\cot\frac{\pi}{6} + 4r\cot\left(\frac{\pi - 2(\pi/6)}{4}\right)$

$$= 2r\sqrt{3} + 4r\cot\left(\frac{2\pi}{12}\right)$$

$$= 2r\sqrt{3} + 4r\sqrt{3}$$

∴ $P = 6\sqrt{3}r$

Q. 20. Find the local maxima and local minima, of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also find the local maximum and local minimum values.

Sol. Given $f(x) = \sin x - \cos x, \quad 0 < x < 2\pi$

$$f'(x) = \cos x + \sin x, \quad 0 < x < 2\pi$$

Put $f'(x) = 0$

⇒ $\cos x + \sin x = 0$ or $\tan x = -1$

∴ $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

Getting $f''(x) = \cos x - \sin x$

At $x = \frac{3\pi}{4}$, $f''\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} - \sin\frac{3\pi}{4}$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$$

i.e., $f''\left(\frac{3\pi}{4}\right)$ is negative, so at $x = \frac{3\pi}{4}$, $f(x)$ is local maxima.

Hence, local maximum value $= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

At $x = \frac{7\pi}{4}$, $f''\left(\frac{7\pi}{4}\right) = \cos\frac{7\pi}{4} - \sin\frac{7\pi}{4}$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

i.e., $f''\left(\frac{7\pi}{4}\right)$ is positive, so at $x = \frac{7\pi}{4}$ is local minima

local minimum value

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

Q. 21. A tank with rectangular base and rectangular sides open at the top is to be constructed so that its depth is 3 m and volume is 75 m^3. If building of tank costs ₹100 per square metre for the base and ₹50 per square metre for the sides, find the cost of least expensive tank.

Sol. Let *l*, *b*, *h* be the length, breadth and depth of the tank, respectively.

∴ $l \times b \times 3 = 75$

⇒ $l \times b = 25$

Let *C* be the cost, then

$$C = 100(l \times b) + 50 \times 2\,[h(b + l)]$$

$$= 100\left(l \times \frac{25}{l}\right) + 300\left(\frac{25}{l} + l\right)$$

$$= 2500 + 300\left(\frac{25}{l} + l\right)$$

Differentiating w.r.t. *l*,

$$\frac{dC}{dl} = 0 + 300\left(\frac{-25}{l^2} + 1\right)$$

For maximum and minimum cost,

$$\frac{dC}{dl} = 0$$

⇒ $300\left(-\frac{25}{l^2} + 1\right) = 0$

⇒ $l^2 = 25$ or $l = 5$

Getting $\left(\frac{d^2C}{dl^2}\right) = 300\left(\frac{50}{l^3}\right)$

⇒ $\left(\frac{d^2C}{dl^2}\right)_{\text{at } l=5} = \frac{15000}{125} > 0$

i.e., *C* is minimum, when $l = 5$

⇒ $b = 5$

∴ $C = 100(25) + 300(10)$

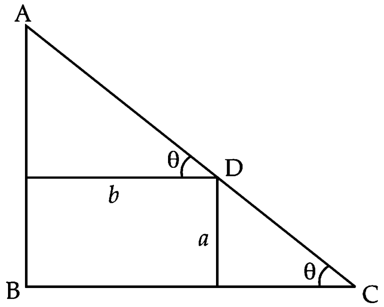
$= 2{,}500 + 3{,}000$

$= 5{,}500$

Hence, the minimum cost is ₹5,500.

Q. 22. A point on the hypotenuse of a right triangle is at distances '*a*' and '*b*' from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Sol. $\therefore AD = b\sec\theta$ and $DC = a\,\text{cosec}\,\theta$



Let L be the length of the hypotenuse

$\therefore \quad L = AC = AD + DC$

$\Rightarrow \quad L = b\sec\theta + a\,\text{cosec}\,\theta$

$\therefore \quad \frac{dL}{d\theta} = b\sec\theta\tan\theta - a\,\text{cosec}\,\theta\cot\theta$

For maximum and minimum length, $\frac{dL}{d\theta} = 0$

$\Rightarrow b\sec\theta\tan\theta - a\,\text{cosec}\,\theta\cot\theta = 0 \Rightarrow \tan^3\theta = \frac{a}{b}$

Now, $\frac{d^2L}{d\theta^2} = b\sec\theta\tan^2\theta + b\sec^3\theta$

$\quad + a\,\text{cosec}\,\theta\cot^2\theta + a\,\text{cosec}^3\theta$

i.e., $\frac{d^2L}{d\theta^2} > 0 \Rightarrow$ Minima

$\therefore \quad L = b\sec\theta + a\,\text{cosec}\,\theta$

$\Rightarrow \quad L = b\frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}} + a\frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}$

$\Rightarrow \quad L = (a^{2/3}+b^{2/3})^{3/2}$

Q. 23. The sum of surface areas of a sphere and a cuboid with sides $\frac{x}{3}, x$ and $2x$ is constant. Show that the sum of their volumes is minimum if x is equal to the three times the radius of sphere.

Sol. Let r be the radius of sphere

Given $\quad S = 4\pi r^2 + 2\left[\frac{x^2}{3} + 2x^2 + \frac{2x^2}{3}\right]$

[$\because$ Surface area of sphere $= 4\pi r^2$ and Surface area of cuboid $= 2(lb + bh + lh)$]

$S = 4\pi r^2 + 6x^2$

$\Rightarrow \quad x^2 = \frac{S-4\pi r^2}{6}$

and $\quad V = \frac{4}{3}\pi r^3 + \frac{2x^3}{3}$

$\therefore \quad V = \frac{4}{3}\pi r^3 + \frac{2}{3}\left(\frac{S-4\pi r^2}{6}\right)^{3/2}$

$$\frac{dV}{dr} = 4\pi r^2 + \left(\frac{S-4\pi r^2}{6}\right)^{1/2}\left(\frac{-8\pi r}{6}\right)$$

For maximum and minimum volume,

$\frac{dV}{dr} = 0$

$\Rightarrow \quad r = \sqrt{\frac{S}{54+4\pi}}$

Now, $\frac{d^2V}{dr^2} = 8\pi r + \left(\frac{-8\pi}{6}\right)\left(\frac{S-4\pi r^2}{6}\right)^{1/2}$

$\quad + \frac{1}{2}\left(\frac{S-4\pi r^2}{6}\right)^{-1/2}\left(\frac{-8\pi r}{6}\right)^2$

At, $r = \sqrt{\frac{S}{54+4\pi}}$; $\frac{d^2V}{dr^2} > 0$

$\therefore$ For $r = \sqrt{\frac{S}{54+4\pi}}$, volume is minimum

i.e., $r^2(54 + 4\pi) = S$

or $r^2(54 + 4\pi) = 4\pi r^2 + 6x^2$

$\Rightarrow \quad 6x^2 = 54r^2$

$\Rightarrow \quad x^2 = 9r^2$

$\Rightarrow \quad x = 3r$

Q. 24. A jet of enemy is flying along the curve $y = x^2 + 2$ and a soldier is placed at the point (3, 2). Find the minimum distance between the soldier and the jet.

Sol. Let $P(x, y)$ be the position of the jet and the soldier is placed at $A(3, 2)$.

$\Rightarrow \quad AP = \sqrt{(x-3)^2 + (y-2)^2}$ ...(i)

As given, $\quad y = x^2 + 2 \Rightarrow y - 2 = x^2$ ...(ii)

$\therefore \quad AP^2 = (x-3)^2 + x^4 = z$ (say)

$\therefore \quad \frac{dz}{dx} = 2(x-3) + 4x^3$

For maximum and minimum value,

$\frac{dz}{dx} = 0$

$\therefore \; 2x - 6 + 4x^3 = 0$

Put $\quad x = 1$

$2 - 6 + 4 = 0$

$\therefore \; x - 1$ is a factor

and $\quad \frac{d^2z}{dx^2} = 12x^2 + 2$

Now, $\frac{d^2z}{dx^2}$ (at $x = 1$) > 0

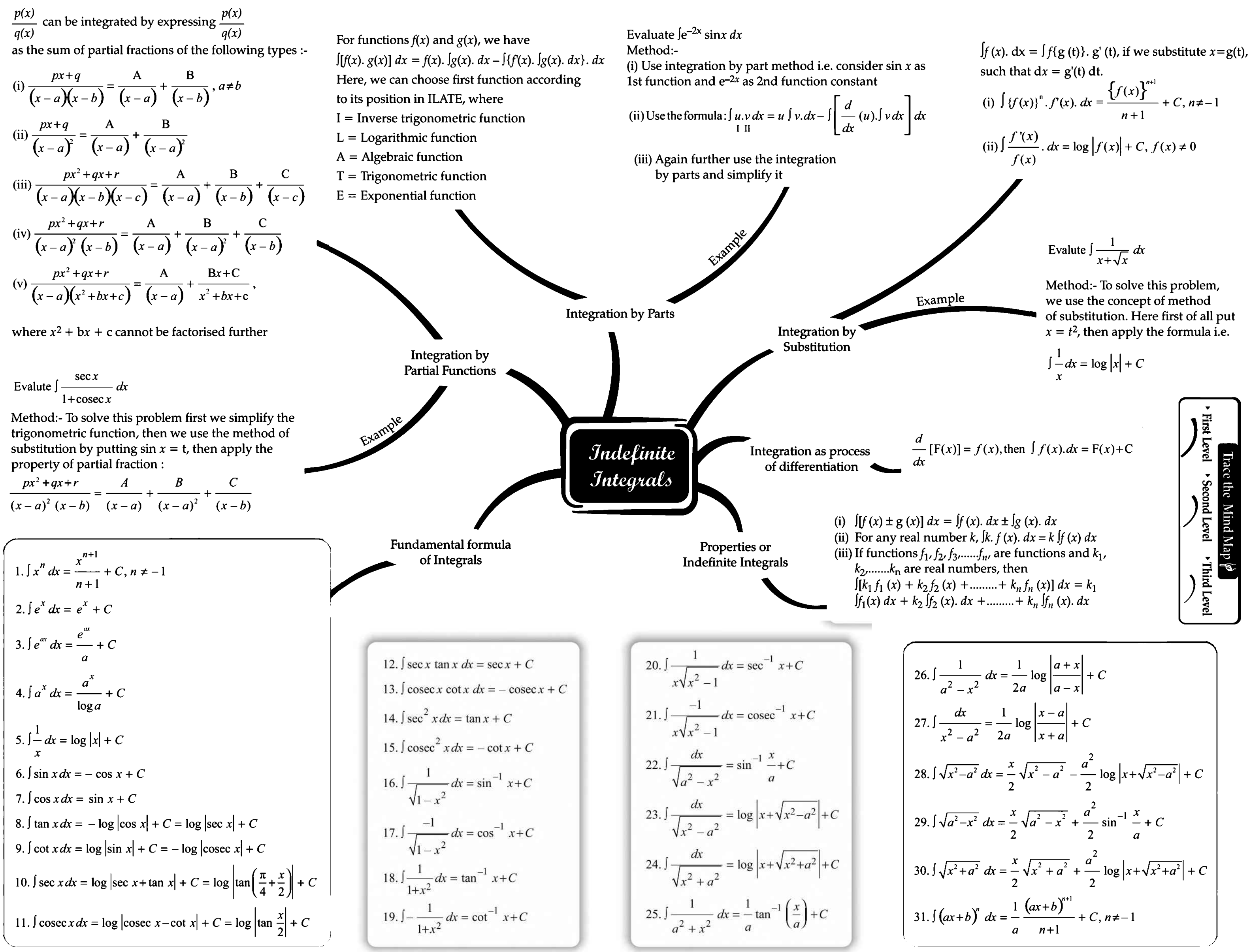
$\therefore \; z$ is minimum, when $x = 1, y = 1 + 2 = 3$

[from (ii)]

$\therefore \quad$ Minimum distance $= \sqrt{(1-3)^2 + (3-2)^2} = \sqrt{5}$

[from (i)

○○



Indefinite Integrals

Trace the Mind Map
▸ First Level ▸ Second Level ▸ Third Level

Integration by Partial Functions

$\frac{p(x)}{q(x)}$ can be integrated by expressing $\frac{p(x)}{q(x)}$ as the sum of partial fractions of the following types :-

(i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}, a \neq b$

(ii) $\frac{px+q}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$,

where $x^2 + bx + c$ cannot be factorised further

Example

Evalute $\int \frac{\sec x}{1+\operatorname{cosec} x}\, dx$

Method:- To solve this problem first we simplify the trigonometric function, then we use the method of substitution by putting $\sin x = t$, then apply the property of partial fraction :

$$\frac{px^2+qx+r}{(x-a)^2\,(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$$

Integration by Parts

For functions $f(x)$ and $g(x)$, we have

$\int [f(x).\,g(x)]\,dx = f(x).\int g(x).\,dx - \int \{f'(x).\int g(x).\,dx\}.\,dx$

Here, we can choose first function according to its position in ILATE, where

I = Inverse trigonometric function
L = Logarithmic function
A = Algebraic function
T = Trigonometric function
E = Exponential function

Example

Evaluate $\int e^{-2x} \sin x\, dx$

Method:-

(i) Use integration by part method i.e. consider $\sin x$ as 1st function and e^{-2x} as 2nd function constant

(ii) Use the formula : $\int \underset{\text{I}}{u}.\underset{\text{II}}{v}\,dx = u\int v.dx - \int\left[\frac{d}{dx}(u).\int v\,dx\right]dx$

(iii) Again further use the integration by parts and simplify it

Integration by Substitution

$\int f(x).\,dx = \int f\{g(t)\}.\,g'(t)$, if we substitute $x=g(t)$, such that $dx = g'(t)\,dt$.

(i) $\int \{f(x)\}^n.f'(x).\,dx = \frac{\{f(x)\}^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int \frac{f'(x)}{f(x)}.\,dx = \log|f(x)| + C, f(x) \neq 0$

Example

Evalute $\int \frac{1}{x+\sqrt{x}}\, dx$

Method:- To solve this problem, we use the concept of method of substitution. Here first of all put $x = t^2$, then apply the formula i.e.

$\int \frac{1}{x}dx = \log|x| + C$

Integration as process of differentiation

$\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x).dx = F(x)+C$

Properties or Indefinite Integrals

(i) $\int [f(x) \pm g(x)]\,dx = \int f(x).\,dx \pm \int g(x).\,dx$
(ii) For any real number k, $\int k.\,f(x).\,dx = k\int f(x)\,dx$
(iii) If functions $f_1, f_2, f_3, \ldots\ldots f_n$, are functions and $k_1, k_2, \ldots\ldots k_n$ are real numbers, then $\int [k_1 f_1(x) + k_2 f_2(x) + \ldots\ldots + k_n f_n(x)]\,dx = k_1 \int f_1(x)\,dx + k_2 \int f_2(x).\,dx + \ldots\ldots + k_n \int f_n(x).\,dx$

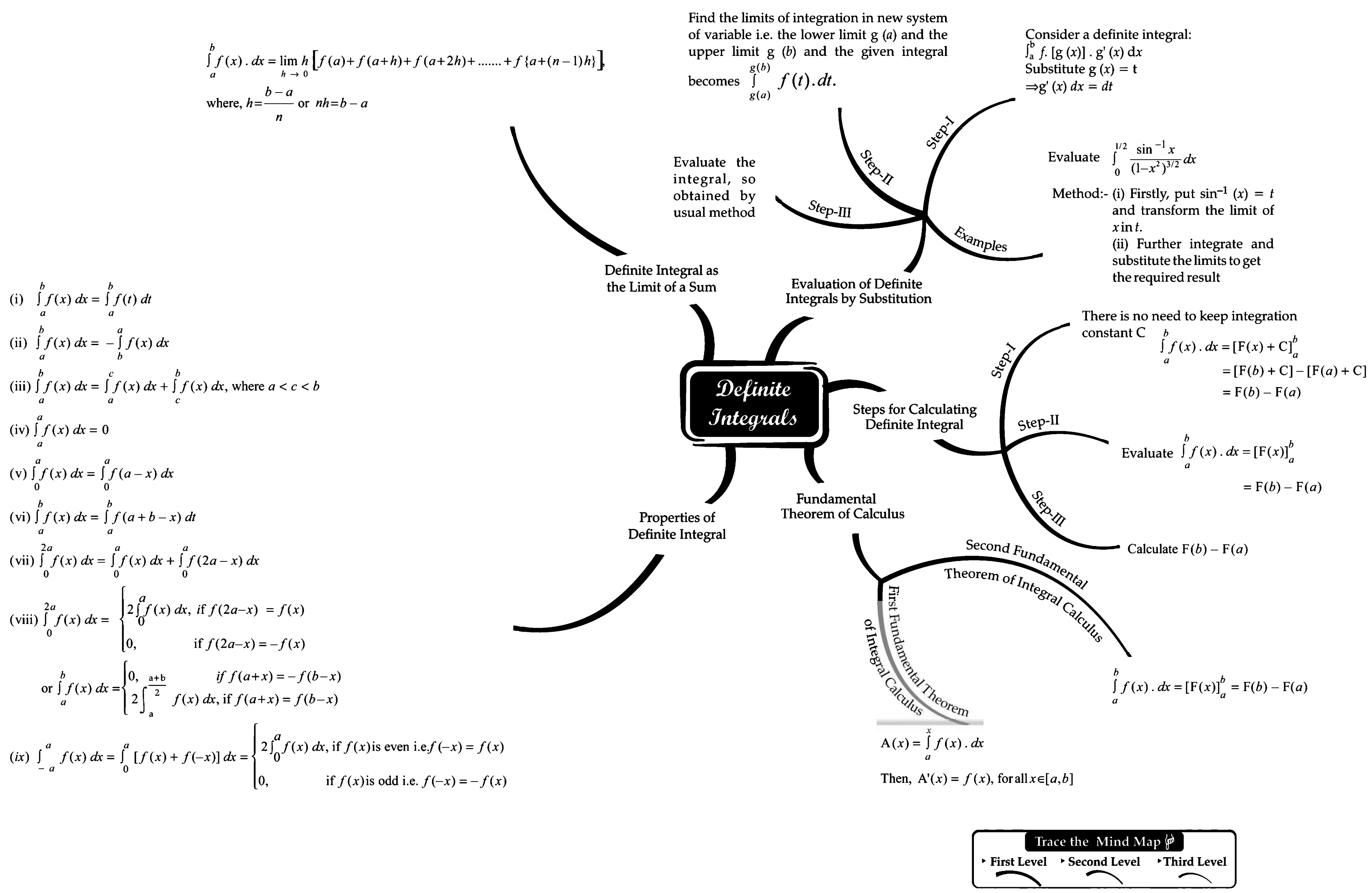
Fundamental formula of Integrals

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
2. $\int e^x\,dx = e^x + C$
3. $\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$
4. $\int a^x\,dx = \frac{a^x}{\log a} + C$
5. $\int \frac{1}{x}\,dx = \log|x| + C$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$
9. $\int \cot x\,dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$
10. $\int \sec x\,dx = \log|\sec x+\tan x| + C = \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$
11. $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x-\cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

12. $\int \sec x \tan x\,dx = \sec x + C$
13. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$
14. $\int \sec^2 x\,dx = \tan x + C$
15. $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$
16. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
17. $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$
18. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
19. $\int -\frac{1}{1+x^2}\,dx = \cot^{-1} x + C$

20. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$
21. $\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \operatorname{cosec}^{-1} x + C$
22. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$
23. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$
24. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$
25. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

26. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
27. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
28. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$
29. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$
30. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$
31. $\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, n \neq -1$



Definite Integrals

Definite Integral as the Limit of a Sum

$\int_a^b f(x)\,.\,dx = \lim_{h \to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots\ldots+f\{a+(n-1)h\}\right]$

where, $h = \frac{b-a}{n}$ or $nh = b-a$

Evaluation of Definite Integrals by Substitution

Step-I: Consider a definite integral: $\int_a^b f.[g(x)].g'(x)\,dx$ Substitute $g(x) = t$ $\Rightarrow g'(x)\,dx = dt$

Step-II: Find the limits of integration in new system of variable i.e. the lower limit g (*a*) and the upper limit g (*b*) and the given integral becomes $\int_{g(a)}^{g(b)} f(t).dt.$

Step-III: Evaluate the integral, so obtained by usual method

Examples: Evaluate $\int_0^{1/2} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}dx$

Method:- (i) Firstly, put $\sin^{-1}(x) = t$ and transform the limit of x in t.
(ii) Further integrate and substitute the limits to get the required result

Steps for Calculating Definite Integral

Step-I: There is no need to keep integration constant C $\int_a^b f(x)\,.\,dx = [F(x)+C]_a^b$
$= [F(b)+C]-[F(a)+C]$
$= F(b)-F(a)$

Step-II: Evaluate $\int_a^b f(x)\,.\,dx = [F(x)]_a^b$
$= F(b)-F(a)$

Step-III: Calculate $F(b)-F(a)$

Fundamental Theorem of Calculus

Second Fundamental Theorem of Integral Calculus: $\int_a^b f(x)\,.\,dx = [F(x)]_a^b = F(b)-F(a)$

First Fundamental Theorem of Integral Calculus: $A(x) = \int_a^x f(x)\,.\,dx$

Then, $A'(x) = f(x)$, for all $x \in [a,b]$

Properties of Definite Integral

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

(iv) $\int_a^a f(x)\,dx = 0$

(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vi) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dt$

(vii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(viii) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

or $\int_a^b f(x)\,dx = \begin{cases} 0, & \text{if } f(a+x) = -f(b-x) \\ 2\int_a^{\frac{a+b}{2}} f(x)\,dx, & \text{if } f(a+x) = f(b-x) \end{cases}$

(ix) $\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x)+f(-x)]\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even i.e. } f(-x) = f(x) \\ 0, & \text{if } f(x) \text{ is odd i.e. } f(-x) = -f(x) \end{cases}$

Trace the Mind Map
- First Level
- Second Level
- Third Level

CHAPTER
7
INTEGRALS

SYLLABUS

Integration as inverse process of differentiation. Integration of a variety of functions by substitution; by partial fractions and by parts. Evaluation of simple integrals of the following types and problems based on them.

Definite integrals as a limit of a sum, Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

- ***Indefinite integral***
  - ***Integration as the inverse of differentiation.***
  - ***Anti-derivatives of polynomials and functions*** $(ax + b)^n$, $\sin x$, $\cos x$, $\sec^2 x$, $\operatorname{cosec}^2 x$ ***etc.***
  - ***Integrals of the type*** $\operatorname{six}^2 x$, $\sin^3 x$, $\sin^4 x$, $\cos^2 x$, $\cos^3 x$, $\cos^4 x$.
  - ***Integration of*** $1/x$, e^x.
  - ***Integration by substitution.***
  - ***Integrals of the type*** $f\ (x)[f\ (x)]n$, $\dfrac{f'(x)}{f(x)}$
  - ***Integration of*** $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$.
  - ***Integration by parts.***
  - ***Integration using partial fractions.***

  Expressions of the form $\dfrac{f(x)}{g(x)}$ ***when degree of*** $f(x) <$ ***degree of*** $g(x)$

  e.g. $$\frac{x+2}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

  $$\frac{x+2}{(x-2)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

  $$\frac{x+1}{(x^2+3)(x-1)} = \frac{Ax+B}{x^2+3} + \frac{C}{x-1}$$

  When degree of $f(x)$ ***degree of*** $g(x)$,

  e.g.

  $$\frac{x^2+1}{x^2+3x+2} = 1 - \left(\frac{3x+1}{x^2+3x+2}\right)$$

- ***Integrals of the type:***

  $$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$$

  and $\int \sqrt{a^2 \pm x^2}\ dx$, $\int \sqrt{x^2 - a^2}\ dx$,

  $\int \sqrt{ax^2+bx+c}\ dx$, $\int (px+q)\sqrt{ax^2+bx+c}\ dx$

  integrations reducible to the above forms.

  $$\int \frac{dx}{a\cos x + b\sin x},$$

$\int \frac{dx}{a+b\cos x}, \int \frac{dx}{a+b\sin x}, \int \frac{dx}{a\cos x+b\sin x+c},$

$\int \frac{(a\cos x+b\sin x)dx}{c\cos x+d\sin x},$

$\int \frac{dx}{a\cos^2 x+b\sin^2 x+c},$

$\int \frac{1\pm x^2}{1+x^4}dx,$

$\int \frac{dx}{1+x^4}, \int \sqrt{\tan x}\ dx, \int \sqrt{\cot x}\ dx$ *etc.*

- ***Definite integral***
  - *Definite integral as a limit of the sum.*
  - *Fundamental theorem of calculus (without proof)*
  - *Properties of definite integrals.*
  - *Problems based on the following properties of definite integrals are to be covered.*

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

where $a < c < b$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, if & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f \text{ is an even function} \\ 0, \text{ if } f \text{ is an odd function} \end{cases}$$

Trend Analysis

List of Concepts	2016	2017	2018	2019	2020
Indefinite Integral	1 Q (2 marks) 1 Q (4 marks) 1 Q (6 marks)	1 Q (6 marks)	1 Q (4 marks) 1 Q (6 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (4 marks) 1 Q (6 marks)
Definite Integral	1 Q (6 marks)	1 Q (2 marks)	1 Q (4 marks)	1 Q (4 marks) 1 Q (6 marks)	1 Q (2 marks)

TOPIC-1
Indefinite Integral

Revision Notes

TOPIC - 1
Indefinite Integral .... **P. 182**
TOPIC - 2
Definite Integral .... **P. 202**

- **Meaning of Integral of Function :**

  If differentiation of a function $F(x)$ is $f(x)$ *i.e.*, if $\frac{d}{dx}[F(x)] = f(x)$, then we say that integral or primitive or anti-derivative of $f(x)$ and in symbols, we write, $\int f(x)dx = F(x) + C$.

  Therefore, we can say that integration is the inverse process of differentiation.

- **Indefinite Integral :** Let $f(x)$ be a function. Then family of all its Primitives / or anti derivatives is called the indefinite integral of $f(x)$ and denoted by $\int f(x)dx$.

- **Methods of Integration :**

  (a) Integration by Substitution Method : In this method, we change the integral $\int f(x)dx$, where independent variable is x, to another integral in which independent variable is t (say different from x) such that x and t are related by $x = g(t)$.

  Let $u = \int f(x)dx$ then, $\frac{du}{dx} = f(x)$

  Again as $x = g(t)$ so we have $\frac{dx}{dt} = g'(t)$

  Now $\frac{du}{dt} = \frac{du}{dx}.\frac{dx}{dt} = f(x)\cdot g'(t)$

  On integrating both sides w.r.t. t, we get

  $$\int\left(\frac{du}{dt}\right)dt = \int f(x)g'(t)dt$$

  $\Rightarrow$ $u = \int f[g(t)]g'(t)dt$

  i.e., $\int f(x)dx = \int f[g(t)]g'(t)dt$, where $x = g(t)$.

  So, it is clear that substituting $x = g(t)$ in $\int f(x).dx$ will give us the same result as obtained by putting $g(t)$ in place of x and $g'(t)dt$ in place of dx.

  Scan to know more about this topic
  Indefinite Integrals

  Scan to know more about this topic
  Integration by Substitution

  (b) Integration by Partial Fractions : Consider $\frac{f(x)}{g(x)}$ defines a rational polynomial function.

  - If the degree of numerator *i.e.*, $f(x)$ is **greater than or equal to** the degree of denominator *i.e.*, $g(x)$ then, this type of rational function is called an **improper rational function**. And if degree of $f(x)$ is smaller than the degree of denominator *i.e.*, $g(x)$, then this type of rational function is called a **proper rational function.**
  - ***In rational polynomial function if the degree*** (*i.e.*, highest power of the variable) of numerator (N) is **greater than or equal to** the degree of denominator (D), then (without any doubt) **always perform the division** *i.e.*, divide the **N** by **D** before doing anything and thereafter use the following :

  $$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

  Scan to know more about this topic
  Integration by partial fraction

Table Demonstrating Partial Fractions of Various Forms

Form of the Rational Functions	Form of the Rational Functions
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$

$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c can't be factorized further.	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$

(c) Integration by Parts :

If u and v be two functions of x, then

$$\int \underset{\text{I}}{u}, \underset{\text{II}}{v}\, dx = u\left(\int v\,dx\right) - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

Scan to know more about this topic

Integration by Parts

Note

We can choose first function as the one whose initials comes first in the word. 'ILATE', where

I - Inverse Trigonometric function
L - Logarithm function
A - Algebraic function
T - Trigonometric function
E - Exponential function

Know the Formulae

(1) $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C, n \neq -1$

(2) $\int \dfrac{1}{x} dx = \log|x| + C$

(3) $\int e^x dx = e^x + C$

(4) $\int a^x dx = \dfrac{a^x}{\log_e a} + C$

(5) $\int \sin x\, dx = -\cos x + C$

(6) $\int \cos x\, dx = \sin x + C$

(7) $\int \sec^2 x\, dx = \tan x + C$

(8) $\int \text{cosec}^2 x dx = -\cot x + C$

(9) $\int \sec x \tan x\, dx = \sec x + C$

(10) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$

(11) $\int \tan x\, dx = -\log|\cos x| + C = \log|\sec x| + C$

(12) $\int \cot x\, dx = \log|\sin x| + C$

(13) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(14) $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C$

(15) $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C; |x| < 1$

(16) $\int \dfrac{dx}{1+x^2} = \tan^{-1} x + C$

(17) $\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C; |x| > 1$

(18) $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

(19) $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + C$

(20) $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + C$

(21) $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

(22) $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + C$

(23) $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + C$

(24) $\int \sqrt{a^2-x^2}dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right) + C$

(25) $\int \sqrt{x^2+a^2}dx = \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right| + C$

(26) $\int \sqrt{x^2-a^2}dx = \dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + C$

➤ Some Standard Substitutions

Following are some substitutions useful in evaluating integrals :

Expression	Substitution
(a^2+x^2), $\sqrt{x^2+a^2}$, $\frac{1}{\sqrt{x^2+a^2}}$	$x = a \tan\theta$ or $a \cot\theta$
(a^2-x^2), $\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$	$x = a \sin\theta$ or $a \cos\theta$
(x^2-a^2), $\sqrt{x^2-a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$	$x = a \sec\theta$ or $a \operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$, $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$, $\sqrt{(x-\alpha)(\beta-x)}$	$x = \alpha \cos^2\theta + \beta \sin^2\theta$
$\sqrt{\frac{x}{a+x}}$, $\sqrt{\frac{a+x}{x}}$, $\sqrt{x(a+x)}$	$x = a \tan^2\theta$ or $a \cot^2\theta$
$\sqrt{2ax-x^2}$	$x = a(1-\cos\theta)$
$\sqrt{\frac{x}{a-x}}$, $\sqrt{\frac{a-x}{x}}$, $\sqrt{x(a-x)}$, $\frac{1}{\sqrt{x(a-x)}}$	$x = a \sin^2\theta$ or $a \cos^2\theta$
$\sqrt{\frac{x}{x-a}}$, $\sqrt{\frac{x-a}{x}}$, $\sqrt{x(x-a)}$, $\frac{1}{\sqrt{x(x-a)}}$	$x = a \sec^2\theta$ or $a \operatorname{cosec}^2\theta$

➤ Solving Integrals of following types :

(i) $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ or $\int \sqrt{ax^2+bx+c}\, dx$

Working Rule :

(a) Make the coefficient of x^2 unity by taking the coefficient of x^2 outside the quadratic.

(b) Express $ax^2 + bx + c$ as sum or difference of two squares

$$i.e., ax^2 + bx + c = \left[\left(x+\frac{b}{2a}\right)^2 \pm \frac{4ac-b^2}{4a^2}\right]$$

(c) Then integrand is converted into known integrals.

(ii) $\int \frac{px+q}{ax^2+bx+c}dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ or $\int (px+q)\sqrt{ax^2+bx+c}\, dx$

Working Rule :

(a) Put $px + q = \lambda \frac{d}{dx}(ax^2 + bx + c) + \mu$

(b) Compare the coefficient of x and constant term on both sides, to obtain values of λ and μ.

(c) Re-substitute the values of λ and μ and integrate it,

(iii) $\int \frac{P(x)}{ax^2+bx+c}dx$, where $P(x)$ is a polynomial in x of degree $n \geq 2$

Working Rule :

(a) Divide $P(x)$ by $ax^2 + bx + c$ and write in the form $\frac{P(x)}{ax^2+bx+c} = Q(x) + \frac{R(x)}{ax^2+bx+c}$

Where $R(x)$ is a linear expression or constant and $Q(x)$ is quotient.

(b) How, integral reduces to the form discussed earlier.

(iv) $\int\frac{dx}{a+b\sin^2 x}, \int\frac{dx}{a+b\cos^2 x}, \int\frac{dx}{a\sin^2 x+b\cos^2 x}, \int\frac{dx}{a\sin^2 x+b\cos^2 x+c}, \int\frac{dx}{(a\sin x+b\cos x)^2}$

(a) Divide numerator and denominator by $\cos^2 x$.

(b) Replace $\sec^2 x$, if any, in denominator by $1 + \tan^2 x$.

(c) Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$, then the given integral becomes standard form of integrals in t variables.

(d) After integration, convert t variables in terms of x variables.

(v) $\int\frac{dx}{a+b\sin x}, \int\frac{dx}{a+b\cos x}, \int\frac{dx}{a\sin x+b\cos x}, \int\frac{dx}{a\sin x+b\cos x+c},$

(a) Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

(b) Replace $1+\tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$.

(c) Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$ *i.e.*, $\sec^2\frac{x}{2}dx = 2\,dt$, then the given integral becomes standard form of integrals in t variables.

(d) After integration, convert t variables in terms of x variables.

(vi) $\int\frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx,$

Working Rule :

(a) Write, numerator = λ (differentiation of denominator) + μ (Denominator, *i.e.*, $a\sin x + b\cos x = \lambda(c\cos x - d\sin x) + \mu(c\sin x + d\cos x)$

(b) Obtain the values of λ and μ by equating the coefficients of $\sin x$ and $\cos x$ on both the sides.

(c) Replace numerator in the integrand and solve it.

(vii) $\int\frac{x^2+1}{x^4+\lambda x^2+1}dx, \int\frac{x^2-1}{x^4+\lambda x^2+1}dx, \int\frac{1}{x^4+\lambda x^2+1}dx$

Working Rule :

(a) Divide numerator and denominator by x^2.

(b) Express the denominator of integrands either of the form of $\left(x+\frac{1}{x}\right)^2 \pm k^2$ or $\left(x-\frac{1}{x}\right)^2 \pm k^2$.

(c) Introduce $d\left(x+\frac{1}{x}\right)$ or $d\left(x-\frac{1}{x}\right)$ or both in numerator.

(d) Put $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$ as the case may be required.

(e) Integral reduced to the form of $\int\frac{1}{x^2+a^2}dx$ or $\int\frac{1}{x^2-a^2}dx$.

(f) For the integrals $\int\frac{x^2+a^2}{x^2+kx^2+a^4}dx$ or $\int\frac{x^2-a^2}{x^4+kx^2+a^4}dx$

After dividing both numerator and denominator by x^2.

Put $x-\frac{a^2}{x}=t$ or $x+\frac{a^2}{x}=t$, then the given integral becomes standard form of integrals in t variables.

(g) After integration, convert t variables in terms of x variables.

(viii) $\int\sqrt{\tan x}\,dx, \int\sqrt{\cot x}\,dx$

Working Rule :

(a) Put $\tan x = t^2 \Rightarrow \frac{d(\tan x)}{dx} = \frac{d(t^2)}{dt} \Rightarrow \sec^2 x\,dx = 2t\,dt$

(b) Given integral becomes standard form of integrals in t variables.

- Special form of Integrals :
  - (a) $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$
  - (d) $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

1 Mark Questions

AI Q. 1. $\int \frac{dx}{e^x+e^{-x}}$ **is equal to**

(a) $\tan^{-1}(e^x)+C$ **(b)** $\tan^{-1}(e^{-x})+C$

(c) $\log(e^x-e^{-x})+C$ **(d)** $\log(e^x+e^{-x})+C$

Sol. Correct option : (a)

Explanation :

Let $I=\int \frac{dx}{e^x+e^{-x}}dx=\int \frac{e^x}{e^{2x}+1}dx$

Also, let $e^x = t$

$\Rightarrow \quad e^x\,dx = dt$

$$\therefore \quad I=\int \frac{dt}{1+t^2}$$

$$=\tan^{-1}t+C$$

$$=\tan^{-1}\left(e^x\right)+C$$

Q. 2. $\int \frac{dx}{\sqrt{9-25x^2}}$

(a) $\sin^{-1}\left(\frac{5x}{3}\right)+C$ **(b)** $\frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right)+C$

(c) $\frac{1}{6}\log\left(\frac{3+5x}{3-5x}\right)+C$ **(d)** $\frac{1}{30}\log\left(\frac{3+5x}{3-5x}\right)+C$

Sol. Correct option : (b)

Explanation :

$$\int \frac{dx}{\sqrt{9-25x^2}}dx = \int \frac{1}{\sqrt{3^2-(5x)^2}}dx$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right)+C$$

Q. 3. If $\frac{d}{dx}f(x) = g(x)$**, then find anti-derivative of** $g(x)$**.**

Sol. $f(x)$, as $\frac{d}{dx}f(x) = g(x)$

$$\Rightarrow \quad \int g(x)dx = f(x)$$

AI Q. 4. Evaluate : $\int \frac{1}{\sin^2 x\cos^2 x}dx$

Sol. $$\int \frac{1}{\sin^2 x\cos^2 x}dx = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx$$

$$= \int(\sec^2 x+\text{cosec}^2 x)dx$$

$$= \tan x - \cot x + C$$

Q. 5. Evaluate : $\int \cot^2 x\,dx$

Sol. $$\int \cot^2 x dx = \int(\text{cosec}^2 x-1)dx$$

$$= -\cot x - x + C$$

Q. 6. Evaluate : $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Sol. Let $\quad I = \int \frac{\cos\sqrt{x}}{x}dx$

Put $\quad \sqrt{x} = t$

$$\Rightarrow \quad \frac{1}{2\sqrt{x}}dx = dt \Rightarrow dx = 2\sqrt{x}dt$$

$$\therefore \quad I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos t\,dt$$

$$= 2\sin t + C = 2\sin\sqrt{x}+C$$

Q. 7. If $\int \frac{1}{\sqrt{4-9x^2}}dx = \frac{1}{3}\sin^{-1}(ax)+C,$ **then find value of** a **.**

Sol. $$\int \frac{1}{\sqrt{4-9x^2}}dx = \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2-x^2}}dx$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+C$$

$$\Rightarrow \quad a = \frac{3}{2}$$

Q. 8. Find : $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$ **.**

Sol. Let, $\quad I = \int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$

$$= \int(\sec^2 x-\text{cosec}^2 x)dx$$

$$= \tan x + \cot x + C$$

AI Q. 9. Find : $\int \frac{\sin^6 x}{\cos^8 x}dx.$

Sol. Let, $\quad I = \int \tan^6 x\sec^2 x\,dx$

Put, $\quad t = \tan x$

$\quad dt = \sec^2 x\,dx$

$$\therefore \quad I = \int t^6\,dt$$

$$= \frac{t^7}{7}+C = \frac{\tan^7 x}{7}+C.$$

Q. 10. Evaluate : $\int \cos^{-1}(\sin x)dx$**.**

Sol. $$\int \cos^{-1}\left[\cos\left(\frac{\pi}{2}-x\right)\right]dx = \int\left(\frac{\pi}{2}-x\right)dx$$

$$= \int \frac{\pi}{2}dx - \int x\,dx$$

$$= \frac{\pi}{2}x - \frac{x^2}{2} + C$$

Q. 11. Find : $\int \frac{3+3\cos x}{x+\sin x}\,dx$

Sol. Let $x + \sin x = t$

So $(1 + \cos x)dx = dt$

$$I = 3\int \frac{dt}{t} = 3\log|t| + c = 3\log|(x+\sin x)| + c$$

or directly by writing formula

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$$

Examiner's Comment

- Some students split the numerator and apply by parts.

Answering Tip

- Put the denominator as t; if numerator is the derivative of denominator.

AI **Q. 12. E valuate :** $\int (\cot x - \operatorname{cosec}^2 x)e^x dx$

Sol. $I = \int (\cot x - \operatorname{cosec}^2 x)e^x dx$

Consider, $f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$

$$\left[\because \int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C\right]$$

$\therefore I = e^x \cot x + C$

Q. 13. Find : $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x}\,dx$

Sol. $\int \frac{\sin^2 x - \cos^2 x}{\sin x\cos x}dt = \int\left(\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}\right)dx$

$$= \int \tan x\,dx - \int \cot x\,dx$$

$$= \log|\sec x| - \log|\sin x| + C$$

Examiner's Comment

- Few candidates attempt it by integration by parts and make the steps complicated they could not proceed further.

Answering Tip

- This needs to be sufficiently practiced by students. Also, make students revise trigonometric and algebraic laws as well as basic integrals.

AI **Q. 14. Find :** $\int \frac{1}{x(1+\log x)}\,dx$

Sol. Let $1 + \log x = t$

$$\left(\frac{1}{x}\right)dx = dt$$

$$dx = x\,dt$$

$$I = \int \frac{1}{x(t)}x\,dt$$

$$= \int \frac{1}{t}dt$$

$$= \log|t| + C$$

$$= \log|1 + \log x| + C$$

Q. 15. Find $\int xe^{(1+x^2)}dx$

Sol. Let $(1 + x^2) = t$

so $2x\,dx = dt$

$$\Rightarrow I = \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t + c = \frac{1}{2}e^{(1+x^2)} + c$$

Q. 16. Find : $\int \frac{3x}{3x-1}\,dx$

Sol. $\int \frac{3x}{3x-1}dx = \int \frac{3x-1+1}{3x-1}dx = \int\left(1 + \frac{1}{3x-1}\right)dx$

$$= x + \frac{1}{3}\log|3x-1| + C$$

2 Marks Questions

Q. 1. Evaluate : $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$

[ISC Specimen Paper, 2021] [ISC, 2019]

Sol. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx = \int \frac{1/\cos^2 x}{1/\sin^2 x}dx$

$$= \int \tan^2 x\,dx = \int \sec^2(x-1)dx$$

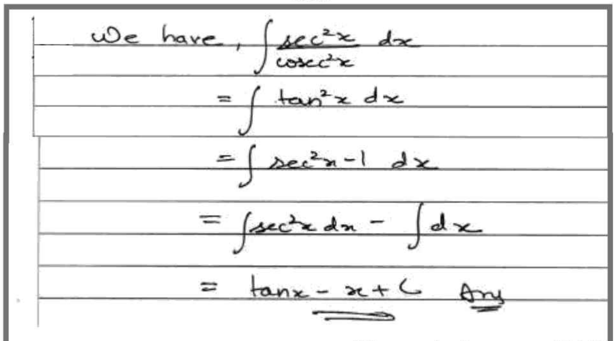
$$[\text{As } \sec^2 x = 1 + \tan^2 x]$$

$$= \int \sec^2 dx - \int 1dx$$

$$= \tan x - x + C$$

[ISC Marking Scheme, 2019]

OR



We have, $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$

$= \int \tan^2 x\,dx$

$= \int \sec^2 x - 1\,dx$

$= \int \sec^2 x\,dx - \int dx$

$= \tan x - x + C$ Ans

[Topper's Answer, 2019]

Q. 2. Evaluate : $\int \frac{x^3+5x^2+4x+1}{x^2}dx$ **[ISC, 2018]**

Sol. Let
$$I = \int \frac{x^3+5x^2+4x+1}{x^2}dx$$
$$= \int\left(x+5+\frac{4}{x}+\frac{1}{x^2}\right)dx$$
$$= \int x\,dx + 5\int 1dx + 4\int\frac{1}{x}dx + \int\frac{1}{x^2}dx$$
$$= \frac{x^2}{2}+5x+4\log x+\left(\frac{-1}{x}\right)+C$$
$$= \frac{x^2}{2}+5x+4\log|x|-\frac{1}{x}+C$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- This question was attempted correctly by most of the candidates some of the candidates made errors in simplifying the expression before integration.

Answering Tip

- Do practice to simplify different types of algebraic expressions in the form of required simplest form for integration.

AI Q. 3. Evaluate : $\int\frac{1}{x^2}\sin^2\left(\frac{1}{x}\right)dx$ **[ISC, 2017]**

Sol. Let $I = \int\frac{1}{x^2}\sin^2\left(\frac{1}{x}\right)dx$

Also, Let $t = \frac{1}{x}$

$\therefore \quad dt = \frac{-1}{x^2}dx$

$\therefore \quad I = -\int\sin^2 t + dt$

$$= -\int\frac{1-\cos 2t}{2}dt$$

$[\because \cos 2\theta = 1-2\sin^2\theta]$

$$= -\frac{1}{2}\left[\int 1dt - \int\cos 2t\,dt\right]$$
$$= -\frac{1}{2}\left[t-\frac{\sin 2t}{2}\right]+C$$
$$= \frac{1}{4}\sin 2t-\frac{1}{2}t+C$$

$\therefore \quad I = \frac{1}{4}\sin\left(\frac{2}{x}\right)-\frac{1}{2x}+C$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Candidates made errors by making inappropriate substitution. Some candidates applied correct substitution $\left(\frac{1}{x}=t\right)$ but made errors in further simplification. Many attempted it by incorrect method of integration by parts so could not get the correct solution.

Answering Tip

- Practice should be needed for the questions of integration by substitution.

Q. 4. Evaluate : $\int\frac{x}{(x+1)^2}dx$. **(ISC, 2010)**

Sol. Given, $\int\frac{x}{(x+1)^2}dx = \int\left[\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]dx$

$$= \log(x+1)+\frac{1}{(x+1)}+C$$

Here, $\frac{x}{(x+1)^2} = \frac{(x+1)-1}{(x+1)^2}$

$$= \frac{x+1}{(x+1)^2}-\frac{1}{(x+1)^2}$$
$$= \frac{1}{x+1}-\frac{1}{(x+1)^2}$$

AI Q. 5. Evaluate : $\int\frac{e^{2x}}{2+e^x}dx$. **[ISC, 2009]**

Sol. Given, $\int\frac{e^{2x}}{2+e^x}dx$

Let $e^x+2 = t$

$e^x = t-2$

$\Rightarrow \quad e^x\,dx = dt$

$\therefore \quad \frac{(t-2)}{t} = t-2\log|t| + C$

$= e^x - 2\log|e^x+2| + C$

[Since $e^{2x}\,dx = e^x.\,e^x\,dx$

$= e^x.e^x\,dx = (t–2)\,dt$]

Q. 6. Evaluate : $\int\frac{x^2}{x^2-4}dx$. **[ISC, 2008]**

Sol. Given, $\int\frac{x^2}{x^2-4}dx$

$\therefore \quad \int\frac{x^2-4+4}{x^2-4}dx = \int\left(\frac{x^2-4}{x^2-4}+\frac{4}{x^2-4}\right)dx$

$\Rightarrow \quad \int\left(1+\frac{4}{x^2-4}\right)dx = x+4\times\frac{1}{2\times 2}\log\left|\frac{x-2}{x+2}\right|+C$

$$= x+\log\left|\frac{x-2}{x+2}\right|+C$$

Q. 7. Solve : $\int \frac{1}{x\cos^2(1+\log x)}dx$. **[ISC, 2007]**

Sol. Given, $\int \frac{1}{x\cos^2(1+\log x)}dx$

Let $1+\log x = t$

On differentiating, we get

$$\frac{1}{x}dx = dt$$

$$\therefore \int \frac{1}{x\cos^2(1+\log x)}dx = \int \frac{dt}{\cos^2 t}$$

$$= \int \sec^2 t\,dt = \tan t + C$$

$$= \tan(1 + \log x) + C$$

Q. 8. Find : $\int \frac{(x^4-x)^{\frac{1}{4}}}{x^5}$

Sol. Let, $\mathrm{I} = \int \frac{(x^4-x)^{\frac{1}{4}}}{x^5}$

$$= \int \left[x^4\left(1-\frac{x}{x^4}\right)\right]^{\frac{1}{4}} \frac{1}{x^5}dx$$

$$= \int x\left(1-\frac{1}{x^3}\right)^{\frac{1}{4}} \frac{1}{x^5}dx$$

$$= \int \left(1-\frac{1}{x^3}\right)^{\frac{1}{4}} \frac{1}{x^4}dx$$

Put $1 - \frac{1}{x^3} = t$

So that, $\frac{3}{x^4}dx = dt \Rightarrow dx = \frac{x^4}{3}dt$

$$= \frac{1}{3}\int t^{\frac{1}{4}}dt = \frac{1}{3}.\frac{4}{5}t^{\frac{5}{4}}$$

$$= \frac{4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}} + C$$

4 Marks Questions

Q. 1 Evaluate : $\int \frac{dx}{\sqrt{5x-4x^2}}$ **[ISC, 2020]**

Sol. $\int \frac{dx}{\sqrt{5x-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{5x}{4}-x^2}}$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{\frac{25}{64}-\left(x^2-\frac{5x}{4}+\frac{25}{64}\right)}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{5}{8}\right)^2-\left(x-\frac{5}{8}\right)^2}}$$

$$= \frac{1}{2}\sin^{-1}\left|\frac{x-\frac{5}{8}}{\frac{5}{8}}\right| + c$$

$$= \frac{1}{2}\sin^{-1}\left|\frac{8x-5}{5}\right| + c$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Some candidates are not able to factorize the expression under radical sign in the required standard form.

Answering Tip

- Revise the concept of factorization *i.e.* factorization using by making whole square and practice more similar type questions.

Q. 2 Evaluate : $\int \sin^3 x\cos^4 x\,dx$. **[ISC, 2020]**

Sol. $\mathrm{I} = \int \sin^3 x\cos^4 x\,dx$

Let $\cos x = t$

$-\sin x\,dx = dt$

$$dx = \frac{-dt}{\sin x}$$

$$I = \int \sin^3 x t^4\left(\frac{-dt}{\sin x}\right)$$

$$= -\int \sin^2 x t^4 dt$$

$$= -\int (1-t^2)t^4 dt$$

$$= -\int (t^4-t^6)dt$$

$$I = \frac{-t^5}{5}+\frac{t^7}{7}+c$$

$$= \frac{-\cos^5 x}{5}+\frac{\cos^7 x}{7}+c$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Most of the candidates made errors in substitution and simplification or applying integration by parts concept for further simplification.

Answering Tip

- Do adequate practice for solving problems based on integration by parts.

Q. 3 Evaluate : $\int \frac{x(1+x^2)}{1+x^4}dx$ **[ISC, 2019]**

Sol. $\int \frac{x(1+x^2)}{1+x^4}dx \qquad x^2 = t \qquad dt = 2x\,dx$

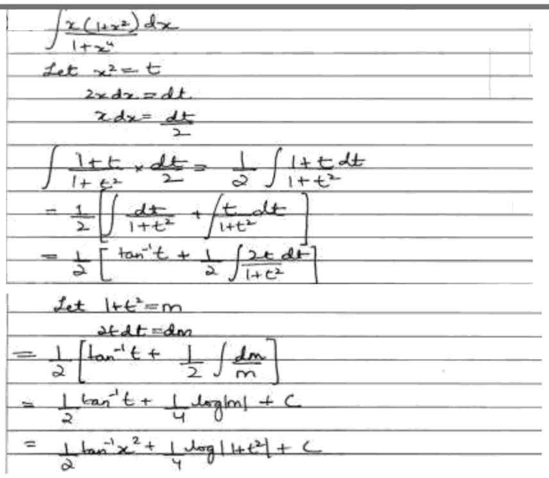
$$\frac{1}{2}\int \frac{1+t}{1+t^2}dt = \frac{1}{2}\int \frac{1}{1+t^2}dt + \frac{1}{4}\int \frac{2t}{1+t^2}dt$$

$$= \frac{1}{2}\tan^{-1}t + \frac{1}{4}\log(1+t^2) + C$$

$$= \frac{1}{2}\tan^{-1}x^2 + \frac{1}{4}\log(1+x^4) + C$$

[ISC Marking Scheme ,2019]

OR



$\int \frac{x(1+x^2)}{1+x^4}dx$

Let $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$\int \frac{1+t}{1+t^2} \times \frac{dt}{2} = \frac{1}{2}\int \frac{1+t}{1+t^2}dt$

$= \frac{1}{2}\left[\int \frac{dt}{1+t^2} + \int \frac{t\,dt}{1+t^2}\right]$

$= \frac{1}{2}\left[\tan^{-1}t + \frac{1}{2}\int \frac{2t\,dt}{1+t^2}\right]$

Let $1+t^2 = m$

$2t\,dt = dm$

$= \frac{1}{2}\left[\tan^{-1}t + \frac{1}{2}\int \frac{dm}{m}\right]$

$= \frac{1}{2}\tan^{-1}t + \frac{1}{4}\log|m| + C$

$= \frac{1}{2}\tan^{-1}x^2 + \frac{1}{4}\log|1+t^2| + C$

[Topper's Answer, 2019]

AI **Q. 4. Evaluate :** $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$ **[ISC Specimen, 2018]**

Sol. Let $I = \int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$

$$= \int \frac{6x+7}{\sqrt{x^2-9x+20}}dx$$

$$= \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$= 3\int \frac{(2x-9)}{\sqrt{x^2-9x+20}}dx + 34\int \frac{dx}{\sqrt{x^2-9x+20}}$$

$$= \frac{3(x^2-9x+20)^{\frac{1}{2}}}{\frac{1}{2}} + 34\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2 + \left(20-\frac{81}{4}\right)}}$$

$$= 6\sqrt{x^2-9x+20} + 34\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= 6\sqrt{x^2-9x+20} + 34\log\left|x-\frac{9}{2}+\sqrt{\left(x-\frac{9}{2}\right)^2-\frac{1}{4}}\right| + C$$

$$= 6\sqrt{x^2-9x+20} + 34\log\left|\frac{2x-9}{2}+\sqrt{x^2-9x+20}\right| + C$$

Q. 5. Evaluate : $\mathbf{I = \int \tan^3 x\,dx}$ **[ISC, 2016]**

Sol. $I = \int \tan^3 x\,dx$

$$= \int \tan x \tan^2 x\,dx$$

$$= \int \tan x(\sec^2 x - 1)\,dx$$

$$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$$= I_1 - (-\log|\cos x|)$$

Now, $I_1 = \int \tan x \sec^2 x\,dx$

Put, $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$I_1 = \int t\,dt = \frac{t^2}{2} = \frac{1}{2}\tan^2 x$$

$$I = \frac{1}{2}\tan^2 x + \log|\cos x| + C$$

[ISC Marking Scheme, 2016]

Q. 6. Evaluate : $\int \frac{2y^2}{y^2+4}dy$ **[ISC, 2015]**

Sol. $\int \frac{2y^2}{y^2+4}dy$

$$= 2\int \frac{y^2+4-4}{y^2+4}dy = 2\int 1dy - 8\int \frac{dy}{y^2+4}dy$$

$$= 2y - 8.\frac{1}{2}\tan^{-1}\frac{y}{2} + C$$

$$= 2y - 4\tan^{-1}\frac{y}{2} + C$$

[ISC Marking Scheme, 2015]

AI **Q. 7. Evaluate :** $\int e^x \frac{(2+\sin 2x)}{\cos^2 x}dx$ **[ISC, 2014]**

Sol. $\int e^x(2+\sin 2x)dx/\cos^2 x$

$$= \int e^x(2\sec^2 x + 2\tan x)dx$$

$$= 2\int e^x \sec^2 x dx + 2(\tan x\, e^x - \int \sec^2 x\, e^x dx)$$

$= 2e^x \tan x + c$ **[ISC Marking Scheme, 2014]**

Detailed answer :

$$I = \int e^x\left(\frac{2}{\cos^2 x} + \frac{2\sin x\cos x}{\cos^2 x}\right)dx$$

$$= \int e^x(2\sec^2 x + 2\tan x)dx$$

$$= 2\int e^x(\sec^2 x + \tan x)dx$$

$$= 2\left[\int e^x \sec^2 x\,dx + \int e^x \tan x\,dx\right]$$

$$= 2\left[e^x \int \sec^2 x\,dx - \int\left\{\frac{d}{dx}e^x \int \sec^2 x\,dx\right\}dx + \int e^x \tan x\,dx\right]$$

$$= 2\left[e^x \tan x - \int e^x \tan x\,dx + \int e^x \tan x\,dx\right]$$

$$= 2e^x \tan x + C$$

Examiner's Comment

- Several candidates could not convert the given integral, $\int e^x.F(x)dx$ into two parts of the form $\int e^x.\{f(x) + f'(x)\}$ and then use by parts rule or formula.

Answering Tip

- $\int e^x\{f(x) + f'(x)\}$ is specific situation in integration by parts. This needs to be sufficiently practiced by students. Also, revise trigonometric and algebraic laws as well as basic integrals.

Q. 8. Evaluate : $\int \frac{1}{x+\sqrt{x}}dx$ **[ISC, 2013]**

Sol. $\int \frac{1}{x+\sqrt{x}}dx$

Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore \quad \int \frac{dx}{x+\sqrt{x}} = \int \frac{2t\,dt}{t^2+t} = \int \frac{2t}{t(t+1)}dt = 2\int \frac{1}{t+1}dt$$

$$= 2 \log |t + 1| + C$$

$$= 2\log(\sqrt{x}+1) + C$$

Q. 9. Evaluate : $\int e^x(\tan x + \log \sec x)dx$ **[ISC, 2012]**

Sol. $\int e^x(\tan x + \log \sec x)dx$

$$= \int e^x \tan x\,dx + \int \underset{\text{II}}{e^x} \underset{\text{I}}{\log \sec x}\,dx$$

$$= \int e^x \tan x\,dx + \log(\sec x)e^x - \int\left[\frac{d}{dx}(\log \sec x).\int e^x\,dx\right]$$

$$= \int e^x \tan x\,dx + \log \sec x\,e^x - \int e^x \tan x\,dx$$

$$= e^x \log \sec x + C$$

Q. 10. Evaluate : $\int \frac{\operatorname{cosec} x}{\log\left[\tan\left(\frac{x}{2}\right)\right]}dx$. **[ISC, 2011]**

Sol. Given, $\int \frac{\operatorname{cosec} x}{\log\left[\tan\left(\frac{x}{2}\right)\right]}dx$

Let $\quad \log\left[\tan\left(\frac{x}{2}\right)\right] = t$

$$\Rightarrow \quad \frac{1}{\tan\frac{x}{2}}.\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

$$\Rightarrow \quad \frac{1}{2}.\frac{\cos x/2}{\sin x/2\cos^2 x/2}dx = dt$$

or $\quad \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx = dt \Rightarrow \frac{1}{\sin x}dx = dt$

or $\quad \operatorname{cosec} x\,dx = dt$

$$\therefore \quad \int \frac{\operatorname{cosec} x\,dx}{\log\tan\left(\frac{x}{2}\right)} = \int \frac{dt}{t}$$

$$= \log t + C = \log\left|\log\tan\frac{x}{2}\right| + C$$

Q. 11. Evaluate : $\int e^{-2x}\sin x\,dx$. **[ISC, 2010]**

Sol. $I = \int \underset{\text{II}}{e^{-2x}}\,\underset{\text{I}}{\sin x}\,dx$

$$\Rightarrow \quad I = \sin x e^{-2x}\left(-\frac{1}{2}\right) - \int \cos x\,e^{-2x}\left(-\frac{1}{2}\right)dx$$

[using integration by parts]

$$= -\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\int \underset{\text{II}}{e^{-2x}}\,\underset{\text{I}}{\cos x}\,dx$$

$$= -\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\left[\cos x\cdot e^{-2x}\left(-\frac{1}{2}\right) - \int -\sin x\,e^{-2x}\left(-\frac{1}{2}\right)dx\right]$$

[using integration by parts]

$$= -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x - \frac{1}{4}\int e^{-2x}\sin x\,dx$$

$$\Rightarrow \quad I = -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x - \frac{1}{4}I + C'$$

$$\Rightarrow I + \frac{1}{4}I = -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x + C'$$

$$\Rightarrow \quad \frac{5}{4}I = -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x + C'$$

$$\Rightarrow \quad I = -\frac{2}{5}e^{-2x}\sin x - \frac{1}{5}e^{-2x}\cos x + C$$

$$\left[\text{put } \frac{4}{5}C' = C\right]$$

AI **Q. 12. Evaluate :** $\int e^x\left[\frac{1+\sin x}{1+\cos x}\right]dx$. **[ISC, 2000]**

Sol. Let $\quad I = \int e^x\left[\frac{1+\sin x}{1+\cos x}\right]dx$

$$= \int \frac{e^x dx}{1+\cos x} + \int \frac{e^x \sin x}{1+\cos x}dx$$

$$= \int \frac{e^x}{2\cos^2\frac{x}{2}}dx + \int \frac{e^x 2\sin x/2.\cos x/2}{2\cos^2 x/2}$$

$$\left[\begin{array}{l}\because \ \sin\theta = 2\sin\theta/2\cos\theta/2 \\ 1+\cos\theta = 2\cos^2\theta/2\end{array}\right]$$

$$= \frac{1}{2}\int \underset{\text{II}}{e^x} \underset{\text{I}}{\sec^2 \frac{x}{2}} dx + \int \underset{\text{II}}{e^x} \underset{\text{I}}{\tan \frac{x}{2}} dx$$

$$= \int e^x \left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2} \right) dx$$

$$= e^x . \tan\frac{x}{2} + C$$

$$\left[\because e^x[f(x) + f'(x)]dx = e^x f(x) + C\right]$$

Q. 13. Evaluate : $\int \frac{\sqrt{x}}{a^3 - x^3} dx$

Sol. $I = \int \frac{\sqrt{x}}{a^3 - x^3} dx$

$$\Rightarrow I = \int \frac{(x)^{1/2}}{\left(a^{3/2}\right)^2 - \left(x^{3/2}\right)^2} dx$$

Put $x^{3/2} = t$

$\therefore \quad \frac{3}{2}\sqrt{x}\, dx = dt$

$\Rightarrow \quad \sqrt{x}\, dx = \frac{2}{3} dt$

$$\therefore \quad I = \frac{2}{3}\int \frac{dt}{\left(a^{3/2}\right)^2 - t^2}$$

$$\Rightarrow \quad I = \frac{2}{3} \times \frac{1}{2a^{3/2}} \log\left|\frac{a^{3/2} + t}{a^{3/2} - t}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{3a^{3/2}} \log\left|\frac{a^{3/2} + x^{3/2}}{a^{3/2} - x^{3/2}}\right| + C$$

Q. 14. Find : $\int \frac{x+3}{\sqrt{5 - 4x - 2x^2}} dx.$

Sol. Let $I = \int \frac{x+3}{\sqrt{5 - 4x - 2x^2}} dx$

Let $x + 3 = \lambda \frac{d}{dx}(5 - 4x - 2x^2) + \mu$. Then

$x + 3 \sim \lambda(-4 - 4x) + \mu$

Comparing the coefficients of like terms of x, we get

$-4\lambda = 1$ and $-4\lambda + \mu = 3 \Rightarrow \lambda = \frac{-1}{4}$ and $\mu = 2$

$$\therefore \quad I = \int \frac{-\frac{1}{4}(-4 - 4x) + 2}{\sqrt{5 - 4x - 2x^2}} dx$$

$$I = -\frac{1}{4}\int \frac{-4 - 4x}{\sqrt{5 - 4x - 2x^2}} dx + 2\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$$

$$= -\frac{1}{4} 2.\sqrt{5 - 4x - 2x^2} + \frac{2}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\frac{7}{2}\right)^2 - (x+1)^2}}$$

$$= -\frac{1}{2}\sqrt{5 - 4x - 2x^2} + \sqrt{2}\sin^{-1}\left(\frac{x+1}{\sqrt{7/2}}\right) + C$$

Q. 15. Evaluate the following indefinite integral :

$$\int \frac{\sin\phi}{\sqrt{\sin^2\phi + 2\cos\phi + 3}} d\phi .$$

Sol. Let $I = \int \frac{\sin\phi}{\sqrt{\sin^2\phi + 2\cos\phi + 3}} d\phi$

$$= \int \frac{\sin\phi}{\sqrt{1 - \cos^2\phi + 2\cos\phi + 3}} d\phi$$

$$= \int \frac{\sin\phi}{\sqrt{-\cos^2\phi + 2\cos\phi + 4}} d\phi$$

Put $\cos\phi = t \Rightarrow -\sin\phi\, d\phi = dt$

$$= \int \frac{-1}{\sqrt{-t^2 + 2t + 4}} dt$$

$$= \int \frac{1}{\sqrt{(\sqrt{5})^2 - (t-1)^2}} dt$$

$$= -\sin^{-1}\left(\frac{t-1}{\sqrt{5}}\right) + C$$

$$= -\sin^{-1}\left(\frac{\cos\phi - 1}{\sqrt{5}}\right) + C$$

AI Q. 16. Integrate the following *w.r.t. x* : $\frac{x^2 - 3x + 1}{\sqrt{1 - x^2}}$

Sol. We have,

$$I = \int \frac{x^2 - 3x + 1}{\sqrt{1 - x^2}} dx$$

$$= \int \frac{2 - 3x - (1 - x^2)}{\sqrt{1 - x^2}} dx$$

$$= 2\int \frac{1}{\sqrt{1 - x^2}} dx - 3\int \frac{x}{\sqrt{1 - x^2}} dx - \int \sqrt{1 - x^2}\, dx$$

Let $(1 - x^2 = t)$ in second integral $\Rightarrow -x\, dx = \frac{dt}{2}$

$$I = 2\int \frac{1}{\sqrt{1 - x^2}} dx + \frac{3}{2}\int \frac{1}{\sqrt{t}} dx - \int \sqrt{1 - x^2}\, dx$$

$$\Rightarrow I = 2\sin^{-1}x + 3\sqrt{t} - \frac{x}{2}\sqrt{1 - x^2} - \frac{1}{2}\sin^{-1}x + C$$

$$\Rightarrow I = 2\sin^{-1}x + 3\sqrt{1 - x^2} - \frac{x}{2}\sqrt{1 - x^2} - \frac{1}{2}\sin^{-1}x + C$$

$[\because 1 - x^2 = t]$

$$= \frac{3}{2}\sin^{-1}x + \frac{1}{2}(6 - x)\sqrt{1 - x^2} + C$$

Q. 17. Evaluate : $\int \sqrt{(x-5)(7-x)}\, dx$

Sol. Let $I = \int \sqrt{(x-5)(7-x)}\, dx$

$$= \int \sqrt{-35 + 12x - x^2}\, dx$$

$$\Rightarrow \quad I = \int \sqrt{(36 - 35) - (x^2 - 12x + 36)}\, dx$$

$$= \int \sqrt{1^2 - (x-6)^2}\, dx$$

$$\therefore\ I = \frac{x-6}{2}\sqrt{1^2-(x-6)^2}+\frac{1}{2}\sin^{-1}\left|\frac{x-6}{1}\right|+C$$

$$\text{or } I = \frac{1}{2}(x-6)\sqrt{-35+12x-x^2}+\frac{1}{2}\sin^{-1}|x-6|+C$$

Q. 18. Evaluate : $\int\sqrt{14x-20-2x^2}\,dx.$

Sol. Let, $I=\int\sqrt{14x-20-2x^2}\,dx$

$$=\sqrt{2}\int\sqrt{-10+7x-x^2}\,dx$$

$$\Rightarrow\quad I=\sqrt{2}\int\sqrt{\left(\frac{49}{4}-10\right)-\left(x^2-7x+\frac{49}{4}\right)}\,dx$$

$$=\sqrt{2}\int\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2}\,dx$$

$$\Rightarrow\quad I=\sqrt{2}\left[\frac{x-\frac{7}{2}}{2}\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2}\right.$$

$$\left.+\frac{9}{2\times 4}\sin^{-1}\left(\frac{x-\frac{7}{2}}{\frac{3}{2}}\right)\right]+C$$

$$\therefore\quad I=\sqrt{2}\left[\frac{2x-7}{4}\sqrt{7x-10-x^2}\right.$$

$$\left.+\frac{9}{8}\sin^{-1}\left(\frac{2x-7}{3}\right)\right]+C$$

6 Marks Questions

AI **Q. 1. Evaluate :** $\int\tan^{-1}\sqrt{x}\,dx$

[ISC Specimen Paper, 2021], [ISC, 2018]

Sol. Let, $\sqrt{x}=t$

$$\Rightarrow\quad \frac{1}{2\sqrt{x}}dx = dt$$

$$=\int\tan^{-1}t.2\sqrt{x}dt$$

$$=2\int t\tan^{-1}t\,dt$$

$$=2\left[\tan^{-1}t\int t\,dt-\int\left[\frac{d}{dt}\tan^{-1}t\int t\,dt\right]dt\right]$$

$$=2\left[\frac{t^2}{2}\tan^{-1}t-\int\frac{1}{1+t^2}.\frac{t^2}{2}d.t\right]$$

$$=t^2\tan^{-1}t-\int\frac{1+t^2-1}{1+t^2}dt$$

$$=t^2\tan^{-1}t-\int\left(1-\frac{1}{1+t^2}\right)dt$$

$$=t^2\tan^{-1}t-\int dt+\int\frac{dt}{1+t^2}$$

$$=t^2\tan^{-1}t-t+\tan^{-1}t+C$$

$$=x\tan^{-1}\sqrt{x}-\sqrt{x}+\tan^{-1}\sqrt{x}+C$$

[ISC Making Scheme, 2018]

Q. 2. Evaluate : $\int\frac{3x+5}{x^2-x^2-x+1}dx$

[ISC Specimen Paper, 2021]

Sol. $\int\frac{3x+5}{x^3-x^2-x+1}dx = \int\frac{3x+5}{(x-1)^2(x+1)}dx$

$$\therefore\quad \frac{3x+5}{(x-1)^2(x+1)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$\therefore\ 3x+5 = A(x-1)2+B(x+1)(x-1)+C(x-1)$

when, $x = 1$

$3.1+5 = c.2$

$\therefore\quad c = 4$

when, $x = -1$

$3(-1)+5 = A(-2)^2$

$\Rightarrow\quad A = 1/2$

$3x+5 = A(x^2-2x+1)+Bx^2-B+Cx+C$

$= x^2(A+B)+x(C-2A)+(A-B+C)$

$\therefore\quad A+B = 0$

$B = 1/2$

$$\therefore\ \int\frac{3x+5}{(x-1)^2(x+1)}dx$$

$$=\frac{1}{2}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{dx}{x-1}+4\int\frac{dx}{(x-1)^2}$$

$$=\frac{1}{2}\ln|x+1|-\frac{1}{2}\ln|x-1|-\frac{4}{x-1}+C$$

$$=\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}+C$$

Q. 3. Evaluate : $\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$ **[ISC, 2020]**

Sol. $I=\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$

Let $x = \cos 2\theta$

$dx = -2\sin 2\theta\, d\theta$

$$I = \int \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\,(-2\sin 2\theta)d\theta$$

$$I = \int \tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\,(-2\sin 2\theta)d\theta$$

$$I = \int \tan^{-1}\tan\theta(-2\sin 2\theta)d\theta$$

$$I = -2\int \theta\sin 2\theta d\theta$$

$$= -2\left[\theta\int \sin 2\theta d\theta - \int\left[\frac{d}{d\theta}\theta\int \sin 2\theta d\theta\right]d\theta\right]$$

$$= -2\left[-\theta\frac{\cos 2\theta}{2} - \int\frac{-\cos 2\theta}{2}d\theta\right]$$

$$= \theta\cos 2\theta - \int \cos 2\theta d\theta$$

$$= \theta\cos 2\theta - \frac{\sin 2\theta}{2} + C$$

$$= \theta\cos 2\theta - \frac{1}{2}\sqrt{1-\cos^2 2\theta} + C$$

$$= \frac{1}{2}x\cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Some candidates made errors in substitution. In most of the cases,candidates who used appropriate substitution did not split the function into partial fraction to find the integral of each term.

Answering Tip

- Thoroughly understand and do adequate practice of integration using substitution, partial fraction and special integrals. Also, do not leave your answer in terms of new variable.

Q. 4. Evaluate : $\int \frac{2x+7}{x^2-x-2}dx$ **[ISC, 2020]**

Sol. $\int \frac{2x+7}{x^2-x-2}dx = \int \frac{2x+7}{(x-2)(x+1)}dx$

$$\frac{2x+7}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$2x+7 = A(x+1) + B(x-2)$$

Put, $x = -1$

$-2 + 7 = B(-1-2)$

$\Rightarrow \quad B = -\frac{5}{3}$

Put, $x = 2$

$4 + 7 = A(1 + 2)$

$\Rightarrow \quad A = \frac{11}{3}$

$$\int \frac{2x+7}{(x-2)(x+1)}dx$$

$$= \frac{11}{3}\int\frac{dx}{x-2} - \frac{5}{3}\int\frac{dx}{x+1}$$

$$= \frac{11}{3}\log|x-2| - \frac{5}{3}\log|x+1| + C$$

$$= \frac{1}{3}\left[\log\left|(x-2)^{11}\right| - \log\left|(x+1)^5\right|\right] + C$$

$$\int \frac{2x+7}{x^2-x-2}dx = \frac{1}{3}\log\left|\frac{(x-2)^{11}}{(x+1)^5}\right| + C$$

Examiner's Comment

- Some students fail to identify partial fraction or are unsuccessful in factorizing the denominator.

Answering Tip

- Partial function rule needs to be understand to apply correctly. Methods of proper substitution need attention.

Q. 5. Evaluate : $\int \frac{x-1}{\sqrt{x^2-x}}dx$ **[ISC, 2018]**

Sol. Let $I = \int \frac{x-1}{\sqrt{x^2-x}}dx$

Take $x - 1 = \lambda\frac{d}{dx}(x^2 - x) + \mu$

$\therefore \quad x - 1 = \lambda(2x - 1) + \mu$

Comparing the coefficients of like terms of x, we get

$2\lambda = 1$ and $-\lambda + \mu = -1$

$\Rightarrow \quad \lambda = \frac{1}{2}$ and $\mu = -\frac{1}{2}$

$$\therefore \quad I = \int\frac{\frac{1}{2}(2x-1) - \frac{1}{2}}{\sqrt{x^2-x}}dx$$

$$= \frac{1}{2}\int\frac{2x-1}{\sqrt{x^2-x}}dx - \frac{1}{2}\int\frac{1}{\sqrt{x^2-x}}dx$$

$$= \frac{1}{2}.2.\sqrt{x^2-x} - \frac{1}{2}\int\frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}dx$$

$$= \sqrt{x^2-x} - \frac{1}{2}\log\left[\left(x-\frac{1}{2}\right) + \sqrt{x^2-x}\right] + C$$

$$\therefore \quad I = \sqrt{x^2-x} - \frac{1}{2}\log\left|\left(x - \frac{-1}{2}\right) + \sqrt{x^2-x}\right| + C$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Many candidates applied incorrect techniques for solution of different types of integrals.

Answering Tip

- Learn rules for all standard methods of integrates. Practice problems based on them.

AI **Q. 6. Evaluate :** $\displaystyle\int \frac{3x+5}{x^3-x^2-x+1}dx$ **[ISC, 2018]**

Sol. $I = \displaystyle\int \frac{3x+5}{x^3-x^2-x+1}dx$

$$\frac{3x+5}{x^2(x-1)-(x-1)} = \frac{3x+5}{(x^2-1)(x-1)}$$

$$= \frac{3x+5}{(x+1)(x-1)^2}$$

Let $\displaystyle\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

$\Rightarrow \quad 3x + 5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$ ...(i)

Put $x = 1$ in equation (i)

$3(1) + 5 = B(2)$

$\Rightarrow \quad B = 4$

Put $x = -1$ in equation (i)

$3(-1) + 5 = C(-1-1)^2$

$\Rightarrow \quad C = \dfrac{1}{2}$

Put $x = 0$ in equation (i)

$5 = -A + B + C$

$\Rightarrow \quad A = \dfrac{-1}{2}$

$$\therefore \quad \frac{3x+5}{(x+1)(x-1)^2} = \frac{-\frac{1}{2}}{x-1} + \frac{4}{(x-1)^2} + \frac{\frac{1}{2}}{x+1}$$

$$\Rightarrow I = \frac{-1}{2}\int\frac{1}{x-1}dx + 4\int\frac{dx}{(x-1)^2} + \frac{1}{2}\int\frac{1}{x+1}dx$$

$$= \frac{-1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + C$$

$$I = \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C$$

Q. 7. Evaluate : $\displaystyle\int \frac{\sin 2x}{(1-\sin x)(2+\sin x)}dx$ **[ISC, 2017]**

Sol. Let $\displaystyle I = \int \frac{\sin 2x}{(1-\sin x)(2+\sin x)}dx$

$$= \int \frac{2\sin x\cos x}{(1+\sin x)(2+\sin x)}dx$$

Let $\sin x = t$

$\Rightarrow \quad \cos x\, dx = dt$

$$\therefore \quad I = \int \frac{2t}{(1+t)(2+t)}dt$$

Now, Let $\displaystyle\frac{t}{(1+t)(2+t)} = \frac{A}{(1+t)} + \frac{B}{(2+t)}$

$t = A(2 + t) + B(1 + t)$

$t = (A + B)t + (2A + B)$

Comparing coefficient of like terms of t both sides, we get

$A + B = 1$ and $2A + B = 0$

$\Rightarrow \quad A = -1$ and $B = 2$

$$\therefore \quad I = 2\int\left(\frac{-1}{1+t} + \frac{2}{2+t}\right)dt$$

$$= 2\left[\int\frac{-1}{1+t}dt + 2\int\frac{1}{2+t}dt\right]$$

$= 2[-\log|1 + t| + 2\log|2 + t|] + C$

$= -2\log|1 + \sin x| + 4\log|2 + \sin x| + C$

[ISC Marking Scheme, 2017]

Q. 8. Evaluate : $\displaystyle\int \frac{\sin x + \cos x}{\sqrt{9+16\sin 2x}}dx$ **[ISC, 2016]**

Sol. Let $\displaystyle I = \int \frac{\sin x + \cos x}{\sqrt{9+16\sin 2x}}dx$

Put $u = \sin x - \cos x$

$du = (\sin x + \cos x)\, dx$

$u^2 = 1 - \sin 2x$

$\therefore \sin 2x = 1 - u^2$

$$\therefore \quad I = \int\frac{du}{\sqrt{9+16(1-u^2)}}$$

$$= \int\frac{du}{\sqrt{25-16u^2}}$$

$$= \int\frac{du}{\sqrt{5^2-(4u)^2}}$$

$$= \frac{1}{4}\sin^{-1}\left(\frac{4u}{5}\right) + C$$

$$= \frac{1}{4}\sin^{-1}\left(\frac{4(\sin x - \cos x)}{5}\right) + C$$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Some candidates made mistakes in correct substitution. In most of the cases, candidates who used appropriate substitution did not give the final answer in terms of x but left in terms of the new variable.

Answering Tip

- Practice needs to be done in Integration using substitution and special integrals. Do not leave the answers in terms of the new variable.

Q. 9. Evaluate : $\int \frac{\sec x}{1+\operatorname{cosec} x}dx$ **[ISC, 2015]**

Sol. Let $I=\int \frac{\sec x}{1+\operatorname{cosec} x}dx$

$$=\int \frac{\sin x}{\cos x(1+\sin x)}dx$$

$$=\int \frac{\sin x\cos x}{\cos^2 x(1+\sin x)}dx$$

Put $t = \sin x$

$dt = \cos x\, dx$

$$=\int \frac{t\,dt}{(1-t^2)(1+t)}$$

$$=\int \frac{t\,dt}{(1-t)(1+t)^2}$$

$$\frac{t}{(1-t)(1+t)^2}\sim\frac{A}{1-t}+\frac{B}{1+t}+\frac{C}{(1+t)^2}$$

$$t\sim A(1+t)^2+B(1-t^2)+C(1-t)$$

$$\Rightarrow\quad t\sim A(1+2t+t^2)+B-Bt^2+C-Ct$$

$$\Rightarrow\quad t\sim t^2(A-B)+t(2A-C)+(A+B+C)$$

$$\therefore\quad A-B=0,\ 2A-C=1,\ A+B+C=0$$

Solving equations, we get

$$\therefore\quad A=\frac{1}{4},\ B=\frac{1}{4},\ C=-\frac{1}{2}$$

$$\therefore\quad I=\frac{1}{4}\int\left(\frac{1}{1-t}+\frac{1}{1+t}-\frac{2}{(1+t)^2}\right)dt$$

$$=\frac{1}{4}\left(-\log|1-t|+\log|1+t|+\frac{2}{1+t}\right)+C$$

$$I=\frac{1}{4}\left(\log\left|\frac{1+\sin x}{1-\sin x}\right|+\frac{2}{1+\sin x}\right)+C$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- Many candidates were not able to integrate the given expression. Some candidates could not decompose the problem into partial fraction. Errors were also made while integrating factors.

Answering Tip

- Partial fraction rule need to be understood and applied correctly. Methods of proper substitution need attention.

Q. 10. Evaluate : $\int \frac{x+\sin x}{1+\cos x}dx$ **[ISC, 2014]**

Sol. $I=\int \frac{x+\sin x}{1+\cos x}dx$

$$=\int \frac{x+\sin x}{2\cos^2\frac{x}{2}}dx$$

$$=\int\left(\frac{x}{2\cos^2\frac{x}{2}}+\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$$

$$=\int\left(\frac{1}{2}x\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\int x\sec^2\frac{x}{2}dx+\int\tan\frac{x}{2}dx$$

$$=\frac{1}{2}\left[x\frac{\tan\frac{x}{2}}{\frac{1}{2}}-\int\left(\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right).1dx\right]+\int\tan\frac{x}{2}dx+C$$

$$=x\tan\frac{x}{2}-\int\tan\frac{x}{2}\,dx+\int\tan\frac{x}{2}\,dx+C$$

$$=x\tan\frac{x}{2}+C$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- After breaking up the integral correctly into two parts, some candidates incorrectly used integration by parts rule. Conversion of sin x as $2\sin\frac{x}{2}\cos\frac{x}{2}$ and $1+\cos x$ as $2\cos^2\frac{x}{2}$ was not remembered or utilised by some. Several candidates committed the mistake of ignoring the constant of integration.

Answering Tip

- Integration by-parts rule needs to be understood and applied correctly. Insufficient practice of such functions leads to errors.

Q. 11. Evaluate : $\int \frac{\cos^{-1}x}{x^2}dx$ **[ISC, 2013]**

Sol. Put $\cos^{-1}x = t$

$\Rightarrow\quad x = \cos t$

$\Rightarrow\quad dx = -\sin t\,dt$

$$\therefore\quad \int\frac{\cos^{-1}x}{x^2}dx=\int\frac{t}{\cos^2 t}(-\sin t)dt$$

$$=-\int \underset{\text{I}}{t}\,(\underset{\text{II}}{\sec t\tan t})dt$$

$$=-t.\sec t-\int 1.\sec t\,dt$$

$$=-t.\sec t+\log|\sec t+\tan t|+C$$

$$=\frac{-t}{\cos t}+\log\left|\frac{1+\sin t}{\cos t}\right|+C$$

$$= \frac{\cos^{-1}x}{x} + \log\left|\frac{1+\sin(\cos^{-1}x)}{x}\right| + C$$

$$= \frac{-\cos^{-1}x}{x} + \log\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C$$

Q. 12. Evaluate : $\int \frac{dx}{x\{6(\log x)^2+7\log x+2\}}$ **[ISC, 2012]**

Sol. Let $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$$\therefore \int \frac{dx}{x\{6(\log x)^2+7\log x+2\}}$$

$$= \int \frac{dt}{6t^2+7t+2}$$

$$= \int \frac{dt}{(3t+2)(2t+1)}$$

Let, $\frac{1}{(3t+2)(2t+1)} = \frac{A}{3t+2} + \frac{B}{2t+1}$

$$1 = A(2t+1) + B(3t+2)$$

On putting $t = -1/2$ and $t = -2/3$ respectively, we get

$$1 = B\left(3\times -\frac{1}{2}+2\right) \text{ and } 1 = A\left(2\times -\frac{2}{3}+1\right)$$

$\Rightarrow \quad B = 2 \quad$ and $\quad A = -3$

$$\int \frac{dt}{(3t+2)(2t+1)} = \int \frac{-3}{3t+2}dt + \int \frac{2}{2t+1}dt$$

$$= -3.\frac{1}{3}\log|3t+2| + 2.\frac{1}{2}\log|2t+1| + C$$

$$= -\log|3t+2| + \log|2t+1| + C$$

$$= \log\frac{|2t+1|}{|3t+2|} + C = \log\left|\frac{2\log x+1}{3\log x+2}\right| + C$$

Q. 13. Evaluate : $\int \frac{x^2-5x-1}{x^4+x^2+1}dx$ **[ISC, 2011]**

Sol. Let $I = \int \frac{x^2-5x-1}{x^4+x^2+1}dx$

$$\Rightarrow \quad I = \int \frac{x^2-1}{x^4+x^2+1}dx + \int \frac{-5x}{x^4+x^2+1}dx \quad \text{...(i)}$$

$\Rightarrow \quad I = I_1 - 5I_2$

Consider

$$I_1 = \int \frac{x^2-1}{x^4+x^2+1}dx$$

$$= \int \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(x^2+1+\frac{1}{x^2}\right)}dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}+1\right)}dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-1}$$

Now, put $x+\frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{t^2-1} = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C_1$$

$$= \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C_1$$

$$= \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + C_1 \quad \text{...(ii)}$$

Now, $I_2 = \int \frac{x}{x^4+x^2+1}dx$

Let, $x^2 = t$

$\Rightarrow \quad 2x\,dx = dt$

$$I_2 = \frac{1}{2}\int \frac{dt}{t^2+t+1}$$

$$\therefore \quad I_2 = \frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$I_2 = \frac{1}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{\left(t+\frac{1}{2}\right)\times 2}{\sqrt{3}}\right] + C_2$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C_2$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C_2 \quad \text{...(iii)}$$

Now, since $I = I_1 - 5I_2$

From (ii) and (iii), we get

$$I = \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| - \frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$$

(Let $C = C_1 + C_2$)

Q. 14. Evaluate : $\int \frac{2\sin 2\theta - \cos\theta}{6-\cos^2\theta - 4\sin\theta}d\theta$ **[ISC, 2010]**

Sol. Let $I = \int \frac{2\sin 2\theta - \cos\theta}{6-\cos^2\theta-4\sin\theta}d\theta$

$$I = \int \frac{4\sin\theta\cos\theta - \cos\theta}{6-1+\sin^2\theta-4\sin\theta}d\theta$$

$$= \int \frac{\cos\theta(4\sin\theta-1)}{\sin^2\theta - 4\sin\theta+5}d\theta$$

Let $\sin\theta = t$

$$\cos\theta\, d\theta = dt$$

$$I = \int \frac{4t-1}{t^2-4t+5}dt$$

$$= \int \frac{4t-8+7}{t^2-4t+5}dt$$

$$= \int \frac{2(2t-4)}{t^2-4t+5}dt + 7\int \frac{dt}{t^2-4t+5}$$

$$= 2\int \frac{2t-4}{t^2-4t+5}dt + 7\int \frac{dt}{(t-2)^2+1}$$

$$= 2\log|t^2-4t+5| + 7\tan^{-1}(t-2) + C$$

$$= 2\log|\sin^2\theta - 4\sin\theta + 5| + 7\tan^{-1}(\sin\theta - 2) + C$$

AI Q. 15. Evaluate : $\int (3x+1)\sqrt{4-3x-2x^2}\,dx$.

Sol. Let $(3x+1) \sim A\frac{d}{dx}(4-3x-2x^2) + B$

$\Rightarrow \quad 3x+1 \sim A(-3-4x) + B$

When $\quad x = -\frac{3}{4}$,

$$3\left(-\frac{3}{4}\right) + 1 = A\left[-3-4\left(-\frac{3}{4}\right)\right] + B$$

$$-\frac{9}{4} + 1 = B$$

$\Rightarrow \quad B = -\frac{5}{4}$

When $\quad x = 0$

$\Rightarrow \quad 3(0)+1 = A(-3-0)+B$

$\Rightarrow \quad 1 = -3A + B$

$\Rightarrow \quad 1 + \frac{5}{4} = -3A \quad \left(\because B = -\frac{5}{4}\right)$

$\Rightarrow \frac{9}{4} \times \left(-\frac{1}{3}\right) = A$

$\Rightarrow \quad A = -\frac{3}{4}$

$\therefore \quad 3x+1 = -\frac{3}{4}(-3-4x) - \frac{5}{4}$

$\therefore \quad I = \int\left[-\frac{3}{4}(-3-4x) - \frac{5}{4}\right]\sqrt{4-3x-2x^2}\,dx$

$$= \int -\frac{3}{4}(-3-4x)\sqrt{4-3x-2x^2}\,dx$$

$$-\int \frac{5}{4}\sqrt{4-3x-2x^2}\,dx$$

$$= -\frac{3}{4}\frac{(4-3x-2x^2)^{3/2}}{\frac{3}{2}} - \frac{5}{4}\int\sqrt{2\left(2-\frac{3x}{2}-x^2\right)}\,dx$$

$$= -\frac{1}{2}(4-3x-2x^2)^{3/2}$$

$$-\frac{5\sqrt{2}}{4}\int\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x+\frac{3}{4}\right)^2}\,dx$$

$$= -\frac{1}{2}(4-3x-2x^2)^{\frac{3}{2}}$$

$$-\frac{5\sqrt{2}}{4}\left[\frac{1}{2}\left(x+\frac{3}{4}\right)\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x+\frac{3}{4}\right)^2}\right.$$

$$\left. + \frac{\frac{41}{16}}{2}\sin^{-1}\frac{x+\frac{3}{4}}{\frac{\sqrt{41}}{4}}\right] + C$$

$\therefore \quad I = -\frac{1}{2}(4-3x-2x^2)^{3/2}$

$$-\frac{5\sqrt{2}}{4}\left[\frac{4x+3}{8}\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x+\frac{3}{4}\right)^2} + \frac{41}{32}\sin^{-1}\frac{4x+3}{\sqrt{41}}\right]$$

Q. 16. Evaluate : $\int \frac{x}{(x^2+1)(x-1)}dx$.

Sol. Let $I = \int \frac{x}{(x^2+1)(x-1)}dx$

$$\therefore \quad \frac{x}{(x^2+1)(x-1)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow \quad x = A(x^2+1) + (x-1)(Bx+C)$

$\Rightarrow \quad x = (A+B)x^2 + (C-B)\,x + (A-C)$

$\Rightarrow A+B = 0, C-B = 1$ and $A-C = 0$

$\Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}$ and $C = \frac{1}{2}$

$$\therefore \quad I = \int \frac{\frac{1}{2}}{(x-1)}dx + \int \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1}dx$$

$$= \frac{1}{2}\int \frac{1}{(x-1)}dx - \frac{1}{2\times 2}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + C$$

AI Q. 17. Evaluate : $\int e^{ax}\cos(bx+c)\,dx$.

Sol. Let $I = \int \underset{\text{I}}{e^{ax}}\,\underset{\text{II}}{\cos(bx+c)}\,dx$

Integrating by parts, we get

$$I = e^{ax}\left(\frac{\sin(bx+c)}{b}\right) - \int ae^{ax}\left(\frac{\sin(bx+c)}{b}\right)dx$$

$$= \frac{e^{ax}\sin(bx+c)}{b} - \frac{a}{b}\int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\sin(bx+c)}\,dx$$

$$= \frac{e^{ax}\sin(bx+c)}{b} - \frac{a}{b}\left[e^{ax}\left(-\frac{\cos(bx+c)}{b}\right)\right.$$

$$\left.-\int ae^{ax}\left(-\frac{\cos(bx+c)}{b}\right)dx\right]$$

$$= \frac{e^{ax}\sin(bx+c)}{b} + \frac{a}{b^2}e^{ax}\cos(bx+c)$$

$$-\frac{a^2}{b^2}\int e^{ax}\cos(bx+c)\,dx$$

$$\therefore\ I = \frac{e^{ax}}{b^2}(b\sin(bx+c) + a\cos(bx+c)) - \frac{a^2}{b^2}I$$

$$\Rightarrow\ \frac{a^2+b^2}{b^2}I = \frac{e^{ax}}{b^2}(a\cos(bx+c) + b\cos(bx+c))$$

$$\Rightarrow\ I = \frac{e^{ax}}{a^2+b^2}(a\cos(bx+c) + b\sin(bx+c))$$

$$\therefore\ \int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[(a\cos(bx+c))$$
$$+ b\sin(bx+c)] + C.$$

Q. 18. Evaluate : $\displaystyle\int \frac{dx}{\sin x + \sin 2x}$.

Sol. We have

$$I = \int \frac{dx}{\sin x + \sin 2x}$$

$$= \int \frac{dx}{\sin x + 2\sin x \cos x}$$

$$= \int \frac{dx}{\sin x(1+2\cos x)}$$

$$= \int \frac{\sin x\,dx}{\sin^2 x(1+2\cos x)}$$

$$= \int \frac{\sin x\,dx}{(1-\cos^2 x)(1+2\cos x)}$$

$$I = \int \frac{\sin x\,dx}{(1-\cos x)(1+\cos x)(1+2\cos x)}$$

Putting $\cos x = t \Rightarrow -\sin x\,dx = dt$

or $\sin x\,dx = -dt$, we get

$$I = \int \frac{-dt}{(1-t)(1+t)(1+2t)}$$

Now, $\displaystyle\frac{1}{(1-t)(1+t)(1+2t)}$

$$= \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1+2t)} \quad \text{(let)}$$

$$1 = A(1+t)(1+2t) + B(1-t)(1+2t)$$
$$+ C(1-t)(1+t)$$

$$1 = A(1+3t+2t^2) + B(1+t-2t^2) + C(1-t^2)$$

$$1 = (2A-2B-C)t^2 + (3A+B)t + (A+B+C)$$

$\therefore\ 2A-2B-C = 0, 3A+B = 0$ and $A+B+C = 1$

Getting $A = \frac{1}{6}, B = -\frac{1}{2}$ and $C = \frac{4}{3}$

$$\therefore\quad I = \int\left\{\frac{-\frac{1}{6}}{1-t} + \frac{\frac{1}{2}}{1+t} - \frac{\frac{4}{3}}{1+2t}\right\}dt$$

$$= \frac{1}{6}\log|1-t| + \frac{1}{2}\log|1+t|$$
$$-\frac{2}{3}\log|1+2t| + C$$

$$= \frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x|$$
$$-\frac{2}{3}\log|1+2\cos x| + C$$

Q. 19. Evaluate : $\displaystyle\int (x+3)\sqrt{3-4x-x^2}\,dx$.

Sol. Let $\quad x+3 = A\dfrac{d}{dx}\{3-4x-x^2\} + B$

$$x+3 = A(-4-2x) + B$$

$$x+3 = (-4A+B) - 2Ax$$

On comparing both sides,

$$A = -\frac{1}{2}$$

and $\quad -4\left(-\frac{1}{2}\right) + B = 3$

$$\Rightarrow\quad B = 1$$

$$\therefore\quad x+3 = -\frac{1}{2}(-4-2x) + 1$$

$$\therefore\quad I = -\frac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}\,dx$$
$$+ \int\sqrt{3-4x-x^2}\,dx$$

$$= -\frac{1}{2}\int(-2x-4)\sqrt{3-4x-x^2}\,dx$$
$$+ \int\sqrt{(\sqrt{7})^2 - (x+2)^2}\,dx$$

$$= -\frac{1}{3}(3-4x-x^2)^{3/2} + \frac{x+2}{2}\sqrt{3-4x-x^2}$$
$$+ \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C$$

Q. 20. Evaluate : $\displaystyle\int \frac{3x+1}{(x+1)^2(x+3)}dx$.

Sol. Let $I=\int\frac{3x+1}{(x+1)^2(x+3)}dx$

Let $\frac{3x+1}{(x+1)^2(x+3)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3}$ ...(i)

$3x+1 = A(x+1)\,(x+3)+B(x+3)+C(x+1)^2$

$\Rightarrow$ $3x+1 = x^2(A + C) + x(4A + B + 2C) + (3A + 3B + C)$

$\Rightarrow$ $A + C = 0,\ 4A + B + 2C = 3,\ 3A + 3B + C = 1$

On solving, we get $A = 2, B = -1, C = -2$

From (i),

$$\frac{3x+1}{(x+1)^2(x+3)}=\frac{2}{x+1}-\frac{1}{(x+1)^2}-\frac{2}{x+3}$$

$$\therefore\quad I=\int\frac{2}{x+1}dx-\int\frac{1}{(x+1)^2}dx-\int\frac{2}{x+3}dx$$

$$=2\log|x+1|+\frac{1}{x+1}-2\log|x+3|+C$$

$$=2\log\frac{(x+1)}{(x+3)}+\frac{1}{x+1}+C$$

Q. 21. Evaluate : $\int\frac{3x+1}{(x^3-x^2-x+1)}dx$**.**

Sol. Let $I=\int\frac{3x+1}{(x^3-x^2-x+1)}dx$

$$\Rightarrow\quad I=\int\frac{3x+1}{(x-1)^2(x+1)}dx$$

Resolving the integrand into partial fractions,

$$\frac{3x+1}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}\quad ...(i)$$

$$3x+1=A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

$\Rightarrow$ $3x+1 \sim x^2(A + C) + x(B - 2C) + (-A + B + C)$

$\Rightarrow$ $A + C \sim 0,\ B - 2C = 3, -A + B + C = 1$

On solving we get $A=-\frac{1}{2}, B=4, C=\frac{1}{2}$

From (i),

$$\int\frac{3x+1}{(x^3-x^2-x+1)}dx=-\frac{1}{2}\left(\frac{1}{x-1}\right)+\frac{4}{(x-1)^2}+\frac{1}{2(x+1)}$$

$$I=\int\left(\frac{-1}{2(x-1)}+\frac{4}{(x-1)^2}+\frac{1}{2}\frac{1}{x+1}\right)dx$$

$$=-\frac{1}{2}\log|x-1|-\frac{4}{x-1}+\frac{1}{2}\log|x+1|+C$$

$$=\frac{1}{2}\log\left(\frac{x+1}{x-1}\right)-\frac{4}{x-1}+C$$

Q. 22. Evaluate : $\int\frac{x^2+1}{(x^2+4)(x^2+25)}dx$**.**

Sol. Let, $I=\int\frac{x^2+1}{(x^2+4)(x^2+25)}dx$

Consider $\frac{x^2+1}{(x^2+4)(x^2+25)}$

Put $x^2 = y$

$$=\frac{y+1}{(y+4)(y+25)}=\frac{A}{y+4}+\frac{B}{y+25}$$

$$\Rightarrow\quad y+1=A(y+25)+B(y+4)$$

On equating the coefficients of like terms, we have

$A + B = 1$

and $25A + 4B = 1$

Solving these equations, we get

$$A=-\frac{1}{7},\ B=\frac{8}{7}$$

$$\frac{x^2+1}{(x^2+4)(x^2+25)}=\frac{\frac{-1}{7}}{x^2+4}+\frac{\frac{8}{7}}{x^2+25}$$

$$\therefore\quad I=-\frac{1}{7}\int\frac{dx}{x^2+4}+\frac{8}{7}\int\frac{dx}{x^2+25}$$

$$=-\frac{1}{7}\int\frac{dx}{x^2+(2)^2}+\frac{8}{7}\int\frac{dx}{x^2+(5)^2}$$

$$=-\frac{1}{7}.\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{8}{7}.\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$$

$$=\frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right)-\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right)+C$$

Q. 23. Evaluate : $\int\frac{dx}{x(x^5+3)}dx$**.**

Sol. $\int\frac{dx}{x(x^5+3)}dx=\int\frac{x^4}{x^5(x^5+3)}dx$

Let, $x^5 + 3 = t$, then

$$d(x^5+3)=dt\Rightarrow 5x^4\,dx=dt$$

$$\Rightarrow\quad x^4\,dx=\frac{dt}{5}$$

$$\Rightarrow\quad I=\int\frac{dt}{5(t-3)t}=\frac{1}{5}\int\frac{dt}{t^2-3t}$$

$$=\frac{1}{5}\int\frac{dt}{t^2-2t\left(\frac{3}{2}\right)+\frac{9}{4}-\frac{9}{4}}$$

$$=\frac{1}{5}\int\frac{dt}{\left(t-\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}$$

$$=\frac{1}{5}\times\frac{1}{2\times\left(\frac{3}{2}\right)}\log\left|\frac{t-\frac{3}{2}-\frac{3}{2}}{t-\frac{3}{2}+\frac{3}{2}}\right|+C$$

$$= \frac{1}{15}\log\left|\frac{t-3}{t}\right| + C = \frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right| + C$$

Q. 24. Evaluate : $\int (2\sin 2x - \cos x)\sqrt{6 - \cos 2x - 4\sin x}\, dx$

Sol. $I = \int (2\sin 2x - \cos x)\sqrt{6-\cos 2x - 4\sin x}\, dx$

$$= \int (4\sin x - 1)\left(\sqrt{\sin^2 x - 4\sin x + 5}\right)\cos x\, dx$$

$$= \int (4t-1)\sqrt{t^2 - 4t + 5}\, dt, \text{ where } \sin x = t$$

$$\Rightarrow \quad \cos x\, dx = dt$$

$$= 2\int (2t-4)\sqrt{t^2-4t+5}\, dt + 7\int \sqrt{(t-2)^2 + 1}\, dt$$

Let $\mu = t^2 - 4t + 5$

$\Rightarrow d\mu = (2t-4)\, dt$

On solving, we get

$$= 2\frac{(t^2-4t+5)^{3/2}}{\frac{3}{2}} + 7\left[\frac{t-2}{2}\sqrt{t^2-4t+5}\right.$$

$$\left. + \frac{1}{2}\log\left|(t-2) + \sqrt{t^2-4t+5}\right|\right] + C$$

$$= \frac{4}{3}\left[\sin^2 x - 4\sin x + 5\right]^{3/2} + 7\left[\frac{(\sin x - 2)}{2}\right.$$

$$\sqrt{\sin^2 x - 4\sin x + 5} + \frac{1}{2}\log|(\sin x - 2)$$

$$\left. + \sqrt{\sin^2 x - 4\sin x + 5}\,|\right] + C$$

Q. 25. Evaluate : $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$**, $x \in [0, 1]$.**

Sol. $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$

$$= \frac{2}{\pi}\int\left[\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)\right]dx$$

$$= \frac{2}{\pi}\int 2\sin^{-1}\sqrt{x}\, dx - \int 1\, dx$$

$$= \frac{4}{\pi}\left[\sin^{-1}\sqrt{x}.x - \int \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}x\, dx\right] - x + C$$

$$= \frac{1}{\pi}\left[4x\sin^{-1}\sqrt{x} - 2\int\frac{\sqrt{x}}{\sqrt{1-x}}dx\right] - x + C$$

Let, $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$$= \frac{1}{\pi}\left[4x\sin^{-1}\sqrt{x} - 2\int\frac{\sin\theta}{\cos\theta}2\sin\theta\cos\theta\, d\theta\right]$$

$$- x + C$$

$$= \frac{1}{\pi}\left[4x\sin^{-1}\sqrt{x} - 2\int(1-\cos 2\theta)\, d\theta\right] - x + C$$

$$= \frac{1}{\pi}\left[4x\sin^{-1}\sqrt{x} - 2\left(\theta - \frac{\sin 2\theta}{2}\right)\right] - x + C$$

$$= \frac{1}{\pi}\left[4x\sin^{-1}\sqrt{x} - 2\left(\theta - \frac{2\sin\theta\cos\theta}{2}\right)\right] - x + C$$

$$= \frac{2}{\pi}\left[2x\sin^{-1}\sqrt{x} - \left(\sin^{-1}\sqrt{x} - \sqrt{x}.\sqrt{1-x}\right)\right] - x + C$$

$$= \frac{2}{\pi}\left[\sqrt{x - x^2} - (1-2x)\sin^{-1}\sqrt{x}\right] - x + C$$

Q. 26. Evaluate : $\int (5x-1)\sqrt{6 + 5x - 2x^2}\, dx$**.**

Sol. Let, $I = \int (5x-1)\sqrt{6+5x-2x^2}\, dx$

Take, $5x - 1 = A\frac{d}{dx}(6 + 5x - 2x^2) + B$

$$= A(5-4x) + B$$

$$\Rightarrow \quad 5 = -4A \text{ and } -1 = 5A + B$$

$$\Rightarrow \quad A = -\frac{5}{4}, B = \frac{21}{4}$$

$$\therefore \quad I = -\frac{5}{4}\int(5-4x)\sqrt{6+5x-2x^2}\, dx$$

$$+ \frac{21}{4}\int\sqrt{2}\sqrt{3 + \frac{5}{2}x - x^2}\, dx$$

$$= -\frac{5}{4}\times\frac{2}{3}(6+5x-2x^2)^{3/2}$$

$$+ \frac{21\sqrt{2}}{4}\sqrt{\left(\frac{\sqrt{73}}{4}\right)^2 - \left(x - \frac{5}{4}\right)^2}\, dx$$

$$= -\frac{5}{6}(6+5x-2x^2)^{3/2}$$

$$+ \frac{21\sqrt{2}}{4}\left[\frac{1}{2}\left(x - \frac{5}{4}\right)\sqrt{\left(\frac{\sqrt{73}}{4}\right)^3 - \left(x - \frac{5}{4}\right)^2}\right.$$

$$\left. + \frac{1}{2}\left(\frac{\sqrt{73}}{4}\right)^2 \sin^{-1}\left(\frac{x - 5/4}{\sqrt{73}/4}\right)\right] + C$$

$$= -\frac{5}{6}(6+5x-2x^2)^{3/2}$$

$$+ \frac{21\sqrt{2}}{8}\left[\left(\frac{4x-5}{4}\right)\sqrt{\left(\frac{73}{4}\right)^2 - \left(x - \frac{5}{4}\right)^2}\right.$$

$$\left. + \frac{73}{16}\sin^{-1}\left(\frac{4x-5}{\sqrt{73}}\right)\right] + C$$

TOPIC-2

Definite Integral

Revision Notes

- **Meaning of Definite Integral of Function :**

  If $\int f(x)dx = F(x)$, be an integral of $f(x)$, then $F(b) - F(a)$ is called the definite integral of $f(x)$ between the limits a and b and in symbols it is written as $\int_a^b f(x)dx$ or $[F(x)]_a^b = F(b) - F(a)$. Moreover, the definite integral given a unique and definite value (numeric value) of anti-derivative of the function between the given intervals. It acts as a substitute for evaluating the area analytically.

- **Definite Integrals as the Limit of a Sum (First Principle of Integrals) :**

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)\right]$$

or

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a+h) + f(a+2h) + f(a+3h) + ... + f(a+nh)\right]$$

i.e.,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh), \text{such that as } n\to\infty, h\to 0 \text{ and } nh = b-a$$

or

$$\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=1}^{n} f(a+rh), \text{such that as } n\to\infty, h\to 0 \text{ and } nh = b-a.$$

Scan to know more about this topic

Definite Integrals

Know the Formulae

- **Properties of definite Integral**

(a) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(b) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(c) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x), a < c < b$

(d) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(e) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(f) $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function } i.e., \ f(x) = f(-x) \\ 0, & \text{if } f(x) \text{ is an odd function } i.e., \ f(x) = -f(x) \end{cases}$

(g) $\int_{-a}^{a} f(x)dx = \int_0^a \{f(x) + f(-x)\}dx$

(h) $\int_0^{2a} f(x)dx = \int_0^{2a} \{f(x) + f(2a-x)\}dx$

(i) $\int_0^{2a} f(x)\,dx = \left[\begin{array}{ll} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{array}\right.$

1 Mark Questions

Q. 1. Evaluate : $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x \, dx$.

Sol. We have, $f(x) = \int_{-\pi}^{\pi} \sin^3 x \cos^2 x \, dx$

$$f(-x) = \int_{-\pi}^{\pi} \sin^3(-x)\cos^2(-x)dx$$

$$= -f(x)$$

Since $f(x)$ is an odd function

So that, $\int_{-\pi}^{\pi} \sin^3 x \cos^2(x)dx = 0$

AI **Q. 2. Evaluate : $\int_{-2}^{2}(x^3+1)dx$.**

Sol.
$$\int_{-2}^{2}(x^3+1)dx = \int_{-2}^{2}(x^3)dx+\int_{-2}^{2}1\,dx = I_1+I_2$$
$$= 0+[x]_{-2}^{2}$$
(As I_1 is odd function)
$$= 2+2 = 4$$

Q. 3. Evaluate : $\int_{2}^{3}3^x dx$.

Sol.
$$\int_{2}^{3}3^x dx = \left[\frac{3^x}{\log 3}\right]_2^3 = \frac{18}{\log 3}$$

Q. 4. Evaluate : $\int_{0}^{2\pi}\cos^5 x\, dx$.

Sol.
$$\int_{0}^{2\pi}\cos^5 x\, dx = 2\int_0^{\pi}\cos^5 x\, dx$$
$$\text{and } 2\int_0^{\pi}\cos^5 x\, dx = 0 \text{ or } \int_0^{2\pi}\cos^5 x\, dx = 0$$

2 Marks Questions

AI **Q. 1. Evaluate : $\int_0^{\pi/2}\frac{\sin^{3/2}x}{\sin^{3/2}x+\cos^{3/2}x}dx$**

[ISC Specimen Paper, 2007 2018, 2020]

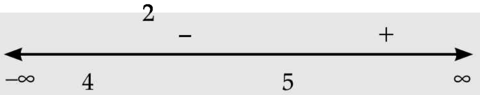
Sol. Let,
$$I = \int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx \qquad \text{...(i)}$$
$$= \int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx$$
$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$
$$I = \int_0^{\frac{\pi}{2}}\frac{\cos^{\frac{3}{2}}x}{\cos^{\frac{3}{2}}x+\sin^{\frac{3}{2}}x}dx \qquad \text{...(ii)}$$

Add equation (i) and (ii),

$$\Rightarrow \qquad 2I = \int_0^{\frac{\pi}{2}}1.dx = [x]_0^{\frac{\pi}{2}}$$
$$2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Q. 2. Evaluate : $\int_4^5|x-5|dx$. **[ISC, 2020]**

Sol.
$$\int_4^5|x-5|dx = \int_4^5-(x-5)dx$$
$$= -\left[\frac{x^2}{2}-5x\right]_4^5$$
$$= -\left[\frac{25}{2}-25-\frac{16}{2}+20\right]$$
$$= \frac{1}{2}$$



[ISC Marking Scheme, 2020]

Examiner's Comment

- Few students have no idea how to solve (take positive or negative sign) the modulus within the given limits.

Answering Tip

- Understand the concept of sign taken for modulus within the given limits by doing various similar questions.

Q. 3. Evaluate : $\int_0^{\pi/4}\log(1+\tan\theta)\,d\theta$ **[ISC 2017]**

Sol. Let
$$I = \int_0^{\pi/4}\log(1+\tan\theta)\,d\theta \qquad \text{...(i)}$$
$$= \int_0^{\pi/4}\log(1+\tan\theta)\,d\theta$$
$$= \int_0^{\frac{\pi}{4}}\log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$
$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$
$$= \int_0^{\frac{\pi}{4}}\log\left[1+\left\{\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}\right]d\theta$$
$$\left[\because \tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}\right]$$
$$\therefore \qquad I = \int_0^{\frac{\pi}{4}}[\log 2-\log(1+\tan\theta)]d\theta \qquad \text{...(ii)}$$

Adding eqs. (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{4}}\log 2\,d\theta$$
$$= \log 2[\theta]_0^{\frac{\pi}{4}} = \log 2.\frac{\pi}{4}$$
$$\therefore \qquad I = \frac{\pi}{8}.\log 2$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates made errors in applying the properties of definite integrals correctly and some candidates attempted it by integration by parts.

Answering Tip

- Learn the properties of definite integrals well and do adequate practice to apply them appropriately.

Q. 4. Evaluate : $\int_0^3 f(x)\,dx;$

where $f(x)=\begin{cases}\cos 2x, & 0\le x\le \dfrac{\pi}{2}\\ 3, & \dfrac{\pi}{2}<x\le 3\end{cases}$ **[ISC, 2015]**

Sol.
$$\int_0^3 f(x)\,dx=\int_0^{\pi/2}\cos 2x\,dx+\int_{\pi/2}^{3}3\,dx$$
$$=\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}+[3x]_{\pi/2}^{3}$$
$$=(0-0)+3\left(3-\frac{\pi}{2}\right)$$
$$=9-\frac{3\pi}{2}$$
[ISC Marking Scheme, 2015]

AI Q. 5. Evaluate : $\int_0^1 \dfrac{xe^x}{(1+x)^2}\,dx.$ **[ISC, 2011]**

Sol.
$$I=\int_0^1\frac{xe^x}{(1+x)^2}dx=\int_0^1\frac{(x+1)-1}{(1+x)^2}.e^x dx$$
$$=\int_0^1\left[\frac{(x+1)e^x}{(x+1)^2}-\frac{e^x}{(x+1)^2}\right]dx$$
$$=\int_0^1 e^x\left[\frac{1}{1+x}-\frac{1}{(1+x)^2}\right]dx\quad=\left[e^x\frac{1}{1+x}\right]_0^1$$
$$[\because\ \int e^x[f(x)+f'(x)]dx=e^x f(x)+c]$$
$$=\frac{e^1}{1+1}-\frac{e^0}{1+0}=\frac{e}{2}-1$$
$$\therefore\quad I=\frac{e}{2}-1$$

Q. 6. Evaluate : $\int_0^{\pi/2}\log(\tan x)\,dx$ **[ISC, 2009]**

Sol. Let $I=\int_0^{\pi/2}\log(\tan x)dx$ ...(i)
$$I=\int_0^{\pi/2}\log\left[\tan\left(\frac{\pi}{2}-x\right)\right]dx$$
$$I=\int_0^{\pi/2}\log(\cot x)\,dx$$ ...(ii)

Add (i) and (ii), we get
$$2I=\int_0^{\pi/2}\log(\tan x.\cot x)\,dx\qquad(\because\ \log 1=0)$$
$$\therefore\quad I=0$$

Q. 7. Evaluate $\int_0^9 f(x)dx$, **where** $f(x)$ **is defined by**

$$f(x)=\begin{cases}\sin x; & 0\le x\le\dfrac{\pi}{2}\\ 1, & \dfrac{\pi}{2}<x\le 5\\ e^{x-5}; & 5<x\le 9\end{cases}$$ **[ISC, 2008]**

Sol.
$$f(x)=\int_0^{\pi/2}\sin x\,dx+\int_{\pi/2}^{5}1\cdot dx+\int_5^9 e^{x-5}dx$$
$$=[-\cos x]_0^{\pi/2}+[x]_{\pi/2}^{5}+[e^{x-5}]_5^9$$
$$=[1]+\left[5-\frac{\pi}{2}\right]+[e^4-1]=\left(e^4+5-\frac{\pi}{2}\right)$$

Q. 8. Evaluate : $\int_1^3\dfrac{dx}{9+x^2}$

Sol.
$$I=\int_1^3\frac{dx}{9+x^2}$$
$$=\int_1^3\frac{dx}{3^2+x^2}$$
$$=\frac{1}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_1^3$$
$$=\frac{1}{3}\left[\tan^{-1}(1)-\tan^{-1}\frac{1}{3}\right]$$
$$=\frac{1}{3}\left[\frac{\pi}{4}-\tan^{-1}\left(\frac{1}{3}\right)\right]$$

Examiner's Comment

- Sometimes students attempt in by other methods of integration and could not reach the proper result.

Answering Tip

- Learn all the special integrals thoroughly.

Q. 9. Evaluate : $\int_0^2\sqrt{4-x^2}\,dx$

Sol.
$$\int_0^2\sqrt{4-x^2}\,dx$$
$$=\int_0^2\sqrt{2^2-x^2}\,dx$$
$$=\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$
$$=[0+2\sin^{-1}(1)]-[0+2\sin^{-1}0]$$
$$=2\times\frac{\pi}{2}-(0+2\times 0)=\pi$$

AI Q. 10. If $\int_0^a\dfrac{1}{4+x^2}dx=\dfrac{\pi}{8}$, **find** a.

Sol. Let
$$I=\int_0^a\frac{1}{4+x^2}dx$$
$$=\int_0^a\frac{1}{2^2+x^2}dx$$
$$=\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^a$$
$$=\frac{1}{2}\left[\tan^{-1}\frac{a}{2}-\tan^{-1}0\right]$$
$$\because\quad I=\frac{\pi}{8}$$
$$\therefore\ \frac{1}{2}\left[\tan^{-1}\frac{a}{2}-\tan^{-1}0\right]=\frac{\pi}{8}$$
$$\tan^{-1}\frac{a}{2}=\frac{\pi}{4}$$

$$\Rightarrow \qquad \frac{a}{2} = \tan\frac{\pi}{4}$$

$$\Rightarrow \qquad \frac{a}{2} = 1 \Rightarrow a = 2$$

Q. 11. Write the value of $\int_0^1 \frac{e^x}{1+e^{2x}}dx$.

Sol. Let $I = \int_0^1 \frac{e^x}{1+e^{2x}}dx$

Let $e^x = y$

$\Rightarrow$ $e^x dx = dy$

When, $x = 0$; $y = 1$ and $x = 1$; $y = e$

$$\Rightarrow \qquad I = \int_1^e \frac{dy}{1+y^2} = \left[\tan^{-1} y\right]_1^e$$

$$= \tan^{-1} e - \tan^{-1} 1$$

$$= \tan^{-1} e - \frac{\pi}{4}$$

Q. 12. Evaluate : $\int_e^{e^2} \frac{dx}{x \log x}$.

Sol. Put, $\log x = t$

or $dt = \frac{1}{x}dx$

Put, $x = e^2$ or $t = 2$ and $x = e$ or $t = 1$

$$\therefore \qquad \int_e^{e^2} \frac{dx}{x \log x} = \int_1^2 \frac{dt}{t}$$

$$= \log 2 - \log 1 = \log 2$$

Q. 13. Evaluate : $\int_0^{\frac{\pi}{2}} e^x(\sin x - \cos x)\, dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} e^x(\sin x - \cos x)\, dx$

or $I = -\int_0^{\frac{\pi}{2}} e^x(\cos x - \sin x)\, dx$

Now, consider, $f(x) = \cos x$

then $f'(x) = -\sin x$

Now, by using

$$\int e^x[f(x) + f'(x)]dx = e^x f(x) + C,$$

we get $I = -[e^x \cos x]_0^{\pi/2}$

$$= -e^{\frac{\pi}{2}} \cos\frac{\pi}{2} + e^0 \cos(0)$$

$$= 0 + 1(1) = 1$$

AI Q. 14. Evaluate $\int_{-1}^{2} \frac{|x|}{x}dx$.

Sol. Let $I = \int_{-1}^{2} \frac{|x|}{x}dx$

Since, $\frac{|x|}{x} = \begin{cases} \frac{-x}{x}, & x < 0 \\ \frac{x}{x}, & x > 0 \end{cases}$

$$= \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$\therefore \qquad I = \int_{-1}^{0} (-1)dx + \int_{0}^{2} (1)dx$$

$$= [-x]_{-1}^{0} + [x]_0^2$$

$$= -[0 - (-1)] + (2 - 0)$$

$$= -1 + 2$$

$$= 1$$

4 Marks Questions

AI Q. 1. Evaluate: $\int_{-6}^{3} |x+3|\, dx$ [ISC Specimen Paper, 2021]

[ISC, 2019]

Sol $\int_{-6}^{3} |x+3|\, dx = \int_{-6}^{-3} -(x+3)dx + \int_{-3}^{3} (x+3)dx$

$$\left[\because \int_a^2 f(x)dx = \int_a^c f(x)dx + \int_a^b f(x)dx\right]$$

$x + 3 > 0 \Rightarrow x > -3$

$x + 3 > 0 \Rightarrow x < -3$

$\Rightarrow -(x + 3) > 0$

$$= \int_{-3}^{-6} (x+3)dx + \int_{-3}^{3} (x+3)dx$$

$$\left[\because \int_a^b f(x)dx = \int_b^a -f(x)dx\right]$$

$$= \left[\frac{x^2}{2} + 3x\right]_{-3}^{-6} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{3}$$

$= 22.5$

[ISC Marking Scheme, 2019]

Examiner's Comment

- Many candidates made errors while splitting the given interval into two parts. Some candidates made calculation errors even after applying correct limits.

Answering Tip

- Understand the concept of splitting the given interval into two or more intervals in case of absolute functions and properties of definite integrals.

Q. 2. Evaluate: $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}dx$

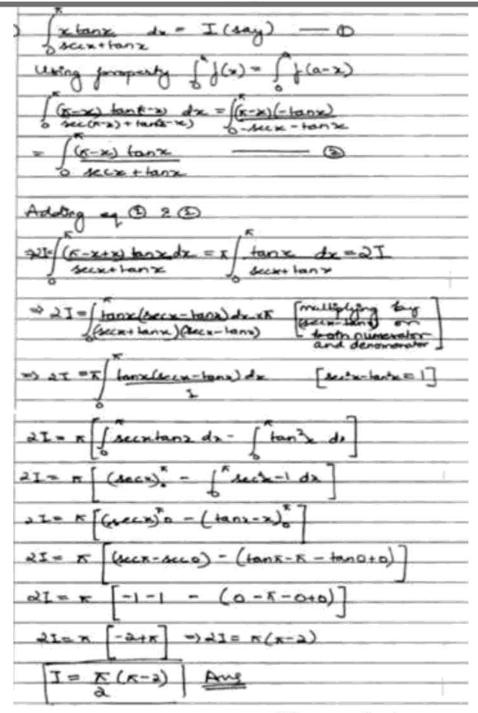
[ISC Specimen Paper, 2021, 2019]

Sol Let,
$$I = \int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}dx \quad \text{...(i)}$$
$$= \int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}dx$$
$$= \int_0^{\pi} \frac{(\pi-x)(\tan x)}{(-\sec x)+(-\tan x)}dx$$
$$I = \int_0^{\pi} \frac{(\pi-x)\tan x}{\sec x+\tan x}dx \quad \text{...(ii)}$$

Adding eq. (i) + (ii)
$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \frac{\tan x}{\sec x+\tan x}dx$$
$$= \pi\int_0^{\pi} \frac{\tan x(\sec x+\tan x)}{\sec^2 x-\tan^2 x}dx$$
$$= \pi\int_0^{\pi} (\sec x\tan x-\tan^2 x)dx$$
$$= \pi\int_0^{\pi} (\sec x\tan x-\sec^2 x+1)dx$$
$$= \pi[\sec x-\tan x+x]_0^{\pi}$$
$$= \pi : (\pi-2)$$
$$\therefore \quad I = \frac{\pi}{2}(\pi-2)$$

[ISC Marking Scheme, 2019]

OR



$\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}dx = I\text{ (say)}$ —— ①

Using property $\int_0^a f(x) = \int_0^a f(a-x)$

$\int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}dx = \int_0^{\pi} \frac{(\pi-x)(-\tan x)}{-\sec x-\tan x}$

$= \int_0^{\pi} \frac{(\pi-x)\tan x}{\sec x+\tan x}$ —— ②

Adding eq ① & ②

$2I = \int_0^{\pi} \frac{(\pi-x+x)\tan x}{\sec x+\tan x}dx = \pi\int_0^{\pi} \frac{\tan x}{\sec x+\tan x}dx = 2I$

$\Rightarrow 2I = \int_0^{\pi} \frac{\tan x(\sec x-\tan x)}{(\sec x+\tan x)(\sec x-\tan x)}dx \times \pi$ [multiplying by $(\sec x-\tan x)$ on both numerator and denominator]

$\Rightarrow 2I = \pi\int_0^{\pi} \frac{\tan x(\sec x-\tan x)}{1}dx$ [$\sec^2 x-\tan^2 x=1$]

$2I = \pi\left[\int_0^{\pi} \sec x\tan x\,dx - \int_0^{\pi} \tan^2 x\,dx\right]$

$2I = \pi\left[(\sec x)_0^{\pi} - \int_0^{\pi} \sec^2 x-1\,dx\right]$

$2I = \pi\left[(\sec x)_0^{\pi} - (\tan x-x)_0^{\pi}\right]$

$2I = \pi\left[(\sec\pi-\sec 0) - (\tan\pi-\pi-\tan 0+0)\right]$

$2I = \pi\left[-1-1 - (0-\pi-0+0)\right]$

$2I = \pi\left[-2+\pi\right] \Rightarrow 2I = \pi(\pi-2)$

$\boxed{I = \frac{\pi}{2}(\pi-2)}$ Ans

[Topper's Answer, 2019]

Q. 3. Evaluate : $\int_1^3 (x^2 + x)dx$, expressing as a limit of sum. **[ISC Specimen, 2018]**

Sol. $\int_1^3 (x^2+x)\,dx$,

Here $nh = 3-1 = 2 \Rightarrow h = \frac{2}{n}$

$$= \lim_{h\to 0} h\ \{f(1) + f(1+h) + f(1+2h) + \text{---} + f(1+(n-1)h\}$$
$$= \lim_{h\to 0} h\ \{(1^2+1) + (1+h)^2 + (1+h) + (1+2h)^2 + (1+2h) + \text{---} + (1+(n-1)h)^2 + 1 + (1+n-1)h)\}$$
$$= \lim_{h\to 0} h\ [(1+1+1+\text{---}+n \text{ terms}) + 2h(1+2+\text{---}+(n-1) \text{ terms} + h^2(1^2+2^2+3^2)+\text{---}+(n-1) \text{ terms})$$
$$+ (1+1+\text{---}+n \text{ terms})$$
$$+ h(1+2+\text{---}+(n-1) \text{ terms})]$$
$$= \lim_{h\to 0} h\left[n + 2h\frac{n(n-1)}{2} + h^2\frac{(n-1)n(2n-1)}{6} + n + h\frac{n(n-1)}{2}\right]$$
$$= \lim_{h\to 0} h\left[n + nh(n-1) + h^2 n\frac{(n-1)(2n-1)}{6} + n + h\frac{n(n-1)}{2}\right]$$
$$= \lim_{h\to 0}\left[nh + n^2h^2\left(1-\frac{1}{n}\right) + \frac{n^3h^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + nh + \frac{n^2h^2}{2}\left(1-\frac{1}{n}\right)\right]$$
$$= \lim_{h\to 0}\left[2 + (4)\left(1-\frac{1}{n}\right) + \frac{8}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + 2 + \frac{4}{2}\left(1-\frac{1}{n}\right)\right] \quad \left[\because h = \frac{2}{n}\right]$$
$$= \lim_{n\to \infty}\left[2 + 4\left(1-\frac{1}{n}\right) + \frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + 2 + 2\left(1-\frac{1}{n}\right)\right]$$
$$= 2 + 4 + \frac{8}{3} + 2 + 2 \qquad (\text{as } h \to 0, n \to \infty)$$
$$= 10 + \frac{8}{3} = \frac{38}{3}.$$

Q. 4. Using properties of definite integrals, Evaluate :

$\int_0^{\pi/2} \frac{\sin x-\cos x}{1+\sin x\cos x}dx$ **[ISC, 2016]**

Sol. $I = \int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x}dx \quad \text{...(i)}$

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)dx}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}$$

$$I=\int\frac{\cos x-\sin x}{1+\sin x\cos x}dx \qquad \text{....(ii)}$$

On adding eq. (i) and (ii), we get

$2I=0\Rightarrow I=0$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Several candidates made mistakes in applying the property $\int_0^a f(x)dx=\int_0^a f(a-x)dx\cdot$ Some attempted other methods of integration and could not reach the proper result.

Answering Tip

- Varieties of questions of integration by substitution need to be practiced.

Q. 5. Using properties of definite integrals, evaluate :

$$\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$ **[ISC, 2014]**

Sol. $$I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad \text{...(i)}$$

$$I=\int_0^{\pi/2}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

$$I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad \text{...(ii)}$$

Adding eqn. (i) and eqn. (ii)

$$2I=\int_0^{\pi/2}1\,dx$$

$$2I=[x]_0^{\pi/2}=\frac{\pi}{2}$$

$$\therefore \quad I=\frac{\pi}{4}$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- Use of the definite integral property $\int_0^a f(x)dx=\int_0^a f(a-x)dx\cdot$ was tried but unsuccessfully by some candidates. Some tried direct integration or integration by substitution and failed after a point.

Answering Tip

- Use properties of integrals to simplify the given integral into easier form. Effective use of properties of definite integrals need to understood.

AI Q. 6. Evaluate : $\int_0^1\log\left(\frac{1}{x}-1\right)dx$. **[ISC, 2013]**

Sol. Let $I=\int_0^1\log\left(\frac{1}{x}-1\right)dx$

$$=\int_0^1\log\left(\frac{1-x}{x}\right)dx$$

Using property,

$\int_0^a f(x)dx=\int_0^a f(a-x)dx$, we have

$$I=\int_0^1\log\left[\frac{1-(1-x)}{1-x}\right]dx$$

$$=\int_0^1\log\left(\frac{x}{1-x}\right)dx$$

$$=\int_0^1\log\left(\frac{1-x}{x}\right)^{-1}dx$$

$$=\int_0^1(-1)\log\left(\frac{1-x}{x}\right)dx$$

$$I=-\int_0^1\log\left(\frac{1-x}{x}\right)dx$$

$$I=-I$$

$$2I=0$$

$$\therefore \quad I=0$$

$$\therefore \quad \int_0^1\log\left(\frac{1}{x}-1\right)dx=0$$

Q. 7. Evaluate : $\int_1^2\frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}dx$. **[ISC, 2012]**

Sol. Let $$I=\int_1^2\frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}dx \qquad \text{...(i)}$$

Using properties,

$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$$I=\int_1^2\frac{\sqrt{3-x}}{\sqrt{3-(3-x)}+\sqrt{3-x}}dx$$

$$I=\int_1^2\frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}dx \qquad \text{...(ii)}$$

Adding eqs. (i) and (ii),

$$2I=\int_1^2\frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}dx=\int_1^2dx$$

$$=[x]_1^2=2-1=1$$

$$\therefore \quad I=\frac{1}{2}$$

Q. 8. Evaluate : $\int_{-3}^{3}|x+2|dx$. **[ISC, 2010]**

Sol. We have $|x+2|=\begin{cases}(x+2) & \text{if } x+2\geq 0\\ -(x+2) & \text{if } x+2\leq 0\end{cases}$

$$=\begin{cases}(x+2) & \text{if } x\geq -2\\ -(x+2) & \text{if } x\leq -2\end{cases}$$

$$\therefore\ \int_{-3}^{3}|x+2|dx=\int_{-3}^{-2}-(x+2)dx+\int_{-2}^{3}(x+2)dx$$

$$=-\left[\frac{x^2}{2}+2x\right]_{-3}^{-2}+\left[\frac{x^2}{2}+2x\right]_{-2}^{3}$$

$$=-\left[2-4-\frac{9}{2}-2(-3)\right]+\left[\frac{9}{2}+6-2+4\right]$$

$$=-\left[4-\frac{9}{2}\right]+\left[\frac{9}{2}+8\right]=4+2\times\frac{9}{2}$$

$$=4+9$$

$$=13$$

Q. 9. Evaluate : $\int_{0}^{5}\left(x+\frac{1}{2}\right)dx$ **as a limit of a sum.** **[ISC, 2019]**

Sol. Let $f(x)=x+\frac{1}{2}$, here $a=0, b=5, nh=b-a$

$\Rightarrow nh=5$

$$\therefore\ I=\int_{0}^{5}\left(x+\frac{1}{2}\right)dx$$

$$=\lim_{h\to 0}h\left[\begin{matrix}f(0)+f(0+h)+f(0+2h)+\\ \ldots+f(0+(n-1)h)\end{matrix}\right]$$

$$=\lim_{h\to 0}h\left[\begin{matrix}\frac{1}{2}+\left(h+\frac{1}{2}\right)+\left(2h+\frac{1}{2}\right)+\ldots\\ +\ldots\left((n-1)h+\frac{1}{2}\right)\end{matrix}\right]$$

$$=\lim_{h\to 0}h\left[\underbrace{\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}}_{n\text{ times}}+h(1+2+\ldots.+(n-1)\right]$$

$$=\lim_{h\to 0}h\left[\frac{1}{2}\times n+h\left(\frac{(n-1)(n-1+1)}{2}\right)\right]$$

$$=\lim_{h\to 0}h\left[\frac{n}{2}+\frac{h}{2}(n(n-1))\right]\quad\left(\because\Sigma n=\frac{n(n+1)}{2}\right)$$

$$=\lim_{h\to 0}\left[\frac{nh}{2}+\frac{hn}{2}.(h(n-1))\right]$$

$$=\lim_{h\to 0}\left[\frac{5}{2}+\frac{5}{2}.(nh-h)\right]\quad(\because nh=5)$$

$$=\lim_{h\to 0}\left[\frac{5}{2}+\frac{5}{2}.(5-h)\right]$$

$$=\frac{5}{2}+\frac{5}{2}.5=\frac{5}{2}+\frac{25}{2}=\frac{30}{2}=15$$

Q. 10. Evaluate : $\int_{0}^{\pi/4}(\tan x+\cot x)^{-1}dx$.

Sol. Let $I=\int_{0}^{\pi/4}(\tan x+\cot x)^{-1}dx$

$$=\int_{0}^{\pi/4}\frac{1}{\tan x+\cot x}dx$$

$$=\int_{0}^{\pi/4}\frac{1}{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}dx$$

$$=\int_{0}^{\pi/4}\frac{\sin x\cos x}{\sin^2 x+\cos^2 x}dx$$

$$=\int_{0}^{\pi/4}\sin x\cos x dx$$

[$\because$ $\sin^2\theta+\cos^2\theta=1$]

On dividing and multiplying by 2, we get

$$I=\frac{1}{2}\int_{0}^{\pi/4}2\sin x\cos x dx$$

$$=\frac{1}{2}\int_{0}^{\pi/4}\sin 2x dx$$

[$\because$ $\sin 2x=2\sin x\cos x$]

$$=\frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_{0}^{\pi/4}$$

$$=-\frac{1}{4}\left[\cos\frac{\pi}{2}-\cos(0)\right]$$

$$=-\frac{1}{4}[0-1]=\frac{1}{4}$$

Q.11. Evaluate : $\int_{0}^{\pi/2}\sin 2x\log\tan x dx$

Sol. Let $I=\int_{0}^{\pi/2}(\log\tan x)\cdot\sin 2x dx$ ...(i)

$$I=\int_{0}^{\pi/2}\log\tan(\pi/2-x)\cdot\sin 2(\pi/2-x)dx$$

[$\because$ $\int_{0}^{a}f(x)dx=\int_{0}^{a}f(a-x)dx$]

$\Rightarrow$ $I=\int_{0}^{\pi/2}\log\cot x\cdot\sin 2x dx$

[$\because$ $\sin(\pi-2x)=\sin 2x$] ...(ii)

On adding eqs. (i) and (ii), we get

$$2I=\int_{0}^{\pi/2}\log\tan x\cdot\sin 2x dx+\int_{0}^{\pi/2}\log\cot x\cdot\sin 2x dx$$

$$=\int_{0}^{\pi/2}\sin 2x\log(\tan x\cdot\cot x)dx$$

[$\because$ $\log m+\log n=\log mn$]

$$=\int_{0}^{\pi/2}\sin 2x\cdot\log 1 dx$$

$\Rightarrow$ $2I=\int_{0}^{\pi/2}0dx=0$ [$\because$ $\log 1=0$]

$\Rightarrow$ $I=0$

AI Q. 12. Evaluate : $\int_{0}^{\pi/2}\frac{\sqrt{\sec x}}{\sqrt{\sec x}+\sqrt{\operatorname{cosec} x}}dx$

Sol. Let $I=\int_{0}^{\pi/2}\frac{\sqrt{\sec x}}{\sqrt{\sec x}+\sqrt{\operatorname{cosec} x}}dx$ ...(i)

Using $\int_{0}^{a}f(x)dx=\int_{0}^{a}f(a-x)dx$

$$\therefore\quad I=\int_{0}^{\pi/2}\frac{\sqrt{\sec\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sec\left(\frac{\pi}{2}-x\right)}+\sqrt{\operatorname{cosec}\left(\frac{\pi}{2}-x\right)}}dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sqrt{\operatorname{cosec} x}}{\sqrt{\operatorname{cosec} x} + \sqrt{\sec x}}\,dx \qquad \ldots\text{(ii)}$$

On adding eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sec x} + \sqrt{\operatorname{cosec} x}}{\sqrt{\operatorname{cosec} x} + \sqrt{\sec x}}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} 1dx = [x]_0^{\pi/2}$$

$$= \left(\frac{\pi}{2} - 0\right)$$

$$\therefore \quad I = \frac{\pi}{4}$$

Q. 13. Evaluate : $\int_a^b \frac{\log x}{x}\,dx.$

Sol. Let $I = \int_a^b \frac{\log x}{x}\,dx$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

When $x = a$, then $t = \log a$

When $x = b$, then $t = \log b$

$$\therefore \quad I = \int_{\log a}^{\log b} t\,dt$$

$$= \left[\frac{t^2}{2}\right]_{\log a}^{\log b}$$

$$= \frac{1}{2}[(\log b)^2 - (\log a)^2]$$

$$I = \frac{1}{2}[(\log b + \log a)\,(\log b - \log a)]$$

$$I = \frac{1}{2}\left[\log ab \cdot \log\frac{b}{a}\right]$$

$\left[\because \log m - \log n = \log mn \right.$

$\left. \text{and } \log m - \log n = \log\frac{m}{n}\right]$

$$I = \frac{1}{2}\log ab \cdot \log\frac{b}{a}.$$

Q. 14. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$.

Sol. Let $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$

$$= \int_0^{\pi} \left\{\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{\sin(2\pi - x)}}\right\}dx$$

$$\left[\because \int_0^{2a} f(x)dx = \int_0^{a}\{f(x)dx + f(2a-x)\}dx\right]$$

$$= \int_0^{\pi} \left\{\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}}\right\}dx$$

$$= \int_0^{\pi} \left\{\frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{1+e^{\sin x}}\right\}dx$$

$$= \int_0^{\pi} \left(\frac{1+e^{\sin x}}{1+e^{\sin x}}\right)dx$$

$$= \int_0^{\pi} dx = [x]_0^{\pi} = \pi$$

Q.15. Evaluate : $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\cot x}}$.

Sol.

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\cot x}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \qquad \text{...(i)}$$

Applying property $\int_a^b f(x)dx = \int_a^b f(a+b-x)$

$$\therefore \text{I} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\,dx$$

$$\text{I} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \qquad \text{...(ii)}$$

Adding eqs. (i) and (ii),

$$\text{I} = \int_{\pi/6}^{\pi/3} 1.dx$$

$$\text{I} = \frac{1}{2}\left[x\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{12}$$

Q. 16. Evaluate : $\int_0^1 x\tan^{-1} x\,dx$. **[ISC, 2002]**

Sol. Let $I = \int_0^1 x\tan^{-1} x\,dx$

$$I = \left[\tan^{-1} x \cdot \frac{x^2}{2}\right]_0^1 - \int_0^1 \left(\frac{1}{1+x^2} \cdot \frac{x^2}{2}\right)dx$$

[Using integration by parts]

$$= \left[\frac{x^2}{2}\tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{1+x^2}\,dx$$

$$= \left(\frac{1}{2}\cdot\frac{\pi}{4}\right) - \frac{1}{2}\left[\int_0^1 \frac{1+x^2}{1+x^2}\,dx - \int_0^1 \frac{1}{1+x^2}\,dx\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 1dx + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{\pi}{8} - \left[\frac{x}{2}\right]_0^1 + \left[\frac{1}{2}\tan^{-1} x\right]_0^1$$

$$\left[\because \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C\right]$$

$$= \frac{\pi}{8} - \frac{1}{2} - 0 + \frac{1}{2}\left(\frac{\pi}{4}\right) - 0$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

6 Marks Questions

AI Q. 1. Evaluate : $\int_0^{\pi/2}\frac{\cos^2 x}{1+\sin x\cos x}dx$ **[ISC, 2018]**

Sol. Let $I = \int_0^{\pi/2}\frac{\cos^2 x}{1+\sin x\cos x}dx$ ...(i)

Apply property

$$\int_0^a f(a)\,dx = \int_0^a f(a-x)\,dx$$

$$I = \int_0^{\pi/2}\frac{\cos^2\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\therefore \quad I = \int_0^{\pi/2}\frac{\sin^2 x}{1+\cos x\sin x}dx \quad \text{...(ii)}$$

Adding eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2}\frac{1}{1+\sin x\cos x}dx$$

$[\because \sin^2\theta + \cos^2\theta = 1]$

Dividing N and D by $\cos^2 x$, we get

$$\therefore \quad 2I = \int_0^{\pi/2}\frac{\sec^2 x}{\sec^2 x+\tan x}dx$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

Also, when $x = 0$, $t = 0$ and $x = \frac{\pi}{2}$, $t = \infty$

$$\therefore \quad 2I = \int_0^{\infty}\frac{dt}{t^2+t+1}$$

$$= \int_0^{\infty}\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$2I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^{\infty}$$

$$2I = \frac{2}{\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right]$$

$$\therefore \quad 2I = \frac{2\pi}{3\sqrt{3}}$$

$$\text{or} \quad I = \frac{\pi}{3\sqrt{3}}$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- A number of candidates could not attempt the question correctly due to incorrect use of properties of definite integrals and using incorrect methods to evaluate integrals.

Answering Tip

- Practice as many problems as you can for successful application of properties of definite integrals and its further simplification.

Q. 2. Prove that : $\int_0^{2\pi}\frac{x\cos x}{1+\cos x}dx = 2\pi^2$ **[ISC, 2006]**

Sol. Let $I = \int_0^{2\pi}\frac{x\cos x}{1+\cos x}dx$ ...(i)

$$I = \int_0^{2\pi}\frac{(2\pi-x)\cos(2\pi-x)}{1+\cos(2\pi-x)}dx$$

$$\left(\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$= \int_0^{2\pi}\frac{(2\pi-x)\cos x}{1+\cos x}dx \quad \text{...(ii)}$$

Add eqs. (i) and (ii), we get

$$2I = \int_0^{2\pi}\frac{(x+2\pi-x)\cos x}{1+\cos x}dx$$

$$\Rightarrow \quad 2I = \int_0^{2\pi}\frac{2\pi\cos x}{1+\cos x}dx \Rightarrow I = \int_0^{2\pi}\frac{\pi\cos x}{1+\cos x}dx$$

$$\Rightarrow \quad I = \pi\int_0^{2\pi}\frac{\cos x+1-1}{1+\cos x}dx$$

$$= \pi\int_0^{2\pi}\left(\frac{\cos x+1}{1+\cos x}-\frac{1}{1+\cos x}\right)dx$$

$$= \pi\left[\int_0^{2\pi}dx-\int_0^{2\pi}\frac{1}{1+\cos x}dx\right]$$

$$= \pi[x]_0^{2\pi}-\pi\int_0^{2\pi}\frac{1}{2\cos^2\frac{x}{2}}dx$$

$$= \pi(2\pi-0)-\frac{\pi}{2}\int_0^{2\pi}\sec^2\frac{x}{2}dx$$

$$= 2\pi^2-\frac{\pi}{2}\left[2\tan\frac{x}{2}\right]_0^{2\pi}$$

$$= 2\pi^2-\pi[\tan\pi-\tan 0]$$

$$I = 2\pi^2-0$$

$$I = 2\pi^2$$

Hence Proved.

Q. 3. Evaluate : $\int_0^{\pi}e^{2x}\sin\left(\frac{\pi}{4}+x\right)dx$. **[ISC, 2005]**

Sol. $\int_0^{\pi}e^{2x}\sin\left(\frac{\pi}{4}+x\right)dx$

Let $I = \int_0^{\pi}e^{2x}\cdot\sin\left(\frac{\pi}{4}+x\right)dx$

Integrating by parts, without limits

$$I = \sin\left(\frac{\pi}{4}+x\right)\cdot\frac{e^{2x}}{2}-\int\cos\left(\frac{\pi}{4}+x\right)\cdot\frac{e^{2x}}{2}dx$$

$$= \sin\left(\frac{\pi}{4}+x\right)\cdot\frac{e^{2x}}{2}-\frac{1}{2}\int e^{2x}\cdot\cos\left(\frac{\pi}{4}+x\right)dx$$

$$= \sin\left(\frac{\pi}{4}+x\right)\cdot\frac{e^{2x}}{2}-\frac{1}{2}\Bigg[\cos\left(\frac{\pi}{4}+x\right)\cdot\frac{e^{2x}}{2}$$

$$-\int -\sin\left(\frac{\pi}{4}+x\right)\cdot\frac{e^{2x}}{2}dx\Bigg]$$

$$= \sin\left(\frac{\pi}{4}+x\right)\cdot\frac{e^{2x}}{2}-\frac{1}{4}e^{2x}\cos\left(\frac{\pi}{4}+x\right)$$

$$-\frac{1}{4}\int e^{2x}\cdot\sin\left(\frac{\pi}{4}+x\right)dx$$

$$\Rightarrow I\left(1+\frac{1}{4}\right)=\frac{1}{4}e^{2x}\left\{2\sin\left(\frac{\pi}{4}+x\right)-\cos\left(\frac{\pi}{4}+x\right)\right\}$$

$$\Rightarrow \qquad I=\frac{1}{5}e^{2x}\left\{2\sin\left(\frac{\pi}{4}+x\right)-\cos\left(\frac{\pi}{4}+x\right)\right\}$$

Now, $\int_0^{\pi} e^{2x}\sin\left(\frac{\pi}{4}+x\right)dx$

$$=\frac{1}{5}\left[e^{2x}\left\{2\sin\left(\frac{\pi}{4}+x\right)-\cos\left(\frac{\pi}{4}+x\right)\right\}\right]_0^{\pi}$$

$$=\frac{1}{5}\left[e^{2\pi}\left\{2\sin\left(\pi+\frac{\pi}{4}\right)-\cos\left(\pi+\frac{\pi}{4}\right)\right\}\right.$$

$$\left.-2\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right]$$

$$=\frac{1}{5}\left[e^{2\pi}\left(-2\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)-\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right]$$

$$=\frac{1}{5}\left[-\frac{e^{2\pi}}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right]$$

$$\therefore\ \int_0^{\pi} e^{2x}\sin\left(\frac{\pi}{4}+x\right)dx$$

$$=-\frac{1}{5\sqrt{2}}[e^{2\pi}+1]$$

AI Q. 4. Evaluate : $\int_0^{\frac{3}{2}}|x\cos\pi x|dx$.

Sol. Let $\qquad I=\int_0^{\frac{3}{2}}|x\cos\pi x|dx$

In $\qquad 0\le x\le\frac{1}{2}, x\cos\pi x\le 0,\ x\ge 0$

In $\qquad \frac{1}{2}\le x\le\frac{3}{2}, x\cos\pi x\le 0,\ x\ge 0$

$$\therefore \qquad I=\int_0^{\frac{1}{2}} x\cos\pi x\,dx-\int_{\frac{1}{2}}^{\frac{3}{2}} x\cos\pi x\,dx$$

$$=\left[x\frac{(\sin\pi x)}{\pi}-\int(1)\cdot\frac{(\sin\pi x)}{\pi}dx\right]_0^{\frac{1}{2}}$$

$$-\left[x\frac{(\sin\pi x)}{\pi}-\int(1)\cdot\frac{(\sin\pi x)}{\pi}dx\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$\therefore \qquad I=\left[\frac{x\sin\pi x}{\pi}+\frac{\cos\pi x}{(\pi)^2}\right]_0^{\frac{1}{2}}$$

$$-\left[\frac{x\sin\pi x}{\pi}+\frac{\cos\pi x}{(\pi)^2}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$=\left[\frac{\frac{1}{2}\sin\frac{\pi}{2}}{\pi}+\frac{\cos\frac{\pi}{2}}{\pi^2}-0-\frac{\cos 0}{\pi^2}\right]$$

$$-\left[\frac{\frac{3}{2}\sin\frac{3\pi}{2}}{\pi}+\frac{\cos\frac{3\pi}{2}}{\pi^2}-\frac{\frac{1}{2}\sin\frac{\pi}{2}}{\pi}-\frac{\cos\frac{\pi}{2}}{\pi^2}\right]$$

$$\therefore \qquad I=\left[\frac{1}{2\pi}-\frac{1}{\pi^2}\right]-\left[\frac{3}{2\pi}(-1)+0-\frac{1}{2\pi}-0\right]$$

$$=\left[\frac{1}{2\pi}-\frac{1}{\pi^2}+\frac{3}{2\pi}+\frac{1}{2\pi}\right]$$

$$\therefore \qquad I=\frac{1}{\pi}\left[\frac{5}{2}-\frac{1}{\pi}\right]$$

Q. 5. Evaluate : $\int_{-\pi}^{\pi}(\cos ax-\sin bx)^2dx$.

Sol. Let $\qquad I=\int_{-\pi}^{\pi}(\cos ax-\sin bx)^2dx$

$$=\int_{-\pi}^{\pi}(\cos^2 ax+\sin^2 bx)dx$$

$$-\int_{-\pi}^{\pi}2\cos ax\sin bx\,dx$$

$$=I_1-I_2$$

$$I_1=\int_0^{\pi}2(\cos^2 ax+\sin^2 bx)dx$$

$I_2=0$ (Being an even function)

$\therefore \quad I=I_1$ (Being an odd function)

$$=\int_0^{\pi}(2\cos^2 ax+2\sin^2 bx)dx$$

$$=\int_0^{\pi}(1+\cos 2ax+1-\cos 2bx)dx$$

$$\int_0^{\pi}2dx+\int_0^{\pi}\cos 2ax\,dx-\int_0^{\pi}\cos 2bx\,dx$$

$$=\left[2x+\frac{\sin 2ax}{2a}-\frac{\sin 2bx}{2b}\right]_0^{\pi}$$

$$=\left[\left\{2\pi+\frac{\sin 2a\pi}{2a}-\frac{\sin 2b\pi}{2b}\right\}-0\right]$$

$$=2\pi+\frac{\sin 2a\pi}{2a}-\frac{\sin 2b\pi}{2b}$$

Q. 6. Evaluate : $\int_2^5[|x-2|+|x-3|+|x-5|]dx$.

Sol. Let $\qquad I=\int_2^5[|x-2|+|x-3|+|x-5|]\,dx$

$$=\int_2^3 f(x)dx+\int_3^5 f(x)dx \qquad \text{...(i)}$$

where $f(x)=|x-2|+|x-3|+|x-5|$

$$=\begin{cases}x-2-x+3-x+5=6-x, & \text{if } 2\le x\le 3\\ x-2+x-3-x+5=x, & \text{if } 3\le x\le 5\end{cases}$$

Now, from (i), we have

$$I=\int_2^3(6-x)dx+\int_3^5 x\,dx$$

$$=\left[6x-\frac{x^2}{2}\right]_2^3+\left[\frac{x^2}{2}\right]_3^5$$

$$=\left(18-\frac{9}{2}\right)-(12-2)+\frac{1}{2}(25-9)$$

$$=\frac{23}{2}$$

Q. 7. Evaluate the following definite integral :

$$\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx$$

Sol. Let $I=\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx$

$$=\int_{-\pi}^{\pi}\frac{2x}{1+\cos^2 x}dx+\int_{-\pi}^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx$$

$$= I_1 + I_2$$

$I_1 = 0$ (Being an odd function)

$I_2 = 2\int_0^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx$ (Being an even function)

∴ $I = I_2$

$$=2\int_0^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx$$

$$=4\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$$

Let $I_3=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$

Apply property $\int_0^a f(x)dx=\int_0^a f(a-x)dx$,

$$=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$$

$$=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx-\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$$

$$=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx-I_3$$

∴ $2I_3=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

Putting $\cos x = t \Rightarrow -\sin x dx = dt$

When $x = 0$; $t = 1$ and $x = \pi$; $t = -1$

$$2I_3=-\pi\int_1^{-1}\frac{dt}{1+t^2}$$

$$=\pi[\tan^{-1}x]_{-1}^{1} \qquad \left[\because \int_0^b f(x)dx=\int_0^a f(x)dx\right]$$

$$=\frac{\pi^2}{2}$$

$\Rightarrow$ $I_3=\frac{\pi^2}{4}$

Hence, $I = 4I_3 = \pi^2$.

AI **Q. 8. Show that : $\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\sin x+\cos x}dx=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$**

Sol. Let $I = \int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\sin x+\cos x}dx$

$$= \int_0^{\frac{\pi}{2}}\frac{\sin^2(\pi/_2-x)}{\sin(\pi/_2-x)+\cos\pi/_2-x}dx$$

$$2I = \int_0^{\frac{\pi}{2}}\frac{\sin^2 x+\cos^2 x}{\cos x+\sin x}dx$$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos 2x}{\sin x+\cos x}dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{dx}{\sin x+\cos x}$$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos^2 x-\sin^2 x}{\cos x+\sin x}dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}$$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}}(\cos x-\sin x)dx$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\operatorname{cosec}\left(x+\frac{\pi}{4}\right)dx$$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos x\,dx+\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin x\,dx$$

$$= \frac{1}{2\sqrt{2}}\left[-\log\left|\operatorname{cosec}\left(x+\frac{\pi}{4}\right)+\cot\left(x+\frac{\pi}{4}\right)\right|\right]_0^{\pi/2}$$

$$-\frac{1}{2}[\sin x]_0^{\pi/2}+\frac{1}{2}[-\cos x]_0^{\pi/2}$$

$$= -\frac{1}{2\sqrt{2}}\left[\log|\sqrt{2}-1|-\log|\sqrt{2}+1|\right]$$

$$-\frac{1}{2}(1-0]-\frac{1}{2}[0-1]$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|-\frac{1}{2}+\frac{1}{2}$$

$$= \frac{1}{2\sqrt{2}}\left|\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$= \frac{1}{2\sqrt{2}}\log|(\sqrt{2}+1)^2|$$

$$= \frac{\log|\sqrt{2}+1|}{\sqrt{2}}$$

∴ $I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ **Hence Proved.**

Q. 9. Evaluate : $\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sqrt{\tan x}}$.

Sol. Let $I=\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sqrt{\tan x}}$ ...(i)

$$=\int_0^{\frac{\pi}{2}}\frac{1}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}dx$$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{1+\sqrt{\cot x}}dx$$

$=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}dx$ ...(ii)

By adding eqs. (i) and (ii),

$\Rightarrow$ $2I=\int_0^{\frac{\pi}{2}}\frac{1+\sqrt{\tan x}}{1+\sqrt{\tan x}}dx$

$$2I=\int_0^{\frac{\pi}{2}}1\cdot dx=\frac{\pi}{2}$$

$$\Rightarrow \quad I=\frac{\pi}{4}$$

AI Q. 10. Evaluate : $\int_{-1}^{2}|x^3-x|\,dx$.

Sol. $\int_{-1}^{2}|x^3-x|\,dx$

$$=\int_{-1}^{0}(x^3-x)dx-\int_{0}^{1}(x^3-x)dx+\int_{1}^{2}(x^3-x)dx$$

$$=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}-\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{0}^{1}+\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{1}^{2}$$

$$=\left[(0-0)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]-\left[\left(\frac{1}{4}-\frac{1}{2}\right)-(0-0)\right]+\left[(4-2)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$=\frac{1}{4}+\frac{1}{4}+2+\frac{1}{4}=\frac{11}{4}$$

Q. 11. Evaluate : $\int_{0}^{\pi}\frac{x}{a^2\cos^2x+b^2\sin^2x}dx$.

Sol. $$I=\int_{0}^{\pi}\frac{x}{a^2\cos^2x+b^2\sin^2x}dx \quad \text{...(i)}$$

Applying property $\int_{0}^{a}f(x)dx=\int_{0}^{a}f(a-x)dx$

$$\text{I}=\int_{0}^{\pi}\frac{(\pi-x)}{a^2\cos^2x+b^2\sin^2x}dx \quad \text{...(ii)}$$

Adding eqs. (i) and (iii), we get

$$2I=\pi\int_{0}^{\pi}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx$$

Dividing numerator and denominator by $\cos^2 x$

$$\Rightarrow \quad I=\pi\int_{0}^{\pi/2}\frac{\sec^2x}{a^2+b^2\tan^2x}dx$$

Let $b\tan x = t$

$\therefore \quad b\sec^2 x\,dx = dt$

When $x = 0, t = 0;$

and $\quad x=\frac{\pi}{2}, t=\infty$

$$\Rightarrow \quad I=\frac{\pi}{b}\int_{0}^{\infty}\frac{dt}{a^2+t^2}$$

$$=\frac{\pi}{b}\left[\frac{1}{a}\tan^{-1}\frac{t}{a}\right]_{0}^{\infty}$$

$$\Rightarrow \quad I=\frac{\pi}{ab}\left(\frac{\pi}{2}-0\right)$$

$$=\frac{\pi^2}{2ab}$$

Q. 12. Evaluate : $\int_{0}^{\pi}\frac{x\tan x}{\sec x\,\text{cosec}\,x}dx$.

Sol. Let $$I=\int_{0}^{\pi}\frac{x\tan x}{\sec x\,\text{cosec}\,x}dx \quad \text{...(i)}$$

Applying property

$$\therefore \quad I=\int_{0}^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)\text{cosec}(\pi-x)}dx$$

$$\Rightarrow \quad I=\int_{0}^{\pi}\frac{(\pi-x)\tan x\,dx}{\sec x\cdot\text{cosec}\,x} \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\pi\int_{0}^{\pi}\frac{\tan x dx}{\sec x\cdot\text{cosec}\,x}$$

$$2I=\int_{0}^{\pi}\pi\sin^2 x dx$$

$$2I=\ 2\pi\int_{0}^{\pi/2}\sin^2 x\,dx$$

$$\left[\because \int_{0}^{2a}f(x)dx=2\int_{0}^{a}f(x)dx,\text{ if } f(x) \text{ is even}\right]$$

$$\Rightarrow \quad 2I=2\pi\int_{0}^{\pi/2}\frac{(1-\cos 2x)dx}{2}$$

$$\Rightarrow \quad 2I=\pi\left[x-\frac{\sin 2x}{2}\right]_{0}^{\pi/2}$$

$$\Rightarrow \quad 2I=\frac{\pi}{2}[\pi]=\frac{\pi^2}{2}$$

$$\Rightarrow \quad I=\frac{\pi^2}{4}$$

AI Q. 13. Evaluate : $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos 2x\log(\sin x)dx$.

Sol. $I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos 2x\ \log(\sin x)dx$

$$=\left[\frac{1}{2}(\log\sin x)\cdot\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{2}\cot x\cdot\sin 2x dx$$

$$=\left[0-\frac{1}{2}\log\left(\frac{1}{\sqrt{2}}\right)\right]-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2 x dx$$

$$=\frac{1}{4}\log 2-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1+\cos 2x)\ dx$$

$$=\frac{1}{4}\log 2-\frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\frac{1}{4}\log 2-\frac{1}{2}\left[\left(\frac{\pi}{2}+0\right)-\left(\frac{\pi}{4}+\frac{1}{2}\right)\right]$$

$$=\frac{1}{4}\log 2-\frac{\pi}{8}+\frac{1}{4}$$

Q. 14. Prove that : $\int_{0}^{\pi/2}(\sqrt{\tan}\ x+\sqrt{\cot}\ x)dx=\sqrt{2}\ \pi$

Sol. Let $I = \int_0^{\pi/2}(\sqrt{\tan}\ x + \sqrt{\cot}\ x)dx$

$$= \int_0^{\pi/2}\left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right)dx$$

$$= \int_0^{\pi/2}\frac{\sin x + \cos x}{\sqrt{\cos x \sin x}}dx$$

Now divide and multiply with $\sqrt{2}$ in N and D

$$= \sqrt{2}\int_0^{\pi/2}\frac{\sin x + \cos x}{\sqrt{2\cos x \sin x}}dx$$

$$= \sqrt{2}\int_0^{\pi/2}\frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}dx$$

Let $\sin x - \cos x = t$

$\Rightarrow$ $(\cos x + \sin x)\,dx = dt$

When $x = 0, t = -1$ and $x = \frac{\pi}{2},\ t = 1$

$$\therefore \quad I = \sqrt{2}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}dt$$

$$= \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{1}$$

$$= \sqrt{2}\ [2\sin^{-1}(1)]$$

$$= 2\sqrt{2} \times \frac{\pi}{2}$$

$$= \sqrt{2}\pi$$

Q. 15. Evaluate : $\int_2^5(x^2+3)dx$ as a limit of sum.

Sol. $\int_2^5(x^2+3)dx$

$$= \int_0^5(x^2+3)dx - \int_0^2(x^2+3dx)$$

By definition,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+\ldots + f\{a+(n-1)h\}$$

where, $h = \frac{b-a}{n}$

For $a = 0, b = 5, f(x) = x^2 + 3,$

$$h = \frac{5-0}{n} = \frac{5}{n}$$

$$\therefore \int_0^5(x^2+3)dx = 5\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right) + f\left(\frac{10}{n}\right)+\ldots+f\left(\frac{5(n-1)}{n}\right)\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[3+\left(\frac{5^2}{n^2}+3\right) + \left(\frac{10^2}{n^2}+3\right)+\ldots+\left(\frac{(5n-5)^2}{n^2}+3\right)\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[(3+3+3+\ldots+n\text{ times}) + \frac{5^2}{n^2}\{1^2+2^2+3^2+\ldots+(n-1)^2\}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[3n+\frac{25}{n^2}\frac{(n-1)n(2n-1)}{6}\right]$$

$$= 5\lim_{n\to\infty}\left[3+\frac{25}{6}\left\{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right\}\right]$$

$$= 5\left[3+\frac{25}{6}\times 2\right] = \frac{170}{3}$$

Similarly,

$$\int_0^2(x^2+3)dx = 2\left[3+\frac{4}{6}\times 2\right] = \frac{26}{3}$$

$$\therefore \quad \int_2^5(x^2+3)dx = \frac{170}{3} - \frac{26}{3} = \frac{144}{3} = 48$$

Q. 16. Evaluate : $\int_1^3(e^{2-3x}+x^2+1)dx$ as a limit of sum.

Sol. Here, $a = 1, b = 3$ and $f(x) = e^{2-3x} + x^2 + 1$

$\therefore nh = b - a = 3 - 1 = 2$

We know that

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\{f(a)+f(a+h)+f(a+2h)+\ldots. + f[a+(n-1)h]\}$$

Now, $f(a) = f(1) = e^{2-3\times 1} + 1^2 + 1$

$f(a+h) = f(1+h) = e^{2-3(1+h)} + (1+h)^2 + 1$

$f(a+2h) = f(1+2h) = e^{2-3(1+2h)} + (1+2h)^2 + 1$

$f[a+(n-1)h] = f[1+(n-1)h] = e^{2-3[1+(n-1)h} + [1+(n-1)h)^2 + 1$

Adding these equations, we get

$$f(a)+f(a+h)+f(a+2h)+\ldots.+f[a+(n-1)h]$$

$$= \left[e^{2-3\times 1}+1^2+1\right]+\left[e^{2-3(1+h)}+(1+h)^2\right] + \left[e^{2-3(1+2h)}+(1+2h)^2+1\right]+\ldots\ldots+ \left\{e^{2-3[1+(n-1)h]}+[1+(n-1)h]^2+1\right\}$$

$$\therefore \int_1^3(e^{2-3x}+x^2+1)\,dx = \lim_{h\to 0} h\,[e^2(e^{-3}+e^{-3}(1+h)\ e^{-3(1+2h)}+\ldots+e^{3[1+(n-1)h]})]$$

$$+\lim_{h\to 0} h\left[1^2+(1+h)^2+(1+2h)^2\ldots.+[1+(n-1)h]^2\right]$$

$$+\lim_{h\to 0} h[1+1+1+\ldots.+1]$$

$$= \lim_{h\to 0} h\left\{e^2\times\left(e^{-3}\frac{1-e^{3nh}}{1-e^{-3h}}\right)\right\}+\lim_{h\to 0} h\ (1+1+1+\ldots.+1)$$

$$+\ 2h[1 + 2 + 3 + \ldots\ldots + (n-1)]$$

$$+\ h^2(1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2)$$

$$+\lim_{h\to 0} h(1+1+1+\ldots\ldots+1)\Big\}$$

$$= \lim_{h\to 0} h\left\{\frac{e^{-1}(1-e^{-6})}{1-e^{-3h}}\right\}$$

$$+\lim_{h\to 0} h\left[n + 2h\frac{(n-1)n}{2} + h^2\frac{(n-1)n(2n+1)}{6}\right] + \lim_{h\to 0} hn$$

$$= \lim_{h\to 0} h\left\{\frac{e^{-1}(1-e^{-6})}{1-e^{-3h}}\right\} + \lim_{h\to 0} h$$

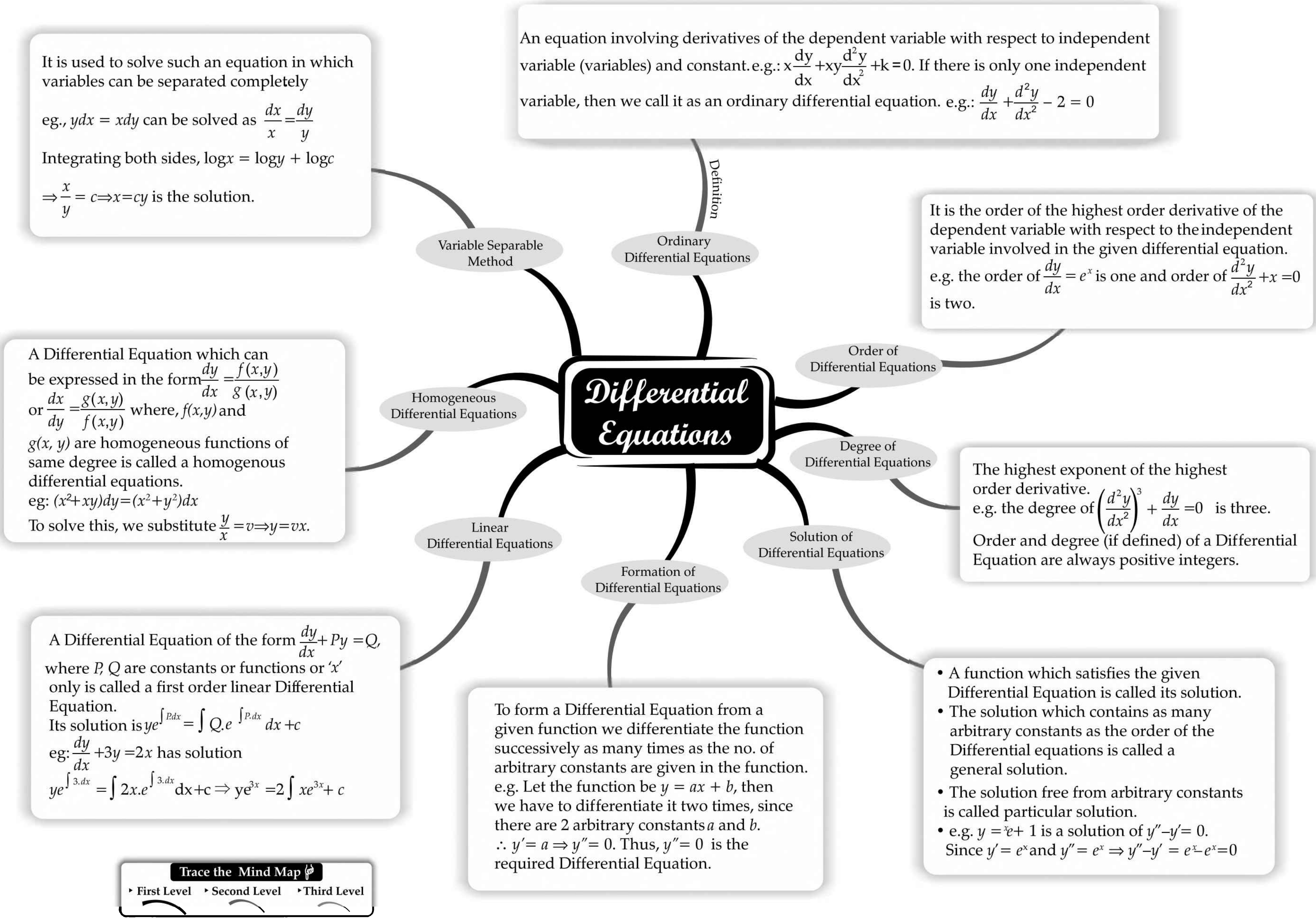
$$\left[2nh + (nh-h)nh + \frac{(nh-1)nh(2nh-1)}{6}\right]$$

$$= \frac{1}{e}\left(1-\frac{1}{e^6}\right)\times\frac{\lim\limits_{h\to 0} e^{3h}}{3\times\lim\limits_{h\to 0}\left(\dfrac{e^{3h}-1}{3h}\right)}$$

$$+\lim_{h\to 0}\left[2\times 2 + (2-h)\times 2 + \frac{(2-h)\times 2\times(2\times 2-h)}{6}\right]$$

$$= \frac{1}{e}\left(1-\frac{1}{e^6}\right)\times\frac{1}{3\times 1} + \left(4+4+\frac{8}{3}\right)$$

$$= \frac{1}{3e}\left(1-\frac{1}{e^6}\right) + \frac{32}{3}$$



Differential Equations

Ordinary Differential Equations

Definition

An equation involving derivatives of the dependent variable with respect to independent variable (variables) and constant. e.g.: $x\frac{dy}{dx}+xy\frac{d^2y}{dx^2}+k=0$. If there is only one independent variable, then we call it as an ordinary differential equation. e.g.: $\frac{dy}{dx}+\frac{d^2y}{dx^2}-2=0$

Order of Differential Equations

It is the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.
e.g. the order of $\frac{dy}{dx}=e^x$ is one and order of $\frac{d^2y}{dx^2}+x=0$ is two.

Degree of Differential Equations

The highest exponent of the highest order derivative.
e.g. the degree of $\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=0$ is three.

Order and degree (if defined) of a Differential Equation are always positive integers.

Solution of Differential Equations

- A function which satisfies the given Differential Equation is called its solution.
- The solution which contains as many arbitrary constants as the order of the Differential equations is called a general solution.
- The solution free from arbitrary constants is called particular solution.
- e.g. $y={}^{x}e+1$ is a solution of $y''-y'=0$.
  Since $y'=e^x$ and $y''=e^x \Rightarrow y''-y'=e^x-e^x=0$

Formation of Differential Equations

To form a Differential Equation from a given function we differentiate the function successively as many times as the no. of arbitrary constants are given in the function.
e.g. Let the function be $y=ax+b$, then we have to differentiate it two times, since there are 2 arbitrary constants a and b.
$\therefore y'=a \Rightarrow y''=0$. Thus, $y''=0$ is the required Differential Equation.

Linear Differential Equations

A Differential Equation of the form $\frac{dy}{dx}+Py=Q$, where P, Q are constants or functions or 'x' only is called a first order linear Differential Equation.
Its solution is $ye^{\int P dx}=\int Q.e^{\int P.dx}\,dx+c$
eg: $\frac{dy}{dx}+3y=2x$ has solution
$ye^{\int 3.dx}=\int 2x.e^{\int 3.dx}dx+c \Rightarrow ye^{3x}=2\int xe^{3x}+c$

Homogeneous Differential Equations

A Differential Equation which can be expressed in the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ or $\frac{dx}{dy}=\frac{g(x,y)}{f(x,y)}$ where, $f(x,y)$ and $g(x,y)$ are homogeneous functions of same degree is called a homogenous differential equations.
eg: $(x^2+xy)dy=(x^2+y^2)dx$
To solve this, we substitute $\frac{y}{x}=v \Rightarrow y=vx$.

Variable Separable Method

It is used to solve such an equation in which variables can be separated completely

eg., $ydx=xdy$ can be solved as $\frac{dx}{x}=\frac{dy}{y}$

Integrating both sides, $\log x=\log y+\log c$

$\Rightarrow \frac{x}{y}=c \Rightarrow x=cy$ is the solution.

CHAPTER
8
DIFFERENTIAL EQUATIONS

SYLLABUS

- ***Definition, order and degree, general and particular solutions of a differential equation. Formation of differential equation whose general solution is given. Solution of differential equations by method of separation of variables solutions of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type: $\frac{dy}{dx} + PY = Q$, where P and Q are functions of x or constants. $\frac{dy}{dx} + PX = Q$, where P and Q are functions of Y or constants.***
  - ***Differential equations, order and degree.***
  - ***Formation of differential equation by eliminating arbitrary constant(s).***
  - ***Solution of differential equations.***
  - ***Variable separable.***
  - ***Homogeneous equations.***
  - ***Linear form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x only. Similarly, for dx/dy.***
  - ***Solve problems of application on growth and decay.***
  - ***Solve problems on velocity, acceleration, distance and time.***
  - ***Solve population-based problems on application of differential equations.***
  - ***Solve problems of application on coordinate geometry.***

NOTE 1: Equations reducible to variable separable type are included.

NOTE 2: The second order differential equations are excluded.

Trend Analysis

List of Concepts	2016	2017	2018	2019	2020
Basic Concepts and Variable Separable Methods	1 Q (2 marks)	1 Q (2 marks) 1 Q (4 marks)		1 Q (4 marks)	1 Q (2 marks)
Linear Differential Equation	1 Q (4 marks)		1 Q (4 marks)		1 Q (4 marks)
Homogeneous Differential Equation			1 Q (6 marks)		
Application of Differential Equations	1 Q (4 marks)				

TOPIC-1
Basic Concepts and Variable Separable Methods

Revision Notes

- **Differential Equation :**

  An equation consisting of an independent variable, dependent variable and differential coefficients of dependent variable with respect to the independent variable is known as differential equation.

TOPIC - 1
Basic Concepts and Variable Separable Methods .... **P. 217**
TOPIC - 2
Linear Differential Equations .... **P. 225**
TOPIC - 3
Homogeneous Differential Equations .... **P. 231**
TOPIC - 4
Application of Differential Equations .... **P. 237**

e.g. : **(i)** $\frac{d^2y}{dx^2} = -a^2y$, **(ii)** $\frac{dy}{dx} = \frac{x+y}{x^2}$, **(iii)** $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = p\frac{dy}{dx}$

Scan to know more about this topic

Differential equation, order and degree

- **Order of Differential Equation :** The order of a differential equation is the order of the highest derivative appearing in the differential equation.

  e.g. : $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^3 + 2 = 0$ is the differential equation of order 3 because highest order derivative of y w.r.t. x is $\frac{d^3y}{dx^3}$.

- **Degree of Differential Equation :** The degree of the differential equation is the degree (power) of the highest order derivative, when the differential coefficient has been made free from the radicals and fractions.

  e.g. : $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}} = 3\frac{d^2y}{dx^2}$ is the differential equation of degree 3, because the power of highest order derivative $\frac{d^2y}{dx^2}$ is 3 (after cubing).

Scan to know more about this topic

General and particular solution of differential equation

- **Formation of the Differential Equation :**

  If the equation of the family of curves is given then its differential equation is obtained by **eliminating arbitrary constants** occurring in equation with the help of equation of the curve and the equations obtained by differentiating the equation of the curve.

- **Algorithm for the Formation of the Differential Equation :**

  Step 1 : Write down the given equation of the curve.

  Step 2 : Differentiate the given equation with respect to the independent variable as many times as the number of arbitrary constants.

  Step 3 : Eliminate the arbitrary constants by using given equation and the equations obtained by the differentiation in step 2.

Scan to know more about this topic

Formation of differential equation

- **Solution of Differential Equations :**

  (a) General solution : The solution which contains as many as arbitrary constants as the order of the differential equations, *e.g.*, $y = \alpha \cos x + \beta \sin x$ is the general solution of $\frac{d^2y}{dx^2} + y = 0$.

  Here, the differential equation is of second order and there are two arbitrary constants *i.e.*, α and β in the general solution.

  (b) Particular solution: Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution *e.g.*, $y = 3 \cos x + 2 \sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

  (c) Solution of Differential by Variable Separable Method : A variable separable form of the differential equation is the one which can be expressed in the form of $f(x)\,dx = g(y)\,dy$. The solution is given by $\int f(x)dx = \int g(y)dy + k$, or $\int g(y)\,dy = \int f(n)\,dn + k$, where k is the constant of integration.

Scan to know more about this topic

Separation of variables method

1 Mark Questions

AI **Q. 1. The degree of the differential equation** $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{dy}{dx}\right)$ **is:**

(a) 1 **(b)** 2

(c) 3 **(d)** not defined

Sol. Correct option : (d)

Explanation : The degree of above differential equation is not defined because when we expand $\sin \frac{dy}{dx}$ we get an infinite series in the increasing powers of $\frac{dy}{dx}$. Therefore its degree is not defined.

Q. 2. The degree of the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}=\frac{d^2y}{dx^2} \text{ is}$$

(a) 4 **(b) $\frac{3}{2}$**

(c) not defined **(d) 2**

Sol. Correct option : (d)

Explanation : Given that,

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=\frac{d^2y}{dx^2}$$

On squaring both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$$

So, the degree of differential equation is 2.

AI **Q. 3. The order and degree of the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}+x^{1/5}=0$ respectively, are :**

(a) 2 and 4 **(b) 2 and 2**

(c) 2 and 3 **(d) 3 and 3**

Sol. Correct option : (a)

Explanation :

Given that, $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{4}}=-x^{1/5}$

$$\Rightarrow \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{4}}=-x^{1/5}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^{\frac{1}{4}}=-\left(x^{1/5}+\frac{d^2y}{dx^2}\right)$$

On squaring both sides, we get

$$\left(\frac{dy}{dx}\right)^{\frac{1}{2}}=\left(x^{1/5}+\frac{d^2y}{dx^2}\right)^2$$

Again, on squaring both sides, we have

$\frac{dy}{dx}=\left(x^{\frac{1}{5}}+\frac{d^2y}{dx^2}\right)^4$ Order = 2, degree = 4

Q. 4. Family $y = Ax + A^3$ of curves is represented by the differential equation of degree :

(a) 1 **(b) 2**

(c) 3 **(d) 4**

Sol. Correct option : (a)

Explanation :

Given that, $y = Ax + A^3$

$\Rightarrow \frac{dy}{dx}=A$

[We can differential above equation only once because it has only one arbitrary constant.]

$\therefore$ Degree = 1

Q. 5. Which of the following is a second-order differential equation?

(a) $(y')^2+x=y^2$ **(b) $y'y''+y=\sin x$**

(c) $y'''+(y'')^2+y=0$ **(d) $y'=y^2$**

Sol. Correct option : (b)

Explanation : The second-order differential equation is $y'y''+y=\sin x$.

Q. 6. Find the sum of order and degree of the differential equation $\left(\frac{dy}{dx}\right)^5+3xy\left(\frac{d^3y}{dx^3}\right)^2+y^2\left(\frac{d^2y}{dx^2}\right)^3=0$

[ISC Specimen Paper, 2021]

Sol. Given differential equation :

$$\left(\frac{dy}{dx}\right)^5+3xy\left(\frac{d^6y}{dx^6}\right)^2+y^2\left(\frac{d^2y}{dx^2}\right)^3=0$$

$\therefore$ Order = 3 ; degree = 2

Sum of order and degree = 3+ 2 = 5

AI **Q. 7. Write the order of the differential equation :**

$$\log\left(\frac{d^2y}{dx^2}\right)=\left(\frac{dy}{dx}\right)^3+x$$

Sol. The highest order derivative occurs in the given differential equation is $\frac{d^2y}{dx^2}$

Hence, order of differential equation = 2

Q. 8. Write the sum of the order and degree of the differential equation $1+\left(\frac{dy}{dx}\right)^4=7\left(\frac{d^2y}{dx^2}\right)^3$.

Sol. Degree of the given differential equation = 3

Order of the given differential equation = 2

Hence, the sum of order and degree = 2 + 3 = 5

Q. 9. Write the order and the degree of the following differential equation :

$$x^3\left(\frac{d^2y}{dx^2}\right)^2+x\left(\frac{dy}{dx}\right)^4=0$$

Sol. Given, $x^3\left(\frac{d^2y}{dx^2}\right)^2+x\left(\frac{dy}{dx}\right)^4=0$

The highest order derivative is $\left(\frac{d^2y}{dx^2}\right)$, hence the order of given differential equation is 2.

Also, the power of the highest order derivative is 2, hence the degree of differential equation is 2.

Therefore, Order = 2 and Degree = 2

Examiner's Comment

- Some students take the degree as the highest power of the derivative.

Answering Tip

- Degree is the highest power of the higher order derivative.

Q. 10. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$.

Sol. Given, differential equation is

$$x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$$

The highest order derivative occurs in the given differential equation is $\frac{d^2y}{dx^2}$

Hence, order of differential equation = 2

Also, the power of highest order derivative is 1.

Hence, degree of differential equation is 1.

AI **Q. 11. Find the order and degree (if defined) of the differential equation.**

$$\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right)$$

Sol. Given differential equation is

$$\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right)$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$ So it is of order 2.

Clearly, the differential equation is not expressible as polynomial in $\frac{d^2y}{dx^2}$. So, its degree is not defined.

Hence, order = 2

degree = not defined

Examiner's Comment

- Mostly students write the degree as 1.

Answering Tip

- Note the questions in which degree is not defined.

Q. 12. If m and n are the order and degree, respectively of the differential equation $y\left(\frac{dy}{dx}\right)^3+x^3\left(\frac{d^2y}{dx^2}\right)^2-xy=\sin x$, then write the value of $m+n$.

Sol. $m + n = 4$

$\because \quad m = 2$ (second order derivative)

$\therefore \quad n = 2$

(degree of the highest order derivative)

2 Marks Questions

AI **Q. 1. Solve** $\frac{dy}{dx} = 1 - xy + y - x$

[ISC Specimen Paper, 2021] [ISC, 2017]

Sol.

$$\frac{dy}{dx} = 1 - xy + y - x$$

$$\frac{dy}{dx} = (1 + y) - x(1 + y)$$

$$= (1 + y)(1 - x)$$

$$\Rightarrow \quad \frac{dy}{1+y} = (1 - x)dx$$

Integrating on both the sides,

$$\int\frac{dy}{1+y} = \int(1-x)dx$$

$$\ln|1 + y| = x-\frac{x^2}{2}+c$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates could not separate the variables to solve the differential equation. Some candidates attempted it by linear equation method but made errors in calculating integrating factor and further. In most of the cases, solution had been written without a constant '*c*'.

Answering Tip

- Do adequate practice of various types of differential equations.

Q. 2. Form a differential equation of the family of the curves $y^2 = 4ax$. **[ISC, 2020]**

Sol.

$$y^2 = 4ax \quad \text{...(i)}$$

$$\frac{d}{dx}y^2 = \frac{d}{dx}(4ax)$$

$$2y\frac{dy}{dx} = 4a \quad (1)$$

$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

$$\frac{y}{2}\frac{dy}{dx} = a \quad \text{...(ii)}$$

Form (i) and (ii),

$$y^2 = 4\frac{y}{2}\frac{dy}{dx}x$$

$$y^2 = 2x\frac{dy}{dx}$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Many candidates did not have an idea of formation of differential equation and made error whole solving it.

Answering Tip

- Understand the fundamentals on formation of differential equations and concepts of identifying the order and degree of the equation.

AI **Q. 3. Find the differential equation of the family of concentric circles $x^2 + y^2 = a^2$. [ISC, 2018]**

Sol. Given equation : $x^2 + y^2 = a^2$

Differentiate *w.r.t.* x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0$$

[ISC Marking Scheme, 2018]

Q. 4. Find the differential equation of the family of curves $y = Ae^x + Be^{-x}$, where A and B are arbitrary concepts [ISC Specimen Paper, 2018]

Sol. Given equation is $y = Ae^x + Be^{-x}$, where A and B are arbitrary constants

Differentiate it *w.r.t.* x, we get

$$\frac{dy}{dx} = Ae^x - Be^{-x}$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$

$\frac{d^2y}{dx^2} = y$ is the required differential equation.

AI **Q. 5. Solve the differential equation :**

$$\log\left(\frac{dy}{dx}\right) = 2x - 3y$$

[ISC, 2013]

Sol. Given, $\log\left(\frac{dy}{dx}\right) = 2x - 3y$

$$\Rightarrow \quad \frac{dy}{dx} = e^{2x-3y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{2x}}{e^{3y}}$$

$$\Rightarrow \quad e^{3y}dy = e^{2x}\,dx$$

Integrating both sides, we get

$$\int e^{3y}\,dy = \int e^{2x}\,dx$$

$$\Rightarrow \quad \frac{e^{3y}}{3} = \frac{e^{2x}}{2} + C$$

is the required solution.

Q. 6. Solve the differential equation :

$$\mathrm{cosec}^3 x\,dy - \mathrm{cosec}\,y\,dx = 0$$

[ISC, 2010]

Sol. Given, $\mathrm{cosec}^3\,x\,dy - \mathrm{cosec}\,y\,dx = 0$

$\Rightarrow \mathrm{cosec}^3\,x\,dy = \mathrm{cosec}\,y\,dx$

$$\sin y\,dy = \sin^3 x\,dx$$

Integrating both sides,

$$\Rightarrow \quad \int \sin y\,dy = \frac{1}{4}\int [3\sin x - \sin 3x]\,dx + C$$

$$[\because \sin 3x = 3\sin x - 4\sin^3 x]$$

$$\Rightarrow \quad -\cos y = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$$

$$\Rightarrow \quad \frac{3}{4}\cos x - \frac{\cos 3x}{12} - \cos y + C_1 = 0 \qquad [C_1 = -C]$$

is the required solution.

Q. 7. Write the differential equation obtained by eliminating the arbitrary constant C in the equation representing the family of curves $xy = C\cos x$.

Sol. On differentiating the given equation, we get

$$x\frac{dy}{dx} + y = -C\sin x$$

Eliminating C using above equation, we get

$$x\frac{dy}{dx} + y + xy\tan x = 0$$

AI **Q. 8. Find the differential equation of the family of lines passing through the origin.**

Sol. Generates equation of family of lines passing through origin

$$y = mx \quad \Rightarrow \quad m = \frac{y}{x}$$

$$\frac{dy}{dx} = m$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$x\frac{dy}{dx} - y = 0$$

Q. 9. Find the differential equation representing the family of curves $V = \frac{A}{r} + B$, where A and B are arbitrary constants.

Sol. On differentiating the given equation, we get

$$\frac{dV}{dr} = -\frac{A}{r^2}$$

$$r^2\frac{dV}{dr} = -A$$

Differentiating again, we get

$$r^2\frac{d^2V}{dr^2} + 2r\frac{dV}{dr} = 0$$

4 Mark Questions

Q. 1. Solve the differential equation: $\frac{dy}{dx} = \frac{x+y+2}{2(x+y)-1}$

[ISC, 2019]

Sol. $\frac{dy}{dx} = \frac{x+y+2}{2(x+y)-1}$

Let $x + y = z$

$$1 + \frac{dy}{dx} = 1 + \frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{z+2}{2z-1} + 1$$

$$\frac{dz}{dx}=\frac{z+2+2z-1}{2z-1}$$

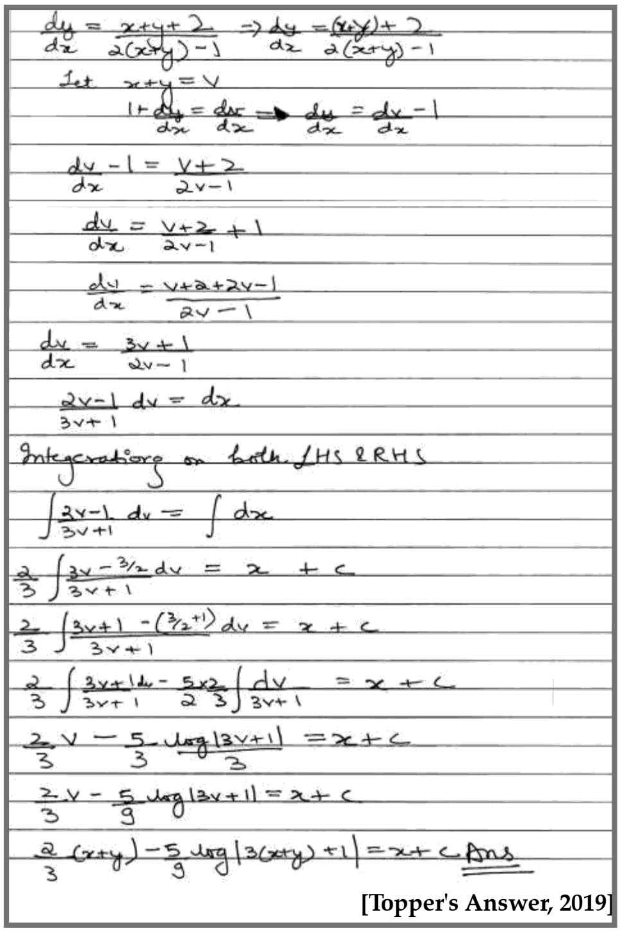
$$\frac{dz}{dx}=\frac{3z+1}{2z-1}$$

$$\int\frac{2z-1}{3z+1}dz=\int dx+c\Rightarrow\frac{2}{3}\left[z-\frac{5}{6}\log(3z+)\right]=x+c$$

$$\frac{2}{3}(x+y)-\frac{5}{9}\log[3x+3y+1]=x+c$$

[ISC Marking Scheme, 2019]

OR



$$\frac{dy}{dx}=\frac{x+y+2}{2(x+y)-1}\Rightarrow\frac{dy}{dx}=\frac{(x+y)+2}{2(x+y)-1}$$

Let $x+y=v$

$$1+\frac{dy}{dx}=\frac{dv}{dx}\Rightarrow\frac{dy}{dx}=\frac{dv}{dx}-1$$

$$\frac{dv}{dx}-1=\frac{v+2}{2v-1}$$

$$\frac{dv}{dx}=\frac{v+2}{2v-1}+1$$

$$\frac{dv}{dx}=\frac{v+2+2v-1}{2v-1}$$

$$\frac{dv}{dx}=\frac{3v+1}{2v-1}$$

$$\frac{2v-1}{3v+1}dv=dx$$

Integrations on both LHS & RHS

$$\int\frac{2v-1}{3v+1}dv=\int dx$$

$$\frac{2}{3}\int\frac{3v-3/2}{3v+1}dv=x+c$$

$$\frac{2}{3}\int\frac{3v+1-(3/2+1)}{3v+1}dv=x+c$$

$$\frac{2}{3}\int\frac{3v+1}{3v+1}dv-\frac{5\times2}{2\ \ 3}\int\frac{dv}{3v+1}=x+c$$

$$\frac{2}{3}v-\frac{5}{3}\frac{\log|3v+1|}{3}=x+c$$

$$\frac{2}{3}v-\frac{5}{9}\log|3v+1|=x+c$$

$$\frac{2}{3}(x+y)-\frac{5}{9}\log|3(x+y)+1|=x+c\ \underline{\underline{\text{Ans}}}$$

[Topper's Answer, 2019]

Examiner's Comment

- Majority of the candidates identified the given differential equation as homogeneous equation whereas, it was the type of equation reducible to simple separable.

Answering Tip

- Understand all types of differential equations and applications and do adequate practice of various types of problems. Clarify the significance of constant in the solution of a differential equation.

AI Q. 2. Solve : $(x^2 - yx^2)\,dy + (y^2 + xy^2)\,dx = 0$ [ISC, 2017]

Sol. Given, $(x^2 - yx^2)\,dy + (y^2 + xy^2)dx = 0$

$$\Rightarrow \quad x^2(1-y)\,dy + y^2(1+x)dx = 0$$

$$\Rightarrow \quad \frac{1-y}{y^2}dy+\frac{1+x}{x^2}dx = 0$$

On integration, we get

$$\int\frac{1-y}{y^2}dy+\int\frac{1+x}{x^2}dx = 0$$

$$\Rightarrow \int\frac{1}{y^2}dy-\int\frac{1}{y}dy+\int\frac{1}{x^2}dx+\int\frac{1}{x}dx = 0$$

$$\Rightarrow \quad -\frac{1}{y}-\log y-\frac{1}{x}+\log x = C$$

$$\Rightarrow \quad \log\left(\frac{x}{y}\right)-\frac{1}{x}-\frac{1}{y} = C$$

$$\Rightarrow \quad \log\left(\frac{x}{y}\right) = C+\frac{1}{x}+\frac{1}{y}$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates attempted it by the method of homogeneous equation by substituting $y = vx$ but could not proceed further. Many candidates made errors in separating the variables x and y.

Answering Tip

- Clearly identify all the forms of differential equations and do sufficient practice of solving problems based on different forms.

Q. 3. Solve the differential equation :

$$\sin^{-1}\frac{dy}{dx}=x+y$$

[ISC, 2015]

Sol. Given, $\quad \sin^{-1}\dfrac{dy}{dx}=x+y$

$$\therefore \quad \frac{dy}{dx}=\sin(x+y)$$

Let $\quad x + y = v$

$$\Rightarrow \quad 1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dv}{dx}-1=\sin v$$

$$\Rightarrow \quad \frac{dv}{dx}=1+\sin v$$

$$\Rightarrow \quad \int\frac{dv}{1+\sin v}=\int dx \text{ (Integrating both sides)}$$

$$\Rightarrow \quad \int\frac{(1-\sin v)dv}{1-\sin^2 v}=\int dx$$

$$\Rightarrow \quad \int\frac{(1-\sin v)dv}{\cos^2 v}=\int dx$$

$$\Rightarrow \quad \int(\sec^2 v-\tan v\sec v)dv=\int dx$$

$$\Rightarrow \quad \tan v-\sec v = x + C$$

$$\Rightarrow \tan(x+y)-\sec(x+y)=x+C$$

[ISC Marking Scheme, 2015]

AI Q. 4. Solve the differential equation :

$$(x+1)dy - 2xy\,dx = 0$$ **[ISC, 2014]**

Sol. $(x+1)\,dy = 2xy\,dx$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{2x}{x+1}dx$$

$$\Rightarrow \log y = 2\int \frac{x+1-1}{x+1}dx = 2\int\left(1-\frac{1}{x+1}\right)dx$$

$$\Rightarrow \log y = 2[x - \log(x+1)] + \log c$$

$$\log y + 2\log(x+1) - \log C = 2x$$

$$\Rightarrow \log\frac{y(x+1)^2}{c} = 2x$$

$$\Rightarrow \frac{y(x+1)^2}{c} = e^{2x}$$

$$\Rightarrow y = \frac{ce^{2x}}{(x+1)^2}$$

[ISC Marking Scheme, 2014]

Q. 5. Solve the differential equation :

$$(xy^2+x)dx + (x^2y+y)\,dy = 0$$ **[ISC, 2012]**

Sol. Given, $(xy^2+x)\,dx + (x^2y+y)\,dy = 0$

$$\Rightarrow x(y^2+1)\,dx + y(x^2+1)\,dy = 0$$

$$\Rightarrow x(y^2+1)dx = -y(x^2+1)\,dy$$

$$\Rightarrow \frac{x}{x^2+1}dx = \frac{-y}{y^2+1}dy$$

$$\Rightarrow \int\frac{2x}{x^2+1}dx = \int\frac{-2y}{y^2+1}dy$$

(Integrating both sides)

$$\log|x^2+1| = -\log|y^2+1| + \log C$$

$$\log|x^2+1| + \log|y^2+1| = \log C$$

$$|x^2+1| \times |y^2+1| = C$$

$$(x^2+1)(y^2+1) = C$$

is the required solution.

Q. 6. Solve the differential equation :

$$\frac{dy}{dx} = e^{x+y} + x^2e^y$$ **[ISC, 2011]**

Sol. Given, $\frac{dy}{dx} = e^{x+y} + x^2e^y$

$$= e^x.e^y + x^2.e^y$$

$$\Rightarrow \frac{dy}{dx} = e^y(e^x + x^2)$$

$$\int\frac{dy}{e^y} = \int(e^x + x^2)dx \text{ (Integrating both sides)}$$

$$\Rightarrow \int e^{-y}\,dy = \int e^x\,dx + \int x^2\,dx$$

$$\Rightarrow -e^{-y} = e^x + \frac{x^3}{3} + c$$

$$\Rightarrow -\frac{1}{e^y} = e^x + \frac{x^3}{3} + c$$

or $e^{x+y} + \frac{e^y.x^3}{3} + ce^y + 1 = 0$

is the required solution.

AI Q. 7. Solve the differential equation :

$$x\cos y\,dy = e^x(x\log x + 1)dx$$ **[ISC, 2009]**

Sol. Given, $x\cos y\,dy = e^x(x\log x + 1)dx$

$$\cos y.dy = e^x\left(\log x + \frac{1}{x}\right).dx$$

On integrating both sides, we get

$$\int\cos y\cdot dy = \int e^x\left(\log x + \frac{1}{x}\right).dx$$

$$\Rightarrow \sin y = \int e^x\log x.dx + \int\frac{e^x}{x}.dx$$

$$\Rightarrow \sin y = \log x\int e^x dx - \int\left[\frac{d}{dx}\log x.\int e^x dx\right]dx + \int\frac{e^x}{x}dx$$

(Integrating $e^x\log x$ by parts)

$$\Rightarrow \sin y = e^x\log x - \int\frac{1}{x}.e^x\,dx + \int\frac{e^x}{x}dx + c$$

$\Rightarrow \sin y = e^x\log x + c$ is the required solution.

Q. 8. Solve the differential equation :

$$x(x^2 - x^2y^2)dy + y(y^2 + x^2y^2)dx = 0$$ **[ISC, 2009]**

Sol. Given, $x(x^2 - x^2y^2)dy + y(y^2 + x^2y^2)dx = 0$

$$\therefore\ x^3(1-y^2)dy + y^3(1+x^2)dx = 0$$

$$\Rightarrow x^3(1-y^2)dy = -y^3(1+x^2)dx$$

$$\Rightarrow \frac{1-y^2}{y^3}dy = -\frac{(1+x^2)}{x^3}dx$$

$$\Rightarrow \left(\frac{1}{y^3} - \frac{1}{y}\right)dy = -\left(\frac{1}{x^3} + \frac{1}{x}\right)dx$$

Integrating both sides

$$\int\left(\frac{1}{y^3} - \frac{1}{y}\right)dy = -\int\left(\frac{1}{x^3} + \frac{1}{x}\right)dx$$

$$\Rightarrow -\frac{1}{2y^2} - \log|y| = -\left[-\frac{1}{2x^2} + \log|x|\right] + c$$

$$\Rightarrow -\frac{1}{2y^2} - \frac{1}{2x^2} = \log|y| - \log|x| + c$$

$$\Rightarrow -\left(\frac{1}{2y^2} + \frac{1}{2x^2}\right) = \log\left|\frac{y}{x}\right| + c$$

$$\Rightarrow \log\left|\frac{y}{x}\right| + \frac{1}{2y^2} + \frac{1}{2x^2} + c = 0$$

is the required solution.

AI Q. 9. Solve the differential equation :

$$\tan y\,dx + \sec^2 y\tan x\,dy = 0$$ **[ISC, 2007]**

Sol. Given, $\tan y\,dx + \sec^2 y\tan x\,dy = 0$

$$\tan y\,dx = -\sec^2 y\tan x\,dy$$

$$\Rightarrow \frac{dx}{\tan x} = \frac{-\sec^2 y}{\tan y}dy$$

Integrating both sides

$$\int\frac{dx}{\tan x}=\int\frac{-\sec^2 y}{\tan y}dy$$

⇒ $\log|\sin x|=-\int\frac{dt}{t}$ (Let $\tan y=t$ ∵ $\sec^2 y\,dy=dt$)

⇒ $\log|\sin x|=-\log|t|+\log c$

$=-\log|\tan y|+\log c$

⇒ $\log|\sin x|+\log|\tan y|=\log c$

⇒ $\log(\sin x.\tan y)=\log c$

⇒ $\sin x\,.\,\tan y=c$ is the required solution.

Q. 10. Solve the differential equation :

$$\frac{dy}{dx}-e^{x+y}=e^{x-y}$$

Sol. Given, $\frac{dy}{dx}-e^{x+y}=e^{x-y}$

⇒ $\frac{dy}{dx}-e^x.e^y=\frac{e^x}{e^y}$

⇒ $\frac{dy}{dx}=\frac{e^x}{e^y}+e^x.e^y$

$=e^x\left(\frac{1}{e^y}+e^y\right)$

⇒ $\frac{e^y dy}{e^{2y}+1}=e^x dx$

Integrating both sides, we get

$$\int\frac{e^y}{e^{2y}+1}dy=\int e^x dx$$

⇒ $\int\frac{dt}{1+t^2}=e^x+c$ (Put $e^y=t\Rightarrow e^y\,dy=dt$)

⇒ $\tan^{-1}t=e^x+c$

⇒ $\tan^{-1}e^y=e^x+c$

⇒ $e^y=\tan(e^x+c)$

is the required solution.

Q. 11. Solve the differential equation :

$$e^y(1+x^2)dy-\frac{x}{y}dx=0$$

Sol. Given, $e^y(1+x^2)dy-\frac{x}{y}dx=0$

⇒ $ye^y\,dy=\frac{x}{1+x^2}\,dx$

Integrating both sides

$$\int ye^y\,dy=\int\frac{x}{1+x^2}dx$$

(Using integration by parts for $\int ye^y dy$)

⇒ $y\int e^y dy-\int\left[\frac{d}{dy}(y)\left\{\int e^y dy\right\}\right]dy=\frac{1}{2}\log|1+x^2|+c$

⇒ $ye^y-\int e^y\,dy=\frac{1}{2}\log|1+x^2|+c$

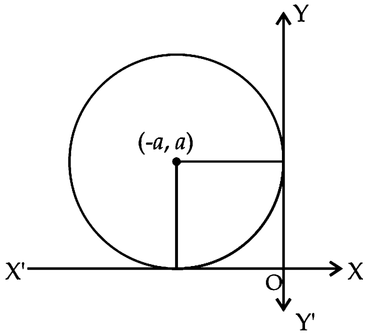
⇒ $ye^y-e^y=\log|\sqrt{1+x^2}|+c$

⇒ $e^y(y-1)=\log|\sqrt{1+x^2}|+c$

is the required solution.

AI Q. 12. From the differential equation of the family of circles in the second quadrant and touching the co-ordinate axes.

Sol.



Equation of family of circle is

$$(x+a)^2+(y-a)^2=a^2$$

or $x^2+y^2+2ax-2ay+a^2=0$ ...(i)

On differentiating, we get

$$2x+2y\frac{dy}{dx}+2a-2a\frac{dy}{dx}=0$$

⇒ $x+y\frac{dy}{dx}=a\left(\frac{dy}{dx}-1\right)$

⇒ $a=\frac{x+yy'}{y'-1}$

where $y'=\frac{dy}{dx}$

Substituting the value of a in (i) and simplifying

$$\left[x-\frac{(x+yy')}{y'-1}\right]^2+\left[y-\frac{(x+yy')}{y'-1}\right]^2=\left(\frac{x+yy'}{y'-1}\right)^2$$

⇒ $(xy'-x+x+yy')^2+(yy'-y-x-yy')^2$

$=(x+yy')^2$

⇒ $(x+y)^2[(y')^2+1]=(x+yy')^2$

Q .13. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$ given that $y=\frac{\pi}{2}$ where $x=1$.

Sol. The given differential equation is :

$$\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$$

⇒ $(\sin y+y\cos y)dy=x(2\log x+1)\,dx$

⇒ $\int(\sin y+y\cos y)dy=\int x(2\log x+1)\,dx$

On integrating both sides

⇒ $-\cos y+y\sin y+\cos y$

$=x^2\log x-\frac{x^2}{2}+\frac{x^2}{2}+c$

⇒ $y\sin y=x^2\log x+c$ ...(i)

It is given that when $y=\frac{\pi}{2}$, $x=1$

So, putting $y=\frac{\pi}{2}$, $x=1$ in (i), we get

$$\Rightarrow \quad \frac{\pi}{2}\sin\frac{\pi}{2} = 1.\log 1 + c$$

$$\Rightarrow \quad c = \frac{\pi}{2}$$

Putting $c = \frac{\pi}{2}$ in (i), we obtain

$$\Rightarrow \quad y\sin y = x^2\log x + \frac{\pi}{2}$$

This is the required solution.

Q. 14. Find the particular solution of the following differential equation:

$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, given that y = 1, when x = 0.

Sol. The given differential equation is :

$$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$$

$$= (1 + x^2)(1 + y^2)$$

$$\Rightarrow \int\frac{dy}{1+y^2} = \int(1+x^2)dx$$

$$\tan^{-1}y = x + \frac{x^3}{3} + c$$

For $x = 0, y = 1, c = \frac{\pi}{4}$

$$\tan^{-1}y = x + \frac{x^3}{3} + \frac{\pi}{4}$$

or $$y = \tan\left(\frac{\pi}{4} + x + \frac{x^3}{3}\right)$$

is the required solution.

Q. 15. Solve the differential equation :

$$3e^x\tan y\,dx + (2-e^x)\sec^2 y\,dy = 0,$$

given that when $x = 0, y = \frac{\pi}{4}$.

Sol. The given differential equation is :

$$3e^x\tan y\,dx + (2-e^x)\sec^2 y\,dy = 0$$

$$\Rightarrow \quad (2-e^x)\sec^2 y\,dy = -3e^x\tan y\,dx$$

$$\Rightarrow \quad \frac{\sec^2 y}{\tan y}dy = \frac{-3e^x}{2-e^x}dx$$

Integrating both sides

$$\Rightarrow \quad \int\frac{\sec^2 y}{\tan y}dy = 3\int\frac{-e^x}{2-e^x}dx$$

$$\Rightarrow \quad \log|\tan y| = 3\log|2-e^x| + \log c$$

$$\Rightarrow \quad \log|\tan y| = \log|c(2-e^x)^3|$$

$$\Rightarrow \quad \tan y = c(2-e^x)^3$$

TOPIC-2
Linear Differential Equations

Revision Notes

- **Linear Differential Equation :**

  A differential equation is said to be linear if dependent variable (say y) and its derivative occurs in the first degree.

- **Linear differential equation in y :** It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are functions of x only.

  - **Solution of Linear Differential Equation in y :**

    Step 1 : Write the given differential equation in the form $\frac{py}{dx} + P(x)y = Q(x)$.

    Step 2 : Find the **integration Factor** (I.F.) $= e^{\int P(x)dx}$.

    Step 3 : The solution is given by, $y(I.F.) = \int Q(x).(I.F.)dx + k$, where k is the constant of integration.

Scan to know more about this topic

Linear differential equation

- **Linear Differential equation in x :** It is of the form $\frac{dx}{dy} + P(y)x = Q(y)$ where $P(y)$ and $Q(y)$ are functions of x only.

  - **Solution of Linear Differential Equation in x :**

    Step 1 : Write the given differential equation in the form $\frac{dx}{dy} + P(y)x = Q(y)$.

    Step 2 : Find the **integration Factor** (I.F.) $= e^{\int P(y)dy}$.

    Step 3 : The **solution** is given by, $x(I.F.) = \int Q(y).(I.F.)dy + \lambda$, where λ is the constant of integration.

Mnemonics

Linear Differential equation $\frac{dy}{dx} + Py = Q$

Its solution can be remember as :

$$\underset{\text{WHY}}{y} \times \underset{\text{IF}}{IF} = \int(\underset{\text{KYON}}{Q} \times \underset{\text{IF}}{IF})dx + C$$

4 Marks Questions

Q. 1. Solve the differential equation

$$(1+x^2)\frac{dy}{dx} = 4x^2 - 2xy$$ **[ISC, 2020]**

Sol. $$(1+x^2)\frac{dy}{dx} = 4x^2 - 2xy$$

$$\frac{dy}{dx} = \frac{4x^2}{1+x^2} - \frac{2xy}{1+x^2}$$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

Compare $\frac{dy}{dx} + Py = Q(x)$

$$\therefore \quad P = \frac{2x}{1+x^2} \text{ and } Q(x) = \frac{4x^2}{1+x^2}$$

$$\text{I.F. } e^{\int Pdx} = e^{\int \frac{2x}{1+x^2}dx}$$

$$= e^{\log|1+x^2|} = (1+x^2)$$

Solution of $\frac{dy}{dx} + Py = Q(x)$ is

$$y \times \text{I.F.} = \int \text{I.F.} \times Q(x)dx$$

$$(1+x^2)y = \int (1+x^2)\frac{4x^2}{1+x^2}dx$$

$$(1+x^2)y = \int 4x^2 dx$$

$$(1+x^2)y = \frac{4x^3}{3} + c$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Most candidates made errors while transforming into linear differential equation and in finding the integrating factor. Many candidates did not express the answer in terms of constant C.

Answering Tip

- Always remember the significance of constant in the solution of differential equation.

AI Q. 2. Solve : $\sin x\frac{dy}{dx} - y = \sin x.\tan\frac{x}{2}$ **[ISC, 2018]**

Sol. Given, $$\sin x\frac{dy}{dx} - y = \sin x.\tan\frac{x}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} - y \operatorname{cosec} x = \tan\frac{x}{2}$$

It is of the form $\frac{dy}{dx} + Py = Q$, where

$P = -\operatorname{cosec} x$ and $Q = \tan\frac{x}{2}$

$$\therefore \quad \text{I.F.} = e^{\int Pdx} = e^{\int(-\operatorname{cosec} x)dx}$$

$$= e^{-\log\tan\frac{x}{2}} = \frac{1}{\tan\left(\frac{x}{2}\right)}$$

The solution is given by

$$y.I.F. = \int Q.\,I.f.\,dx + C$$

$$\Rightarrow \quad y.\frac{1}{\tan\left(\frac{x}{2}\right)} = \int \tan\left(\frac{x}{2}\right).\frac{1}{\tan\left(\frac{x}{2}\right)}dx + C$$

$$\Rightarrow \quad y.\cot\left(\frac{x}{2}\right) = x + C$$

[ISC Marking Scheme, 2018]

Q. 3. Solve the following differential equation :

$x\frac{dy}{dx} + 2y = x^2 \log x$ **[ISC Specimen Paper, 2018]**

Sol. $$x\frac{dy}{dx} + 2y = x^2 \log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y = x\log x$$

This is a linear differential equation of the type

$$\frac{dy}{dx} + Py = Q$$

where $P = \frac{2}{x}$ and $Q = x\log x$

$$\therefore \quad I.F = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

$\therefore$ Solution of the given equation is

$$y \times I.F = \int (Q \times I.F)\,dx + C$$

$$\Rightarrow \quad y\,.\,x^2 = \int x\log x \times x^2 dx + C$$

$$\Rightarrow \quad y\,.\,x^2 = \int x^3 \log x\,dx + C$$

Let $$I = \int x^3 \log x\,dx$$

$$I = \log x \int x^3\,dx - \int\left(\frac{d}{dx}\log x \int x^3 dx\right)dx + C$$

[Using integration by parts]

$$= \log x\frac{x^4}{4} - \int\frac{1}{x}.\frac{x^4}{4}dx + C$$

$$= \frac{x^4}{4}\ \log x - \frac{1}{4}.\frac{x^4}{4}\ + C$$

Therefore, solution is

$$yx^2 = \frac{x^4}{4}\ \log x - \frac{1}{16}\ x^4 + C$$

$$\Rightarrow \qquad y = \frac{x^2}{4}\ \log x - \frac{x^2}{16} + \frac{C}{x^2}$$

is the required solution.

AI Q. 4. Solve the differential equation :

$$x\frac{dy}{dx} + y = 3x^2 - 2$$ **[ISC Specimen Paper, 2016]**

Sol. Given, $x\ \dfrac{dy}{dx}\ + y = 3x^2 - 2$

$$\Rightarrow \qquad \frac{dy}{dx} + \frac{y}{x}\ = 3x - \frac{2}{x}$$

which is of the form $\dfrac{dy}{dx} + Py\ = Q$

Here, $P = \dfrac{1}{x}$ and $Q = 3x - \dfrac{2}{x}$

$$\text{I.F.} = e^{\int pdx} = e^{\int\frac{1}{x}dx}$$

$$= e^{\log x} = x$$

∴ The solution is given by

$$y \times IF = \int Q \times IF\ dx + C$$

$$\Rightarrow \qquad y.x = \int\left(3x - \frac{2}{x}\right).x\ dx + C$$

$$\Rightarrow \qquad y.x = \int(3x^2 - 2)dx + C$$

$$\Rightarrow \qquad y.x = \frac{3x^3}{3} - 2x + C$$

$$\Rightarrow \qquad y.x = x^3 - 2x + C$$

$$\Rightarrow \qquad y = x^2 - 2 + \frac{C}{x}$$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Many candidates made errors while identifying the differential equation and attempted to solve it by variable-separable form.

Answering Tip

- Do ample practice of various types of differential equations.

Q. 5. Solve the differential equation :

$$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$$

Sol. Given equation is,

$$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$$

Dividing the whole equation by $(x+1)$

$$\frac{dy}{dx} - \frac{1}{(x+1)}y = e^{3x}(x+1)$$

It is of the form $\dfrac{dy}{dx} + Py\ = Q$

Here, $P = -\dfrac{1}{x+1}, Q = e^{3x}.(x+1)$

$$\therefore \qquad I.F. = e^{\int Pdx} = e^{-\int\frac{1}{x+1}dx} = e^{-\log|x+1|} = \frac{1}{x+1}$$

∴ The solutions is

$$y \cdot I.F. = \int Q \cdot I.F. + C$$

$$y \cdot \frac{1}{(x+1)} = \int e^{3x}.(x+1).\frac{1}{(x+1)}\ + C$$

$$\Rightarrow \qquad \frac{y}{(x+1)} = \frac{e^{3x}}{3} + c$$

$$\Rightarrow \qquad y = \frac{e^{3x}}{3}(x+1) + c(x+1)$$

is the required solution.

Q. 6. Solve the differential equation :

$$(1-x^2)\frac{dy}{dx} - xy = x' \text{ given } y = 2 \text{ when } x = 0.$$

Sol. Given equation is,

$$(1-x^2)\frac{dy}{dx} - xy = x$$

$$\Rightarrow \qquad \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{x}{1-x^2}$$

Given equation is in the form $\dfrac{dy}{dx} + Py = Q$,

where, $P = \dfrac{-x}{1-x^2}, Q = \dfrac{x}{1-x^2}$

$$IF = e^{\int P\,dx} = e^{\int\frac{-x}{1-x^2}dx}$$

$$= e^{\frac{1}{2}\int\frac{-2x}{1-x^2}dx} = e^{\frac{1}{2}\log|1-x^2|}$$

$$= \sqrt{1-x^2} \qquad (\because e^{\log_e x} = x)$$

Solution of given linear differential equation is

$$y.IF = \int(Q \times IF)dx + c$$

$$i.e., \qquad y\sqrt{1-x^2} = \int\left(\frac{x}{1-x^2} \times \sqrt{1-x^2}\right)dx + c$$

$$= \int\frac{x}{\sqrt{1-x^2}}dx + c$$

Put $\qquad t = \sqrt{1-x^2}$

$$\Rightarrow \qquad dt = \frac{1}{2\sqrt{1-x^2}}(-2x)\ dx$$

$$\Rightarrow \qquad -dt = \frac{x}{\sqrt{1-x^2}}\ dx$$

$$\therefore \qquad y\sqrt{1-x^2} = \int -dt + c$$

$$\Rightarrow \qquad y\sqrt{1-x^2} = -t + c$$

$$\Rightarrow \quad y\sqrt{1-x^2} = -\sqrt{1-x^2} + c$$

Now given, $y = 2$ when $x = 0$

$$\therefore \quad 2 = -1 + c \Rightarrow c = 3$$

$$\therefore \quad y\sqrt{1-x^2} + \sqrt{1-x^2} = 3$$

is the required particular solution.

AI **Q. 7. Solve : $(1 + y + x^2y)dx + (x + x^3)dy = 0$**

Sol. Given equation is

$$(1 + y + x^2y)dx + (x + x^3)dy = 0$$

$$\Rightarrow \quad 1 + y + x^2y + (x + x^3)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 1 + y(1 + x^2) + x(1 + x^2)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{x(1+x^2)} + \frac{y}{x} + \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = \frac{-1}{x(1+x^2)}$$

On comparing above equation with the linear differential $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{1}{x} \text{ and } Q = \frac{-1}{x(1+x^2)}$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\log|x|} = x \quad [\because a^{\log_a x} = x]$$

$\therefore$ The solution is

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\,dx + C$$

$$\Rightarrow \quad yx = \int \frac{-1}{x(1+x^2)} \times x\,dx + C$$

$$\Rightarrow \quad yx = -\int \frac{dx}{1+x^2} + C$$

$$\Rightarrow \quad yx = -\tan^{-1}x + C$$

Q. 8. Solve : $\cos^2 x\frac{dy}{dx} + y = \tan x$

Sol. Given equation is

$$\cos^2 x\frac{dy}{dx} + y = \tan x$$

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

[dividing both sides by $\cos^2 x$]

$$\Rightarrow \quad \frac{dy}{dx} + y\sec^2 x = \tan x \cdot \sec^2 x$$

On comparing above equation with linear differential equation $\frac{dy}{dx} + Py = Q$, we get

and $\quad Q = \tan x.\sec^2 x$

$\quad P = \sec^2 x$

$$IF = e^{\int P\,dx} = e^{\int \sec^2 x\,dx} = e^{\tan x}$$

$\therefore$ The solution is

$$y \times IF = \int (Q \times IF)\,dx + C$$

$$\Rightarrow \quad ye^{\tan x} = \int \tan x \cdot \sec^2 x\, e^{\tan x}\,dx + C$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore \quad ye^{\tan x} = \int \underset{\text{I}}{t}\,.\,\underset{\text{II}}{e^t}\,dt + C$$

$$\therefore \quad ye^{\tan x} = t\int e^t dt - \int\left[\frac{d(t)}{dt}\int e^t\,dt\right]dt + C$$

[using integration by parts]

$$= te^t - \int 1 \cdot e^t\,dt + C$$

$$= te^t - e^t + C = e^t(t-1) + C$$

$$ye^{\tan x} = e^{\tan x}(\tan x - 1) + C \qquad [\text{put } t = \tan x\,]$$

AI **Q. 9. Solve : $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$.**

Sol. Given equation is,

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$$

On comparing above equation with linear differential equation $\frac{dy}{dx} + Py = Q$

We get, $P = \frac{2x}{1+x^2}$ and $Q = \frac{1}{(1+x^2)^2}$

$$\therefore \quad IF = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\int \frac{1}{1+x^2}\frac{d}{dx}(1+x^2)dx}$$

$$= e^{\log|1+x^2|} \qquad [\because a^{\log_a x} = x]$$

$$= (1 + x^2)$$

$\therefore$ The solution is

$$y \times IF = \int (Q \times IF)\,dx + C$$

$$\Rightarrow \quad y(1+x^2) = \int \frac{1}{(1+x^2)^2} \times (1+x^2)\,dx + C$$

$$\Rightarrow \quad y(1+x^2) = \int \frac{1}{1+x^2}dx + C$$

$$\Rightarrow \quad y(1+x^2) = \tan^{-1}x + C$$

AI **Q. 10. Solve : $(1 + y^2)\frac{dx}{dy} = \tan^{-1}y - x$.**

Sol. Given, $(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

On comparing above equation with linear differential equation $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1+y^2},\ Q = \frac{\tan^{-1}y}{1+y^2}$$

$$\therefore \quad IF = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

The solution is

$$x \times IF = \int (Q \times IF)dy + C$$

$$\Rightarrow \quad xe^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy + C \qquad \text{...(i)}$$

Let $I = \int \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy$

Put $\tan^{-1}y = t$

Then, $\frac{1}{1+y^2}dy = dt$

$$\therefore \quad I = \int \underset{\text{I}}{t}\,\underset{\text{II}}{e^t}\,dt$$

$$= t\int e^t\,dt - \int\left\{\frac{d(t)}{dt}\int e^t\,dt\right\}dt$$

[using integration by parts]

$$= te^t - \int 1\cdot e^t dt = te^t - e^t + C$$

$$= (\tan^{-1}y - 1)e^{\tan^{-1}y} + C$$

∴ Eq. (i) becomes

$$xe^{\tan^{-1}y} = (\tan^{-1}y - 1)e^{\tan^{-1}y} + C$$

$$\Rightarrow \quad x = (\tan^{-1}y - 1) + Ce^{-\tan^{-1}y}$$

[dividing both sides by $e^{\tan^{-1}y}$]

Q. 11. Solve the following differential equation :

$$\boldsymbol{x\frac{dy}{dx} + y - x + xy\cot x = 0, x \neq 0}$$

Sol. $x\frac{dy}{dx} + y - x + xy\cot x = 0, x \neq 0$

$$\Rightarrow x\frac{dy}{dx} + y(1 + x\cot x) = x$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right)y = 1$$

Comparing with $\frac{dy}{dx} + Px = Q$, we get

$P = \frac{1}{x} + \cot x$ and $Q = 1$

$$\therefore \quad I.F. = e^{\int P dx}$$

$$= e^{\int\left(\frac{1}{x}+\cot x\right)dx}$$

$$= e^{\log|x| + \log|\sin x|}$$

$$= e^{\log|x\sin x|} = x\sin x$$

∴ Now the solution of given differential equation is

$y.I.F. = \int Q.IF\,dx + C$

$$\Rightarrow \quad y.(x\sin x) = \int 1.x\sin x\,dx + c$$

$$\Rightarrow \quad xy\sin x = -x\cos x + \int 1.\cos x\,dx + c$$

[using integration by parts]

$$\Rightarrow \quad xy\sin x = -x\cos x + \sin x + c$$

AI **Q. 12. Solve the differential equation given as :**

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1, x \neq 0$$

Sol. Given, $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1, x \neq 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{\sqrt{x}}.y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is a linear equation of the form $\frac{dy}{dx} + Py = Q$

where, $\quad P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \quad I.F. = e^{\int P dx}$$

$$= e^{\int \frac{1}{\sqrt{x}}dx} = e^{\int x^{-1/2}\,dx}$$

$$= e^{2\sqrt{x}}$$

Therefore, required solution is

$$y.I.F. = \int Q \times I.F.dx + c$$

$$\Rightarrow \quad y.e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}.e^{2\sqrt{x}}dx + c$$

$$\Rightarrow \quad y.e^{2\sqrt{x}} = \int \frac{dx}{\sqrt{x}} + c$$

$$\Rightarrow \quad y.e^{2\sqrt{x}} = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow \quad y.e^{2\sqrt{x}} = 2\sqrt{x} + c$$

6 Marks Questions

AI **Q. 1. Solve the following differential equation :**

$\boldsymbol{ye^y dx = (y^3 + 2xe^y)dy}$ **,given that** $\boldsymbol{x = 0, y = 1}$

[ISC, 2013]

Sol. Given, $\quad ye^y\,dx = (y^3 + 2xe^y)dy$

$$\Rightarrow \quad ye^y\frac{dx}{dy} = y^3 + 2xe^y$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{2}{y}.x + y^2e^{-y}$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{2}{y}.x = y^2e^{-y} \qquad \text{...(i)}$$

It is a linear differential equation in of the form $\frac{dx}{dy} + Px = Q.$

$$P=-\frac{2}{y} \text{ and } Q=y^2e^{-y} \quad (\because IF=e^{\int P dy})$$

$$I.F.=e^{\int -\frac{2}{y}.dy}$$

$$=e^{-2\log|y|}$$

$$=e^{\log|y|^{-2}}$$

$$=|y^{-2}|=\frac{1}{y^2}$$

Therefore, general solution of (i) is

$$x.IF=\int Q\times IF\ dy+C$$

$$x.\frac{1}{y^2}=\int y^2e^{-y}.\frac{1}{y^2}dy+c$$

$$x.\frac{1}{y^2}=\int e^{-y}dy+c$$

$$\Rightarrow \quad x.\frac{1}{y^2}=\frac{e^{-y}}{-1}+c \qquad \text{...(ii)}$$

Given that $x=0$ when $y=1$

$\Rightarrow \quad 0=-e^{-1}+c$

$\Rightarrow \quad c=e^{-1}$

Substituting the value of c in (ii), the required particular solution of the given equation is

$$\frac{x}{y^2}=-e^{-y}+e^{-1}$$

i.e., $\quad x=y^2(-e^{-y}+e^{-1})$

Q. 2. Solve the differential equation :
$\frac{dy}{dx}-3y\cot x=\sin 2x$, given $y=2$ when $x=\frac{\pi}{2}$.
[ISC, 2012]

Sol. Given, $\frac{dy}{dx}-3y\cot x=\sin 2x$

This is a linear equation of the form $\frac{dy}{dx}+Py=Q$

where, $\quad P=-3\cot x,\ Q=\sin 2x$

$\therefore \quad$ I.F. $=e^{\int P dx}$

I.F. $=e^{\int -3\cot x dx}=e^{-3\int \cot x dx}$

$$=e^{-3\log|\sin x|}=e^{\log|\sin x|^{-3}}$$

$$=(\sin x)^{-3}=\frac{1}{\sin^3 x}$$

$\therefore$ Solution is given as,

$$y.\times(\text{I.F.})=\int Q\times(\text{I.F.})dx$$

$$y.\frac{1}{\sin^3 x}=\int \sin 2x.\frac{1}{\sin^3 x}dx$$

[$\because$ $\sin 2x = 2\sin x\cos x$]

$$y.\frac{1}{\sin^3 x}=\int \frac{2\sin x\cdot\cos x}{\sin^3 x}dx$$

$$\frac{y}{\sin^3 x}=2\int \operatorname{cosec} x.\cot x.\ dx$$

$$\frac{y}{\sin^3 x}=-2\operatorname{cosec} x+c$$

Putting $y=2$, $x=\pi/2$

$$\frac{2}{\sin^3 \pi/2}=-2\operatorname{cosec}(\pi/2)+c$$

$$2=-2+c\Rightarrow c=4$$

$$\therefore \quad \frac{y}{\sin^3 x}=-\frac{2}{\sin x}+4$$

$$\Rightarrow \quad y=-2\sin^2 x+4\sin^3 x$$

Q. 3. Solve the differential equation :
$(y+\log x)dx-x\,dy=0$, given that $y=0$, when $x=1$
[ISC, 2010]

Sol. Given, $(y+\log x)dx-x\,dy=0$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y}{x}+\frac{\log x}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}-\frac{y}{x}=\frac{\log x}{x}$$

It is linear differential equation in of the form $\frac{dy}{dx}+Py=Q$.

Here, $P=-\frac{1}{x},\ Q=\frac{\log x}{x}$

I.F. $=e^{\int P dx}$

$\Rightarrow \quad$ I.F. $=e^{-\int \frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}$

$\therefore$ Solution is given as,

$$y\cdot\text{I.F.}=\int Q\cdot\text{I.F.}\ dx$$

i.e., $\quad y.\frac{1}{x}=\int \frac{\log x}{x}.\frac{1}{x}dx$

$$\frac{y}{x}=\int \log x.\frac{1}{x^2}dx$$

$$\Rightarrow \quad \frac{y}{x}=\log x\int x^{-2}dx-\int\left[\frac{d}{dx}\log x\int x^{-2}dx\right]dx$$

[using integration by parts]

$$\Rightarrow \quad \frac{y}{x}=\log x\left(-\frac{1}{x}\right)-\int\frac{1}{x}\left(-\frac{1}{x}\right)dx$$

$$\Rightarrow \quad \frac{y}{x}=\log x\left(-\frac{1}{x}\right)+\left(-\frac{1}{x}\right)+c$$

$$\Rightarrow \quad y=-\log_e x-1+cx$$

$y=0$, when $x=1$ is given,

$$0=-\log_e 1-1+c$$

$\Rightarrow \quad c=1 \qquad (\because \log_e 1=0)$

$\therefore \quad y=-\log_e x-1+x$

is the required solution.

AI Q. 4. Solve the following differential equation for a particular solution:
$dy=(5x-4y)\,dx$, when $y=0$ and $x=0$ [ISC, 2009]

Sol. Given, $\quad dy=(5x-4y)\,dx$

$\frac{dy}{dx}+4y=5x$, is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$.

Here, $P=4,\ Q=5x$

Now, $\quad$ I.F. $=e^{\int 4dx}=e^{4x}$

Solution of given equation is given by

$$y.(I.F.) = \int Q.(I.F.)dx$$

$$y.e^{4x} = \int 5x \cdot e^{4x}\,dx$$

$$= 5x \cdot \frac{e^{4x}}{4} - \frac{5}{4}\int e^{4x}dx + c$$

[using integration by parts]

$$= \frac{5}{4}x \cdot e^{4x} - \frac{5}{16}e^{4x} + c$$

Given, when $y = 0,\ x = 0,\ \therefore\ c = \frac{5}{16}$

$$\therefore \quad y = \frac{5}{4}x - \frac{5}{16} + \frac{5}{16}e^{-4x}$$

Q. 5. Solve the differential equation

***ydx* – (*x* + 2*y*²) *dy* = 0**

Sol. Given, $y\,dx - (x + 2y^2)dy = 0$

$$\Rightarrow \quad y\,dx = (x + 2y^2)dy$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x + 2y^2}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x}{y} + 2y$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{x}{y} = 2y$$

It is linear differential equation of the form,

$$\frac{dx}{dy} + Px = Q$$

Here, $P = -\frac{1}{y}$ and $Q = 2y$

$$\therefore \quad \text{IF} = e^{\int P dy} = e^{-\int \frac{1}{y}dy}$$

$$\Rightarrow \quad \text{IF} = e^{-\log|y|} = e^{\log \frac{1}{|y|}} = \frac{1}{y}$$

The solution is

$$x \times \text{IF} = \int (Q \times \text{IF})dy + C$$

$$\Rightarrow \quad \frac{x}{y} = \int \frac{2y}{y}dy + C$$

$$\Rightarrow \quad \frac{x}{y} = 2\int dy + C$$

$$\Rightarrow \quad \frac{x}{y} = 2y + C$$

$$\Rightarrow \quad x = 2y^2 + yC$$

Q. 6. Find the general solution of the following differential equation :

$$\mathbf{(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0}$$

Sol. Given differential equation is

$$(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$$

$$i.e., \quad \frac{dx}{dy} + \frac{1}{1 + y^2}x = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

Above differential equation is in the form of

$\frac{dx}{dy} + Px = Q$, where $P = \frac{1}{1 + y^2} = Q = \frac{e^{\tan^{-1} y}}{1 + y^2}$

Now, $\quad$ I.F. $= e^{\int P dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$

Solution is given by

$$x(\text{I.F.}) = \int e^{\tan^{-1} y} \cdot \frac{e^{\tan^{-1} y}}{1 + y^2}dy + C_1$$

Consider $I_1 = \int e^{\tan^{-1} y} \cdot \frac{e^{\tan^{-1} y}}{1 + y^2}dy + C_1$

$$\left[\text{put } \tan^{-1} y = t \Rightarrow \frac{1}{1 + y^2}dy = dt\right]$$

$$= \int e^t \cdot e^t\,dt + C_1$$

$$= \int e^{2t}\,dt + C_1$$

$$= \frac{e^{2t}}{2} + C_1 = \frac{e^{2\tan^{-1} y}}{2} + C_1$$

Therefore, solution is given by

$$x e^{\tan^{-1} y} = \frac{1}{2}e^{2\tan^{-1} y} + C_1$$

$$\Rightarrow 2x e^{\tan^{-1} y} - e^{2\tan^{-1} y} = 2C_1$$

$$\Rightarrow \quad x = \frac{C + e^{2\tan^{-1} y}}{2e^{\tan^{-1} y}}, \text{ where } C = 2C_1$$

TOPIC-3

Homogeneous Differential Equations

Scan to know more about this topic

Homogeneous Differential equation

Revision Notes

- **Homogeneous Differential Equations and Their Solution**
  - **Identifying a Homogeneous Differential Equation :**

    Step 1 : Write down the given differential equation in the form $\frac{dy}{dx} = F(x, y)$.

    Step 2 : If $f(kx, ky) = k^n f(x, y)$, then the given differential equation is homogeneous of degree '*n*'.

- **Solving a homogeneous differential equation :**

Case I: If $\frac{dy}{dx} = f(x,y)$

Put $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Case II: If $\frac{dx}{dy} = f(x,y)$

Put $x = vy$

$\Rightarrow \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$

Then, we separate the variables to get the required solution.

4 Marks Questions

AI **Q. 1. Solve $x(x-y)dy + y^2\,dx = 0$.**

Sol. Given, $x(x-y)dy + y^2\,dx = 0$

$$\Rightarrow \quad x(x-y)\frac{dy}{dx} + y^2 = 0 \qquad [\because \text{ divide by } dx]$$

$$\Rightarrow \quad x(x-y)\frac{dy}{dx} = -y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y^2}{x^2 - xy} \qquad [\text{Homogeneous form}]$$

Since, degree of numerator and denominator is same, thus it is a homogeneous differential equation.

Put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, $v + x\frac{dv}{dx} = \frac{-(vx)^2}{x^2 - x.vx}$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2(1-v)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{-v^2}{(1-v)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^2}{(1-v)} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^2 - v + v^2}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v}{1-v} \Rightarrow \frac{(1-v)dv}{v} = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int\left(\frac{1}{v} - 1\right)dv = -\int\frac{dx}{x}$$

$$\Rightarrow \quad \log|v| - v = -\log|x| + C$$

$$\Rightarrow \quad \log\left|\frac{y}{x}\right| - \frac{y}{x} + \log|x| = C \qquad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \log|y| - \log|x| - \frac{y}{x} + \log|x| = C$$

$$\Rightarrow \quad \log|y| - \frac{y}{x} = C$$

Q. 2. Solve the following differential equation :

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x},\ x > 0$$

Sol. Given differential equation is

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x},\ x > 0 \qquad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right)$$

Hence, homogeneous.

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{...(ii)}$$

On comparing (i) & (ii), the differential equation becomes

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

or $\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

On integrating, we get

$$\log|v + \sqrt{1+v^2}| = \log|x| + \log c$$

$$\Rightarrow \quad \log|v + \sqrt{1+v^2}| = \log|x|c$$

$$\Rightarrow \quad v + \sqrt{1+v^2} = \pm\, cx$$

$$\Rightarrow \quad \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = \pm\, cx$$

$$\Rightarrow \quad y + \sqrt{x^2+y^2} = cx^2,$$

which is the required solution.

Q. 3. Find the particular solution of the differential equation: $2ye^{\frac{x}{y}}dx + (y - 2xe^{\frac{x}{y}})dy = 0$, given that $x = 0$ when $y = 1$.

Sol. Given differential equation is

$$2ye^{\frac{x}{y}}dx + (y - 2xe^{\frac{x}{y}})dy = 0 \qquad \text{...(i)}$$

Also, when $x = 0$ and $y = 1$ ...(ii)

From (i), $2e^{\frac{x}{y}}dx + \left(1-2\frac{x}{y}e^{\frac{x}{y}}\right)dy = 0$ (divided by y) ...(iii)

Put, $\frac{x}{y} = v$

$\Rightarrow \quad x = vy$

$\Rightarrow \quad dx = v\,dy + y\,dv$

From (iii),

$$2e^v(v\,dy + y\,dv) + (1-2ve^v)dy = 0$$

$$\Rightarrow 2e^v y\,dv + 2ve^v dy + dy - 2ve^v dy = 0$$

$$\Rightarrow \quad 2e^v y dv + dy = 0$$

$$\Rightarrow \quad 2e^v\,dv + \frac{dy}{y} = 0$$

Integrating both sides, we get

$$-\int 2edv + \int \frac{dy}{y} = 0$$

$$2e^v + \log|y| = c$$

$$\Rightarrow \quad 2e^{\frac{x}{y}} + \log|y| = c$$

From (ii),

$$2e^0 + \log 1 = c$$

$$\Rightarrow \quad c = 2$$

$\therefore 2e^{\frac{x}{y}} + \log|y| = 2$ is the required solution.

6 Marks Questions

Q. 1. Solve the following differential equation :

$x^2dy + (xy + y^2)dx = 0$, when $x = 1$ and $y = 1$.

[ISC, 2016]

Sol. $\frac{dy}{dx} = -\frac{(xy+y^2)}{x^2} = \frac{-y}{x} - \left(\frac{y}{x}\right)^2$

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = -v - v^2$$

$$x\frac{dv}{dx} = -2v - v^2$$

$$\int \frac{dv}{(v^2+2v)} = \int -\frac{dx}{x}$$

$$\frac{1}{2}\log\left|\frac{v}{v+2}\right| = -\log x + \log c$$

$$\frac{v}{(v+2)} = \frac{c^2}{x^2}$$

$$\frac{y}{(y+2x)} = \frac{c^2}{x^2}$$

Given, when $x = 1, y = 1$

$$\therefore \quad c = \frac{1}{\sqrt{3}}$$

Hence, solution is $3x^2 y = y + 2x$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Candidates made errors while solving the integration part due to lack of clear understanding about the formula of special integrals and sufficient practice.

Answering Tip

- Post differentiation and integration solving of differential equation needs a lot of practice

Detailed Answer :

The given differential equation is,

$$x^2dy + (xy + y^2)dx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y(x+y)}{x^2}$$

which is homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, the given equation becomes

$$v + x\frac{dv}{dx} = \frac{-vx(x+vx)}{x^2}$$

$$x\frac{dv}{dx} = -v(1+v) - v = -(2v+v^2) = -v(v+2)$$

$$\Rightarrow \quad \frac{dv}{v(v+2)} = -\frac{1}{x}dx$$

$$\therefore \quad \int \frac{dv}{v(v+2)} = -\int \frac{dx}{x}$$

$$\int \frac{1}{2}\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}[\log|v| - \log|v+2|] = -\log|x| + \log C$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{v}{v+2}\right| = \log\left(\frac{C}{x}\right)$$

$$\Rightarrow \quad \sqrt{\frac{v}{v+2}} = \frac{C}{x}$$

$$\Rightarrow \quad \frac{v}{v+2} = \frac{C^2}{x^2}$$

$$\Rightarrow \quad \frac{y/x}{y/x+2} = \frac{C^2}{x^2}$$

$$\Rightarrow \qquad \frac{y}{y+2x} = \frac{C^2}{x^2}$$

$$\Rightarrow \qquad x^2y = C^2(y+2x) \qquad \text{...(i)}$$

When $x = 1$ and $y = 1$, equation (i) becomes

$$1 = C^2(1+2) \Rightarrow C^2 = \frac{1}{3}$$

Putting the value of C^2 in equation (i), we get

$$3x^2y = (y+2x)$$

AI Q. 2. Solve the following differential equation :

$$e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)+\left(1+e^{\frac{x}{y}}\right)\frac{dx}{dy} = 0, \text{ when } x = 0, y = 1$$

[ISC, 2015]

Sol. Given, $e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)+\left(1+e^{\frac{x}{y}}\right)\frac{dx}{dy} = 0$

Substitute $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$e^v - ve^v + v + ve^v + (1+e^v)y\frac{dv}{dy} = 0$$

$$\Rightarrow \qquad (v+e^v)+(1+e^v)y\frac{dv}{dy} = 0$$

$$\Rightarrow \qquad \left(\frac{1+e^v}{v+e^v}\right)dv + \frac{dy}{y} = 0$$

Integrating

$$\int\left(\frac{1+e^v}{v+e^v}\right)dv = -\int\frac{dy}{y}$$

$$\Rightarrow \qquad \log|v+e^v| = -\log|y| + \log c$$

$$\Rightarrow \log|v+e^v| + \log|y| = \log c$$

$$\Rightarrow \quad \log|\{(v+e^v)y\}| = \log c$$

$$\Rightarrow \qquad \log\left|\left\{\left(\frac{x}{y}+e^{\frac{x}{y}}\right)y\right\}\right| = \log c$$

$$\Rightarrow \qquad x + e^{\frac{x}{y}}y = c$$

When $x = 0$ and $y = 1 \Rightarrow c = 1$

$$x + e^{\frac{x}{y}}y = 1$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- Many candidates substituted $y = vx$ and proceeded further to solve the given equation using the rules of homogeneous equation, which was an incorrect approach. The subsequent integrals were not correctly understood by some. A few candidates did not find the value of C (constant) under the given condition *i.e.*, $x = 0, y = 1$.

Answering Tip

- All forms of integration need rigorous practice. The constant of integration should not be ignored.

Detailed Answer :

$$e^{x/y}(1-x/y)+(1+e^{x/y})\frac{dx}{dy} = 0$$

$$\frac{dx}{dy} = \frac{-e^{x/y}(1-x/y)}{1+e^{x/y}} \qquad \text{...(i)}$$

It is a homogeneous differential equation.

Put $\qquad x = vy$

$$\Rightarrow \qquad \frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the values of x and $\frac{dx}{dy}$ in (i), we get

$$v + y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v}$$

$$\Rightarrow \qquad y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v} - v$$

$$= \frac{-v-e^v}{1+e^v}$$

$$\Rightarrow \qquad \frac{1+e^v}{v+e^v}dv = -\frac{dy}{y}$$

On integrating both sides, we get

$$\log|v+e^v| = -\log|y| + c$$

$$\Rightarrow \qquad \log|y(v+e^v)| = c$$

$$\Rightarrow \qquad y\left(\frac{x}{y}+e^{x/y}\right) = \pm\, e^c$$

$$\Rightarrow \qquad x + ye^{x/y} = c_1 \qquad \text{...(i)}$$

Where, c_1 is arbitrary constant.

When $x = 0$, $y = 1$,

$c_1 = 1$

$$[\because 0 + 1e^{0/1} = 0 + 1\times 1 = 1]$$

Substituting this value of c_1 in equation (i), we get

$$x + ye^{x/y} = 1,$$

which is the required particular solution.

Q. 3. Solve the following differential equation :

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

[ISC, 2014]

Sol. $\frac{dy}{dx} = -\left(\frac{(3xy+y^2)}{(x^2+xy)}\right)$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \qquad v + x\frac{dv}{dx} = -\left(\frac{3x\cdot vx + v^2x^2}{x^2 + x\cdot vx}\right)$$

$$x\frac{dv}{dx} = -\frac{-3v-v^2}{1+v} - v$$

$$x\frac{dv}{dx} = \frac{-3v-v^2-v-v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{-2v^2-4v}{1+v}$$

$$\frac{1+v}{2v^2+4v}dv = -\frac{1}{x}dx$$

$$\int \frac{1+v}{2v^2+4v}dv = \int -\frac{1}{x}dx$$

$$\frac{1}{4}\int \frac{2+2v}{2v+v^2}dv = -\int \frac{1}{x}dx$$

$$\frac{1}{4}\log\left|v^2+2v\right| = -\log|x|+c$$

$$\frac{1}{4}\log\left(\frac{y^2}{x^2}+2\frac{y}{x}\right)\cdot x = c$$

$$\log\left(\frac{y^2}{x}+2y\right) = 4c$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- Many candidates proceeded to solve by correctly using the rules for homogeneous equations but after putting $y = vx$ made error in noting that $\frac{dy}{dx} = v + y\frac{dv}{dx}$ or made error while simplifying and separating the v and x functions. The subsequent integrals were not correctly understood by some.

Answering Tip

- Homogenous differential equations and their solution needs practice and drill.

Q. 4. Solve the following differential equation for a particular solution :

$$y - x\frac{dy}{dx} = x + y\frac{dy}{dx}, \text{ when } y = 0 \text{ and } x = 1.$$

[ISC, 2011]

Sol. Given, $y - x = (x+y)\frac{dy}{dx}$

or $\frac{dy}{dx} = \frac{y-x}{y+x}$, which is homogeneous ...(i)

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

eg. (i) becomes,

$$v + x\frac{dv}{dx} = \frac{vx-x}{vx+x} = \frac{x(v-1)}{x(v+1)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - \frac{v}{1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1-v^2-v}{v+1} = \frac{-(v^2+1)}{v+1}$$

$$\Rightarrow \frac{v+1}{v^2+1}dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{v+1}{v^2+1}dv + \int \frac{dx}{x} = 0$$

or $$\frac{1}{2}\int \frac{2v}{v^2+1}dv + \int \frac{1}{v^2+1}dv + \int \frac{dx}{x} = c$$

$$\Rightarrow \frac{1}{2}\log|v^2+1| + \tan^{-1}v + \log|x| = c$$

$$\Rightarrow \frac{1}{2}\log\left|\frac{y^2}{x^2}+1\right| + \tan^{-1}\left(\frac{y}{x}\right) + \log|x| = c$$

[replacing $v = \frac{y}{x}$]

Now, when $y = 0$ and $x = 1$

$$c = \frac{1}{2}\log 1 + \tan^{-1}0 + \log 1$$

$$c = 0 \qquad [\because \log 1 = 0]$$

$\therefore$ Required solution is

$$\frac{1}{2}\log\left|\frac{y^2}{x^2}+1\right| + \tan^{-1}\left(\frac{y}{x}\right) + \log|x| = 0$$

or $$\log\left|\frac{x^2+y^2}{x^2}\right| + 2\tan^{-1}\left(\frac{y}{x}\right) + 2\log|x| = 0$$

or $$\log\left|\frac{x^2+y^2}{x^2}\right| + \log|x^2| + 2\tan^{-1}\left(\frac{y}{x}\right) = 0$$

or $$\log\left|\frac{x^2+y^2}{x^2}\right|.x^2 + 2\tan^{-1}\left(\frac{y}{x}\right) = 0$$

or $$\log|x^2+y^2| + 2\tan^{-1}\frac{y}{x} = 0$$

AI **Q. 5. Solve the differential equation $(x^2+y^2)dx - 2xy\,dy = 0$ and $y = 0$ when $x = 1$.**

[ISC, 2008]

Sol. Given equation may be written as :

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

Which is a homogeneous equation.

$$\Rightarrow \quad y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ Given equation becomes :

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x^2v}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow \frac{2v}{1-v^2}dv = \frac{dx}{x}$$

Integrating both sides

$$\int \frac{2v}{1-v^2}dv = \int \frac{dx}{x}$$

$$-\log|1-v^2| = \log|x| + c$$

$$\Rightarrow -\log\left|1-\left(\frac{y}{x}\right)^2\right| = \log|x| + c \quad \left(\text{putting } v = \frac{y}{x}\right)$$

$$\Rightarrow \qquad c = 0 \qquad [\because y = 0 \text{ when } x = 1]$$

$$\therefore \qquad \frac{1}{1-\dfrac{y^2}{x^2}} = x$$

$$\Rightarrow \qquad x = x^2 - y^2,$$

which is the required solution.

Q. 6. Solve the differential equation

$$2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2} \textbf{ by substituting } y = vx.$$

Sol. Given : $2\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$ ...(i)

It is a homogeneous differential equation.

Put $\quad y = vx$

Then, $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

∴ Equation (i) becomes,

$$2\left[v + x\frac{dv}{dx}\right] = v + v^2$$

$$\Rightarrow 2v + 2x\frac{dv}{dx} = v + v^2$$

$$\Rightarrow \qquad 2x\frac{dv}{dx} = v + v^2 - 2v$$

$$\Rightarrow \qquad 2x\frac{dv}{dx} = v^2 - v$$

$$\Rightarrow \qquad \frac{dv}{v^2 - v} = \frac{1}{2} \times \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{dv}{v^2 - v} = \frac{1}{2}\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{\left(v^2 - 2.\dfrac{v}{2} + \dfrac{1}{4} - \dfrac{1}{4}\right)} = \frac{1}{2}\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{\left(v - \dfrac{1}{2}\right)^2 - \left(\dfrac{1}{2}\right)^2} = \frac{1}{2}\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2\times\dfrac{1}{2}}\log\left|\frac{v - \dfrac{1}{2} - \dfrac{1}{2}}{v - \dfrac{1}{2} + \dfrac{1}{2}}\right| = \frac{1}{2}\log x + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$\Rightarrow \qquad \log\left|\frac{v-1}{v}\right| = \frac{1}{2}\log x + \log C$$

$$\Rightarrow \qquad \log\left|\frac{\dfrac{y}{x} - 1}{\dfrac{y}{x}}\right| = \frac{1}{2}\log x + \log C \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \qquad \log\left|\frac{y-x}{y}\right| = \frac{1}{2}\log x + \log C$$

$$\Rightarrow \qquad \frac{y-x}{y} = C\sqrt{x}$$

$$\Rightarrow \qquad y - x = C\sqrt{x}y$$

Q. 7. Find the particular solution of the differential equation $x\dfrac{dy}{dx} - y + \sin\left(\dfrac{y}{x}\right) = 0$, given that when $x = 2, y = \pi$.

Sol. We have,

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right) \qquad \text{...(i)}$$

It is evident that the given differential equation is homogeneous.

So, put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

(On differentiating w.r.t. x both sides)

Substituting these in equation (i), we have

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \sin\left(\frac{vx}{x}\right)$$

$$\Rightarrow \qquad v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \qquad x\frac{dv}{dx} = -\sin v$$

$$\Rightarrow \qquad \int \operatorname{cosec} v\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|\operatorname{cosec} v - \cot v| = -\log|x| + \log|k|$$

$$\Rightarrow \quad \log|\operatorname{cosec} v - \cot v| = \left|\log\frac{k}{x}\right|$$

$$\Rightarrow \operatorname{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right) = \frac{k}{x}$$

$$\Rightarrow \qquad k\sin\left(\frac{y}{x}\right) = x\left[1 - \cos\left(\frac{y}{x}\right)\right]$$

It is given that when $x = 2, y = \pi$.

$$\text{So,} \quad k\sin\left(\frac{\pi}{2}\right) = 2\left[1 - \cos\left(\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \qquad k(1) = 2[1 - 0]$$

$$\Rightarrow \qquad k = 2$$

$$\text{Thus,} \quad x\left[1 - \cos\left(\frac{y}{x}\right)\right] - 2\sin\left(\frac{y}{x}\right) = 0$$

This is the required particular solution of the given differential equation.

AI **Q. 8. Find the particular solution of the differential equation $xe^{y/x} - y\sin\left(\dfrac{y}{x}\right) + x\dfrac{dy}{dx}\sin\left(\dfrac{y}{x}\right) = 0$, given that when $x = 1, y = 0$.**

Sol. Given differential equation is homogeneous.

∴ Putting $y = vx$ to get $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\frac{dy}{dx} = \frac{y\sin\left(\dfrac{y}{x}\right) - xe^{y/x}}{x\sin\left(\dfrac{y}{x}\right)}$$

$$\Rightarrow \qquad v + x\frac{dv}{dx} = \frac{v\sin v - e^v}{\sin v}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \frac{e^v}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{e^v}{\sin v}$$

$$\therefore \int \sin v e^{-v} dv = -\int \frac{dx}{x} \qquad \begin{bmatrix} I_1 = \int \frac{dx}{x} \\ I_2 = \int e^{-v} \sin v dv \end{bmatrix}$$

$$\Rightarrow \quad I_1 = -\log x + c_1 \qquad \text{...(i)}$$

$$\Rightarrow \quad I_2 = -\sin v e^{-v} + \int \cos v e^{-v} dv$$

$$\Rightarrow \quad I_2 = -\frac{1}{2}(\sin v + \cos v)e^{-v}$$

Putting (i),

$$(\sin v + \cos v)e^{-v} = \log x^2 + c$$

$$\Rightarrow \left[\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right] e^{-\frac{y}{x}} = \log x^2 + c$$

For $x = 1, y = 0 \Rightarrow c = 1$

Hence, solution is

$$\left[\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right] e^{-\frac{y}{x}} = \log x^2 + 1$$

Q. 9. Solve the following differential equation,

$$x\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)$$

Sol. $x\cos\left(\frac{y}{x}\right)(ydx + xdy) = y\sin\left(\frac{y}{x}\right)(xdy - ydx)$

$$\Rightarrow \left(y + x\frac{dy}{dx}\right)x\cos\left(\frac{y}{x}\right) = \left(x\frac{dy}{dx} - y\right)y\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}} \qquad \text{...(i)}$$

Eqn . (i) is homogeneous. So, putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in equation (i), we get

$$v + x\frac{dv}{dx} = \frac{vx[x\cos v + vx\sin v]}{x[vx\sin v - x\cos v]}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v[\cos v + v\sin v]}{(v\sin v - \cos v)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v[\cos v + v\sin v]}{(v\sin v - \cos v)} - v$$

$$= \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad \frac{1.\cos v - v\sin v}{v\cos v}dv = -\frac{2dx}{x}$$

$$\Rightarrow \quad \int \frac{1.\cos v - v\sin v}{v\cos v}dv = -2\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|v\cos v| = -2\log|x| + \log c$$

$$\Rightarrow \quad x^2 v\cos v = c$$

$$\Rightarrow \quad xy\cos\left(\frac{y}{x}\right) = c \qquad [\text{as } y = vx]$$

TOPIC-4

Application of Differential Equations

Revision Notes

Differential equations have wide application in physical sciences, biological sciences, social sciences and geometry.

- **Applications on Growth and Decay**

Law of Natural Growth : Let $x(t)$ be the population at any time t. Assume that population grows at a rate directly proportional to the amount of present population. Then the differential equation governing this phenomenon is the linear equation.

$$\frac{dx}{dt} = kx, \text{ where } k \text{ is the proportionality constant}$$

Here, $k > 0$, since this is phenomenon.

The solution of the D.E. is

$$x(t) = Ce^{kt},$$

Where C is constant of integration. Here, C, k are determined from the two given (initial).

Law of Natural Decay :

The differential equation

$$\frac{dm}{dt} = -km, k > 0$$

describes the decay phenomenon, where it is assumed that the material $m(t)$ at any timet decays at a rate which is proportional to the amount present. The solution is

$$m(t) = ce^{-kt}$$

It initially at, $t = 0$, m_0 is the amount present then

Scan to know more about this topic

Growth and decay model

$$m(t) = m_0^{e^{-kt}}$$

- **Applications on velocity, acceleration, distance and time**

  Velocity : The rate of change of position of a particle with respect to time.

  i.e., $$v = \frac{dx}{dt}$$

  Acceleration : Rate of change of velocity with respect to time at any point along a given direction is called the acceleration of the particle along the direction.

  i.e., $$a = \frac{d^2x}{dt^2}$$

4 Marks Questions

AI **Q. 1. The population of a town grows at the rate of 10% per year. Using differential equation, find how long will it take for the population to grow 4 times. [ISC, 2018]**

Sol. Let initial population be : P_0 and population after t years be P.

$$\frac{dP}{dt} = 10\% \text{ of } P$$

$$\Rightarrow \quad \frac{dP}{dt} = \frac{1}{10}P$$

$$\Rightarrow \quad \frac{dP}{P} = \frac{1}{10}dt$$

Integrating both sides, we get

$$\int\frac{dP}{P} = \int\frac{1}{10}dt$$

$$\log P = \frac{1}{10}t + C$$

When $t = 0, P = P_0 \Rightarrow \log P_0 = C$

$$\log P = \frac{1}{10}.t + \log P_0$$

$$\log\left(\frac{P}{P_0}\right) = \frac{1}{10}.t$$

Since, population grows 4 times, $P = 4P_0$

$$\therefore \quad \log 4 = \frac{1}{10}.t$$

$$\Rightarrow \quad t = 10\log 4$$

$\Rightarrow$ 20 log 2(years) **[ISC Marking Scheme, 2018]**

Examiner's Comment

- Not many candidates attempted this question correctly. Some candidates did not have any basic knowledge of applications of differential equations while a few candidates did not use the initial value correctly.

Q. 2. Assume that a spherical raindrop evaporates at a rate proportional to its surface area. If its radius originally in 3 mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the rain drop at an time.

Sol. Let $r(t)$ denotes the radius of the raindrop after t minutes.

Since, given the radius is decreasing as increases, the rate of change of r must be negative.

Let V be the volume of the raindrop and S be its surface area

i.e., $$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2$$

Also given, $\frac{dv}{dt} \propto S$

$$\Rightarrow \quad \frac{dv}{dt} = -kS$$

[$\therefore$ as the rate of change of radius with time is negative]

$$\therefore \quad \frac{4}{3}\pi 3r^2\frac{dr}{dt} = -k\,4\pi r^2$$

$$\left[\because \frac{dv}{dt} = \frac{4}{3}\pi(3r^2)\right]$$

$$\text{or} \quad \frac{dr}{dt} = -k \quad \text{...(i)}$$

Integrating (i), we get

$$\int\frac{dr}{dt} = -\int k$$

$$\Rightarrow \quad \int dr = -\int k\,dt$$

$$\Rightarrow \quad r = -kt + C \quad \text{...(ii)}$$

Let at time $t = 0$, radius is r_0

$\therefore$ From (ii), $r_0 = 0 + C = C$

Hence, $r = -kt + r_0$ is the required expression.

Q. 3. A radio active substance has a half life of h days. Find a formula for its mass m in terms of t the time, if the initial mass is m_0. What is the initial decay rate ?

Sol. According to question,

The equation of decay is $\frac{dm}{dt} = -km$ ...(i)

$$\therefore \quad \frac{dm}{m} = -k\,dt$$

Integrating both sides, we get

$$\int\frac{dm}{m} = -k\int dt$$

$$\therefore \quad \log m = -kt + C \quad \text{...(ii)}$$

Initially, $t = 0, m = m_0 \therefore \log m_0 = C$

and $\log m = -kt + \log m_0$

Again when $t = h$ (half life), $m = \frac{m_0}{2}$

[$\because$ half life means – half mass decays in h days]

$$\Rightarrow \quad \log\left|\frac{m_0}{2}\right| = -kh + \log|m_0|$$

$$\Rightarrow \quad kh = \log|m_0| - \log\left|\frac{m_0}{2}\right|$$

$$= \log\left|\frac{m_0}{m_0/2}\right| = \log 2$$

$$\Rightarrow \quad k = \frac{\log 2}{h}$$

From eq. (ii),

$$\log m = \frac{-\log 2}{h}.t + \log m_0$$

$$[\therefore C = \log m_0]$$

$$\therefore \quad \log|m| - \log|m_0| = \frac{-t}{h}\log 2$$

$$\Rightarrow \quad \log\left|\frac{m}{m_0}\right| = \log 2^{-\left(\frac{t}{h}\right)}$$

$$\Rightarrow \quad \frac{m}{m_0} = 2^{-\left(\frac{t}{h}\right)}$$

$$\Rightarrow \quad m = m_0.2^{-\left(\frac{t}{h}\right)} \qquad \text{...(ii)}$$

Which is the formula for its mass m in terms of t.

The initial decay rate is given by $\frac{dm}{dt}$ when $m = m_0$ and $k = \frac{\log 2}{h}$

Hence, $\frac{dm}{dt} = -km = \frac{-\log 2}{h} \times m_0$

AI Q. 4. The acceleration after t seconds of a particle which starts from rest and moves in a straight line is $\left(8 - \frac{t}{5}\right)$ cm/sec^2. Find the velocity of the particle at this instant when the acceleration is zero.

Sol. Let v be the velocity of the particle at time t.

$$\therefore \quad \frac{dv}{dt} = \left(8 - \frac{t}{5}\right)$$

Integrating both sides, we get

$$\int dv = \int\left(8 - \frac{t}{5}\right)dt$$

$$\Rightarrow \quad v = 8t - \frac{t^2}{10} + C \qquad \text{...(i)}$$

When $t = 0$, we have $v = 0$

From eq (i), $C = 0$

$$\therefore \quad v = 8t - \frac{t^2}{10} \qquad \text{...(ii)}$$

Given Acceleration is zero

i.e., $\frac{dv}{dt} = 0$

or $8 - \frac{2t}{10} = 0$

$$\Rightarrow \quad t = 40$$

Put $t = 40$ in eq. (ii), we get

$V = 320 - 160 = 160$

Hence, the required velocity is 160 cm/sec

6 Marks Questions

AI Q. 1. Suppose the growth of a population is proportional to the number present. If the population of a colony doubles in 25 days, in how many days will the population become triple ?

Sol. Let P_0 be the initial population and P be the population at any time t.

Given, $\frac{dP}{dt} \propto P$

$$\Rightarrow \quad \frac{dP}{dt} = kP, k \text{ is constant}$$

$$\Rightarrow \quad \frac{dP}{P} = k\,dt$$

$$\Rightarrow \quad \int\frac{dP}{P} = k\int dt$$

$$\Rightarrow \quad \log|P| = kt + C \qquad \text{...(i)}$$

At $t = 0$, we have $P = P_0$

$\therefore$ From eq (i) $\log|P_0| = 0 + C$

$$\Rightarrow \quad C = \log|P_0|$$

Therefore, eq (i) becomes $\log P = kt + \log P_0$

$$\Rightarrow \quad \log\left|\frac{P}{P_0}\right| = kt$$

Also, given that $P = 2P_0$ when $t = 25$ days.

$\therefore$ From eq (ii), we have

$$\log\left|\frac{2P_0}{P_0}\right| = 25\,k$$

$$\Rightarrow \quad k = \frac{1}{25}\log 2$$

Put value of k in eq (ii), we get

$$\log\left|\frac{P}{P_0}\right| = \left(\frac{1}{25}\log 2\right)t \qquad \text{...(ii)}$$

Suppose the population is tripled in t_1 days *i.e.,* $P = 3P_0$ when $t = t_1$

Put $P = 3P_0$ and $t = t_1$ in eq (ii), we get

$$\log\left|\frac{3P_0}{P_0}\right| = \left(\frac{1}{25}\log 2\right)t_1$$

$$\Rightarrow \quad t_1 = 25\left(\frac{\log 3}{\log 2}\right) \text{days}$$

Hence, population is tripled in $25\left(\dfrac{\log 3}{\log 2}\right)$ days.

Q. 2. It is given that radius decomposes at a rate proportional to the amount present. If *P*% of the original amount of radium disappears in *l* years, what percentage of it will remain after 2*l* years ?

Sol. Let A_0 be the original amount of radium and A be the amount of radium at any time t. Then, the rate of decompose of radium is $\dfrac{dA}{dt}$.

Given, $\dfrac{dA}{dt} \propto A$

$$\Rightarrow \quad \frac{dA}{dt} = -kA, \text{ where } k \text{ is positive constant}$$

$$\Rightarrow \quad \frac{dA}{A} = -kdt$$

$$\Rightarrow \quad \int \frac{dA}{A} = -k\int dt$$

$$\Rightarrow \quad \log|A| = -kt + C \qquad \text{...(i)}$$

At $t = 0$, we have $A = A_0$, therefore from eq (i), we get

$$\log|A_0| = 0 + C \Rightarrow C = \log A_0$$

∴ Eq (i) becomes,

$$\log A = -kt + \log A_0$$

$$\Rightarrow \quad \log\left|\frac{A}{A_0}\right| = -kt \qquad \text{...(ii)}$$

Given that $p\%$ of the original amount of radium disintegrates in l years.

∴ Amount of radium present at $t = l$ is

$$\left(A_0 - \frac{p}{100}\times A_0\right) = \left(1-\frac{p}{100}\right)A_0$$

Put $A = A_0\left(1-\dfrac{p}{100}\right)$ and $t = l$ in eq. (ii), we get

$$\log\left(1-\frac{p}{100}\right) = -kl$$

$$\Rightarrow \quad k = \frac{-1}{l}\log\left(1-\frac{p}{100}\right)$$

Substitute value of k in eq (ii), we get

$$\log\left(\frac{A}{A_0}\right) = \frac{t}{l}\log\left(1-\frac{p}{100}\right) \qquad \text{...(iii)}$$

Let A be the amount of radium available after $2l$ years

Put $t = 2l$ in eq (iii), we get

$$\log\left|\frac{A}{A_0}\right| = 2\log\left|1-\frac{p}{100}\right|$$

$$\Rightarrow \quad \left(\frac{A}{A_0}\right) = \left(1-\frac{p}{100}\right)^2$$

Multiplying both sides by 100

$$\Rightarrow \quad \frac{A}{A_0}\times 100 = \left(1-\frac{p}{100}\right)^2\times 100$$

$$\Rightarrow \quad \frac{A}{A_0}\times 100 = \left(10-\frac{p}{10}\right)^2$$

Hence, required percentage is $\left(10-\dfrac{p}{10}\right)^2$.

AI Q. 3. The population of a village increases at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25,000 in the year 2004, what was its population in 2009 ?

Sol. Let the population at time t be x.

Given, rate of population increase is proportional to number of inhabitants.

i.e., $\dfrac{dx}{dt} \propto x$

$$\Rightarrow \quad \frac{dx}{dt} = kx, \text{ where } k \text{ is constant of proportionality}$$

$$\Rightarrow \quad \frac{dx}{x} = kdt$$

$$\Rightarrow \quad \int\frac{dx}{x} = k\int dt$$

$$\Rightarrow \log|x| = kt + C \qquad \text{...(i)}$$

In 1999, $t = 0$ and population = 20,000

∴ $\log 20{,}000 = C$

Put value of C in eq (i), we get

$\log|x| = kt + \log 20{,}000$

$\Rightarrow \log|x| - \log 20{,}000 = kt$

$$\Rightarrow \quad \log\left(\frac{x}{20{,}000}\right) = kt \qquad \text{...(ii)}$$

In 2004, $t = 5$, $x = 25{,}000$

$$\therefore \quad \log\left(\frac{25000}{20000}\right) = k\times 5$$

$$\Rightarrow \quad k = \frac{1}{5}\log\left(\frac{5}{4}\right)$$

∴ From eq (ii) becomes

$$\log\left|\frac{x}{20{,}000}\right| = \left(\frac{1}{5}\log\frac{5}{4}\right)t$$

In 2009, $t = 10$

$$\therefore \quad \log\left|\frac{x}{20{,}000}\right| = \left(\frac{1}{5}\log\frac{5}{4}\right)\times 10$$

$$\Rightarrow \quad \log\left|\frac{x}{20{,}000}\right| = 2\log\left(\frac{5}{4}\right)$$

$$= \log\left(\frac{5}{4}\right)^2 = \log\left(\frac{25}{16}\right)$$

$$\Rightarrow \quad \frac{x}{20{,}000} = \frac{25}{16}$$

$$\Rightarrow \quad x = \frac{25\times 20{,}000}{16} = 31{,}250$$

Hence, in 2009 the population was 31,250.

SELF ASSESSMENT TEST

UNIT–III

Maximum Marks : 25 **Timing : 1 Hour**

Q. 1. **Find the value of k for which the function.**

$$f(x) = \begin{cases} \dfrac{x^2+3x-10}{x-2} & ,x \neq 2 \\ k & ,x = 2 \end{cases}$$

is continuous at $x = 2$.

Q. 2. **Differentiate $\tan^{-1}\left(\dfrac{1+\cos x}{\sin x}\right)$ with respect to x.**

Q. 3. **Evaluate : $\int x^2 (e^{x^3}) \cos (2e^{x^3})\, dx$.**

Q. 4. **Solve the differential equation :**

$$\frac{dy}{dx} = \frac{(1+\cos^2 x)\sin^2 y}{(1+\sin^2 y)\cos^2 x}$$

Q. 5. **Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing or R.**

Q. 6. **Find : $\lim\limits_{x\to 0} \dfrac{\sin x - x + x^3/6}{x^3}$**

Q. 7. **If $x = a \sin^3 t$ and $y = a \cos^3 t$, then find dy/dx.**

Q. 8. **If the sum of three positive numbers is 26. The second number is thrice as large as the first. If the sum of the squares of number is least, then find the numbers.**

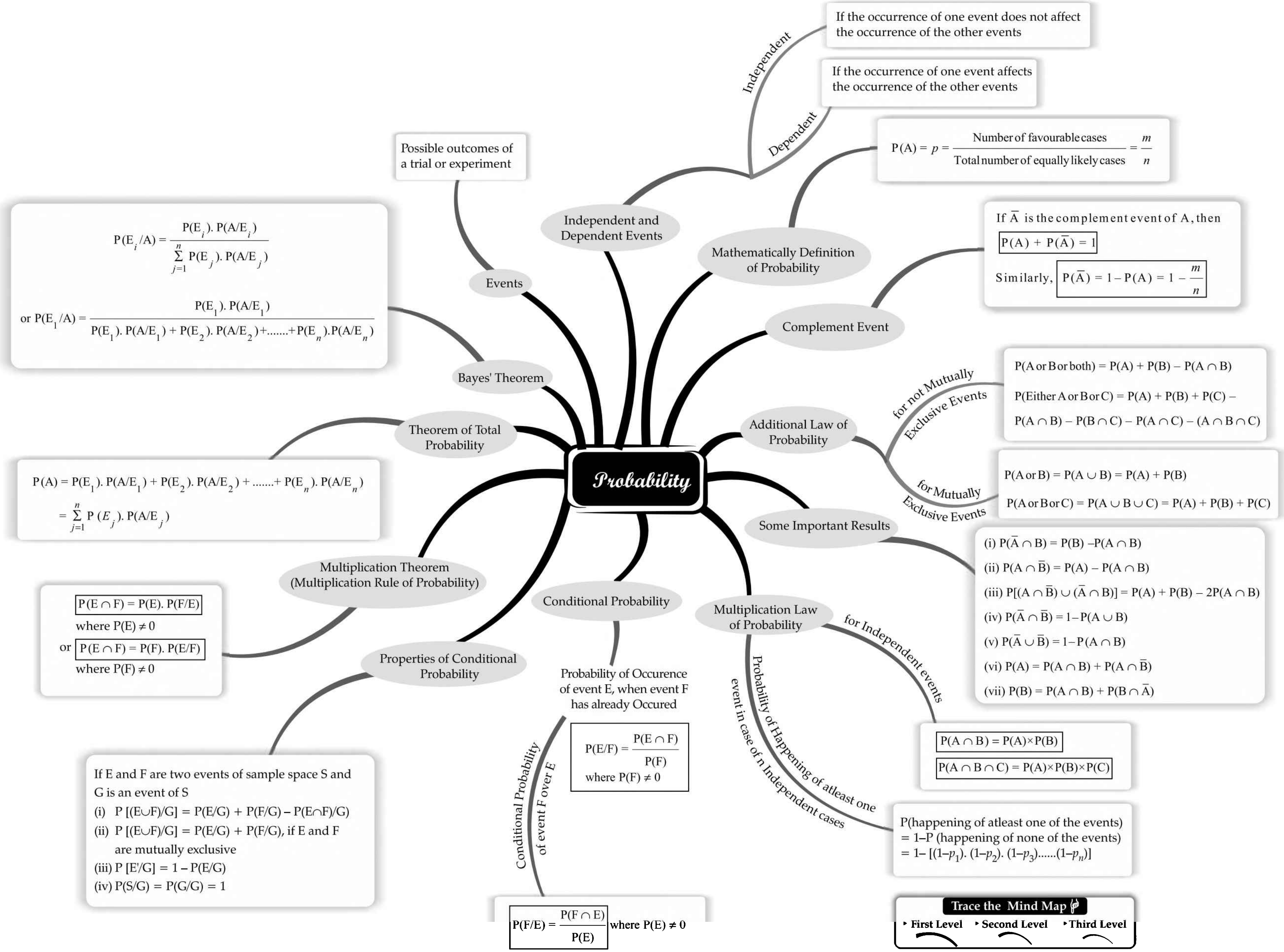
Q. 9. **Solve the differential equation**

$$\tan x \frac{dy}{dx} + 2y = \sec x.$$

OSWAAL PREPARATION TOOLS

Solutions for Self-assessment Test

Visit : *http://qr.w69b.com/g/pi8gdC0qk* OR Scan the Code



Probability

Independent and Dependent Events

Independent: If the occurrence of one event does not affect the occurrence of the other events

Dependent: If the occurrence of one event affects the occurrence of the other events

Mathematically Definition of Probability

$$P(A) = p = \frac{\text{Number of favourable cases}}{\text{Total number of equally likely cases}} = \frac{m}{n}$$

Complement Event

If $\bar{A}$ is the complement event of A, then $\boxed{P(A) + P(\bar{A}) = 1}$

Similarly, $\boxed{P(\bar{A}) = 1 - P(A) = 1 - \frac{m}{n}}$

Additional Law of Probability

for not Mutually Exclusive Events

P(A or B or both) = $P(A) + P(B) - P(A \cap B)$

P(Either A or B or C) = $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) - (A \cap B \cap C)$

for Mutually Exclusive Events

P(A or B) = $P(A \cup B) = P(A) + P(B)$

P(A or B or C) = $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Some Important Results

(i) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

(ii) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

(iii) $P[(A \cap \bar{B}) \cup (\bar{A} \cap B)] = P(A) + P(B) - 2P(A \cap B)$

(iv) $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

(v) $P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B)$

(vi) $P(A) = P(A \cap B) + P(A \cap \bar{B})$

(vii) $P(B) = P(A \cap B) + P(B \cap \bar{A})$

Multiplication Law of Probability

for Independent events

$\boxed{P(A \cap B) = P(A) \times P(B)}$

$\boxed{P(A \cap B \cap C) = P(A) \times P(B) \times P(C)}$

Probability of Happening of atleast one event in case of n Independent cases

P(happening of atleast one of the events)
= 1–P (happening of none of the events)
= $1 - [(1-p_1).(1-p_2).(1-p_3)......(1-p_n)]$

Conditional Probability

Probability of Occurence of event E, when event F has already Occured

$$\boxed{P(E/F) = \frac{P(E \cap F)}{P(F)} \quad \text{where } P(F) \neq 0}$$

Conditional Probability of event F over E

$$\boxed{P(F/E) = \frac{P(F \cap E)}{P(E)}} \text{ where } P(E) \neq 0$$

Properties of Conditional Probability

If E and F are two events of sample space S and G is an event of S

(i) $P[(E \cup F)/G] = P(E/G) + P(F/G) - P(E \cap F)/G)$

(ii) $P[(E \cup F)/G] = P(E/G) + P(F/G)$, if E and F are mutually exclusive

(iii) $P[E'/G] = 1 - P(E/G)$

(iv) $P(S/G) = P(G/G) = 1$

Multiplication Theorem (Multiplication Rule of Probability)

$\boxed{P(E \cap F) = P(E). P(F/E)}$
where $P(E) \neq 0$

or $\boxed{P(E \cap F) = P(F). P(E/F)}$
where $P(F) \neq 0$

Theorem of Total Probability

$$P(A) = P(E_1). P(A/E_1) + P(E_2). P(A/E_2) + ...+ P(E_n). P(A/E_n)$$
$$= \sum_{j=1}^{n} P(E_j). P(A/E_j)$$

Bayes' Theorem

$$P(E_i/A) = \frac{P(E_i). P(A/E_i)}{\sum_{j=1}^{n} P(E_j). P(A/E_j)}$$

$$\text{or } P(E_1/A) = \frac{P(E_1). P(A/E_1)}{P(E_1). P(A/E_1) + P(E_2). P(A/E_2)+.......+P(E_n).P(A/E_n)}$$

Events

Possible outcomes of a trial or experiment

Trace the Mind Map
- First Level
- Second Level
- Third Level

UNIT-IV
Probability

CHAPTER
9
PROBABILITY

SYLLABUS

- *Conditional probability, multiplication theorem on probability, independent events, total probability, Bayes' Theorem, Random variable and its probability distribution, mean and variance of random variable. Repeated independent (Bernoulli) trials and Binomial distribution.*
  - *Independent and dependent events conditional events.*
  - *Laws of Probability, addition theorem, multiplication theorem, conditional probability.*
  - *Theorem of Total Probability.*
  - *Bayes' Theorem*
  - *Theoretical probability distribution, probability distribution function; mean and variance of random variable, Repeated independent (Bernoulli trials), binomial distribution - its mean and variance.*

Trend Analysis

List of Concepts	2016	2017	2018	2019	2020
Conditional Probability and Multiplication Theorem on Probability	1 Q (2 marks) 2 Q (6 marks)	1 Q (4 marks) 2 Q (6 marks)	2 Q (2 marks) 1 Q (4 marks)	2 Q (2 marks) 1 Q (4 marks)	2 Q (2 marks) 1 Q (4 marks)
Random Variable and its Probability Distributions, Bernoulli Trials and Binomial Distribution	1 Q (6 marks)	1 Q (6 marks)	1 Q (6 marks)		1 Q (6 marks)
Bayes' Theorem	1 Q (6 marks)	1 Q (6 marks)		1 Q (6 marks)	

TOPIC-1
Conditional Probability and Multiplication Theorem on Probability

Revision Notes

- **Basic Definition :**
  - **Random Experiment** : An experiment, whose all possible outcomes are known in advance.

    e.g., Tossing of a coin, Throwing a dice selecting a card from a deck of cards.
  - **Sample-Space :** A set of all possible outcomes associated with a random experiment. It is denoted by '*S*'.

    e.g., (i) In the experiment of tossing a coin.

TOPIC - 1
Conditional Probability and Multiplication Theorem on Probability .... **P. 243**
TOPIC - 2
Random Variable and its Probability Distributions, Bernoulli Trials and Binomial Distribution .... **P. 258**
TOPIC - 3
Bayes' Theorem .... **P. 268**

Sample Space, $S = \{H, T\}$

(ii) In the experiment of throwing a dice.

Sample, $S = \{1, 2, 3, 4, 5, 6\}$

- **Event :** An event is a subset of a sample space. If an event is a set containing only one element of a sample space is called a 'simple event'. A 'compound event' is one that can be represented as a union of sample points.

  e.g., In a throw of a single dice, getting number 4 is a simple event where as getting an even number is a compound event.

- **Probability :**

  Let S and E be the sample space and event of an experiment respectively.

  Then,

$$\text{Probability} = \frac{\text{Number of favourable events}}{\text{Total number of elementary events}} = \frac{n(E)}{n(S)}$$

$$0 \le n(E) \le n(S) \Rightarrow 0 \le \frac{n(F)}{n(S)} \le 1$$

or $$0 \le P(E) \le 1$$

Hence, if $P(E)$ denotes the probability of occurrence of an event E then,

$0 \le P(E) \le 1$ and $P(\bar{E}) = 1 - P(E)$ such that $P(\bar{E})$ denotes the probability of non-occurrence of the event E.

Scan to know more about this topic

Conditional probability

Scan to know more about this topic

Multiplicative theorem on probability

- Note that $P(\bar{E})$ can also be represented as $P(E')$.

➢ Mutually Exclusive or Disjoint Events :

Two events A and B are said to be mutually exclusive, if occurrence of one prevents the occurrence of the other *i.e.*, they can't occur simultaneously.

In this case, sets A and B are disjoint, *i.e.*, $A \cap B = \phi$.

Consider an example of throwing a die. We have the sample spaces as, $S = \{1, 2, 3, 4, 5, 6\}$

Suppose A = the event of occurrence of a number greater than $4 = \{5, 6\}$

B = the event of occurrence of an odd number $= \{1, 3, 5\}$

and C = the event of occurrence of an even number $= \{2, 4, 6\}$

In these events, the events B and C are mutually exclusive events but A and B are not mutually exclusive events because they can occur together (when the number 5 comes up). Similarly, A and C are not mutually exclusive events as they can also occur together (when the number 6 comes up).

- If A and B are mutually exhaustive events, then we always have :

$$P(A \cap B) = 0 \qquad [\text{As } n(A \cap B) = n(\phi) = 0]$$
$$P(A \cup B) = P(A) + P(B)$$

- If A, B and C are mutually exhaustive events, then we always have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

➢ Independent Events :

Two events are independent if the occurrence of one does not affect the occurrence of the other.

Consider an example of drawing two balls one by one with replacement from a bag containing 3 red and 2 black balls.

Suppose A = the event of getting a red ball in first draw.

B = the event of getting of a black ball in the second draw.

Then $$P(A) = \frac{3}{5}, \qquad P(B) = \frac{2}{5}$$

Here, probability of occurrence of event B is not affected by the occurrence or non-occurrence of the event A. Hence, events A and B are independent events.

➢ Exhaustive Events :

Two or more events say A, B and C of an experiment are said to be exhaustive events, if

(a) their union is the total sample space, *i.e.*, $A \cup B \cup C = S$

(b) the event A, B and C are disjoint in pairs, *i.e.*, $A \cap B = \phi$, $B \cap C = \phi$ and $C \cap A = \phi$.

(c) $P(A) + P(B) + P(C) = 1$

Consider an example of throwing a die. We have $S = \{1, 2, 3, 4, 5, 6\}$.

Suppose A = the event of getting of an even number $= \{2, 4, 6\}$

B = the event of getting of an odd number $= \{1, 3, 5\}$

and C = the event of getting a number multiple of $3 = \{3, 6\}$

In these events, the events A and B are exhaustive events as $A \cup B = S$, but the events A and C or the events B and C are not exhaustive events as $A \cup C \neq S$ and similarly, $B \cup C \neq S$.

- **Conditional Probability :**

  By the conditional probability, we mean the probability of occurrence of event A when B has already occurred. The conditional probability of occurrence of event A when B has already occurred' is sometimes also called as probability of occurrence of event A w.r.t. B.

  - $P(A/B) = \dfrac{P(A \cap B)}{P(B)}, B \neq \phi, i.e., P(B) \neq 0$
  - $P(B/A) = \dfrac{P(A \cap B)}{P(A)}, A \neq \phi, i.e., P(A) \neq 0$
  - $P(\bar{A}/B) = \dfrac{P(\bar{A} \cap B)}{P(B)}, P(B) \neq 0$
  - $P(A/\bar{B}) = \dfrac{P(A \cap \bar{B})}{P(\bar{B})}, P(\bar{B}) \neq 0$
  - $P(\bar{A}/\bar{B}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}, P(\bar{B}) \neq 0$
  - $P(A/B) + P(\bar{A}|B) = 1, P(B) \neq \phi$

Know the Formulae

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, *i.e.*, $P(A$ or $B) = P(A) + P(B) - P(A$ and $B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
- $P(\bar{A} \cap B) = P($only $B) = P(B - A) = P(B$ but not $A) = P(B) - P(A \cap B)$
- $P(A \cap \bar{B}) = P($only $A) = P(A - B) = P(A$ but not $B) = P(A) - P(A \cap B)$
- $P(\bar{A} \cap \bar{B}) = P($neither A nor $B) = 1 - P(A \cup B)$

Events and Symbolic Representations

Description of the event	Equivalent set notation
Event A	A
Not A	$\bar{A}$ or A'
A or B (occurrence of atleast one A or B)	$A \cup B$ or $A + B$
A and B (simultaneous occurrence of both A and B)	$A \cap B$ or AB
A but not B (A occurs but B does not)	$A \cap \bar{B}$ or $A - B$
Neither A nor B	$\bar{A} \cap \bar{B}$
Atleast one A, B or C	$A \cup B \cup C$
All the three A, B and C	$A \cap B \cap C$

Mnemonics

Independent Events and Mutually Exclusive events

Mnemonics : I is not ME and ME is not I

Interpretation :

I : Independent Events
ME : Mutually Exclusive Events

1 Mark Questions

Q. 1. If $P(A) = \dfrac{4}{5}$ and $P(A \cap B) = \dfrac{7}{10}$, then the value of $P(B/A)$ is :

(a) $\dfrac{1}{8}$ **(b)** $\dfrac{4}{7}$

(c) $\dfrac{5}{7}$ **(d)** $\dfrac{7}{8}$

[ISC Specimen Paper, 2021]

Sol. Correct options : (d)

Explanation :

Given that $P(A) = \dfrac{4}{5}$ & $P(A \cap B) = \dfrac{7}{10}$

$$P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{7}{10}}{\frac{4}{5}} = \frac{7}{8}$$

AI **Q. 2. If *A* and *B* are two events such that *P(A)* ≠ 0 and *P(B/A)* = 1, then**

(a) $A \subset B$ **(b)** $B \subset A$

(c) $B = \phi$ **(d)** $A = \phi$

Sol. Correct options : (a)

Explanation :

$P(A) \neq 0$ and $P(B|A)=1$

$$P(B/A)=\frac{P(B\cap A)}{P(A)}$$

$$1=\frac{P(B\cap A)}{P(A)}$$

$$P(A)=P(B\cap A)$$

$$\therefore A\subset B$$

Q. 3. If *P(A/B)* > *P(A)*, then which of the following is correct :

(a) *P(B/A)* < *P(B)*

(b) *P(A∩B)* < *P(A)* . *P(B)*

(c) *P(B/A)* > *P(B)*

(d) *P(B/A)* = *P(B)*

Sol. Correct option : (c)

Explanation :

$P(A/B)>P(A)$

$\Rightarrow \frac{P(A\cap B)}{P(B)} > P(A)$

$\Rightarrow P(A\cap B)>P(A).P(B)$

$\Rightarrow \frac{P(A\cap B)}{P(A)} > P(B)$

$\Rightarrow P(B/A)>P(B)$

AI **Q. 4. If *A* and *B* are any two events such that *P(A)* + *P(B)* – *P(A* and *B)* = *P(A)*, then**

(a) *P(B/A)* = 1 **(b) *P(A/B)* = 1**

(c) *P(B/A)* = 0 **(d) *P(B/A)* = 0**

Sol. Correct option : (b)

Explanation :

$P(A)+P(B)-P(A \text{ and } B)=P(A)$

$\Rightarrow P(A)+P(B)-P(A\cap B)=P(A)$

$\Rightarrow \quad P(B)-P(A\cap B)=0$

$\Rightarrow \quad P(A\cap B)=P(B)$

$$\therefore \quad P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{P(B)}{P(B)}=1$$

Q. 5. If *A* and *B* are two events such that *A* ⊂ *B* and *P(B)* ≠ 0, then which of the following is correct?

(a) *P(A/B)* = *P(B)/P(A)* **(b) *P(A/B)* < *P(A)***

(c) *P(A/B)* ≥ *P(A)* **(d) *P(A/B)* ≤ *P(A)***

Sol. Correct option : (c)

Explanation :

If $A\subset B$, then $A\cap B=A$

$\Rightarrow P(A\cap B)=P(A)$

Also, $P(A)<P(B)$ ...(i)

Consider,

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A)}{P(B)} \qquad \text{...(ii)}$$

It is known that, $P(B) \leq 1$

$\Rightarrow \frac{1}{P(B)} \geq 1$

$\Rightarrow \frac{P(A)}{P(B)} \geq P(A)$

From equation (ii), we obtain

$\Rightarrow P(A/B) \geq P(A)$

$\therefore$ *P(A/B)* is not less than *P(A)*.

Q. 6. If *P(A)* = 1/2, *P(B)* = 0, then *P(A/B)* is

(a) 0 **(b)** $\frac{1}{2}$

(c) not defined **(d) 1**

Sol. Correct option : (c)

Explanation : It is given that,

$P(A)=\frac{1}{2}$ and $P(B) = 0$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A\cap B)}{0}$$

Therefore, *P(A/B)* is not defined.

Q. 7. If *A* and *B* are events such that *P(A/B)* = *P(B/A)*, then

(a) $A\subset B$ **but** $A \neq B$ **(b)** $A = B$

(c) $A\cap B = \phi$ **(d) *P(A)* = *P(B)***

Sol. Correct option : (d)

Explanation : It is given that,

$P(A/B) = P(B/A)$

$\Rightarrow \frac{P(A\cap B)}{P(B)}=\frac{P(A\cap B)}{P(A)}$

$\Rightarrow \quad P(A)=P(B)$

AI **Q. 8. If *A* and *B* are two events such that *P*(*A*) = 0.2, *P*(*B*) = 0.4 and *P*(*A* ∪ *B*) = 0.5, then value of *P*(*A*/*B*) is ?**

(a) 0.1 **(b) 0.25**

(c) 0.5 **(d) 0.08**

Sol. Correct option : (b)

Explanation :

$$\begin{aligned} \text{...} \quad P(A\cap B) &= P(A)+P(B)-P(A\cup B)\\ P(A\cap B) &= 0.2+0.4-0.5\\ &= 0.1\\ P(A/B) &= \frac{P(A\cap B)}{P(B)}=\frac{0.1}{0.4}=\frac{1}{4}=0.25 \end{aligned}$$

Q. 9. A bag contains 6 balls of which two are red and four are black. Two balls are drawn at random. Probability that they are of the different colours is.

(a) $\frac{2}{5}$ **(b)** $\frac{1}{15}$

(c) $\frac{8}{15}$ **(d)** $\frac{4}{15}$

Sol. Correct option : (c)

Explanation :

Required probability $= \frac{{}^2C_1 \times {}^4C_1}{{}^6C_2}$

$$= \frac{2\times 4}{\frac{6\times 5}{2}} = \frac{16}{30} = \frac{8}{15}$$

AI Q. 10. A and B appeared for an interview for two vacancies. The probability of A's selection is $\frac{4}{5}$ and B's selection is $\frac{1}{3}$. Find the probability that one of them will be selected.

[ISC Specimen Paper, 2021]

Sol. $A \rightarrow A$ will be selected.

$B \rightarrow B$ will be selected.

$$\therefore \quad P(A') = \frac{4}{5} \ \& \ P(B') = \frac{1}{3}$$

$\therefore$ Probability that none of then will be selected

$$= P(A').P(B')$$

$$= [1 - P(A)][1 - P(B)]$$

$$= \left(1-\frac{4}{5}\right)\left(1-\frac{1}{3}\right) = \frac{2}{15}$$

Q. 11. If A and B are two events such that $P(A/B) = p$, $P(A) = p$, $P(B) = 1/3$ and $P(A\cup B) = 5/9$, then find value of p.

Sol. Here,

$$P(A) = p, P(B) = \frac{1}{3} \text{ and } P(A \cup B) = \frac{5}{9}$$

$$\because \quad P(A/B) = \frac{P(A\cap B)}{P(B)} = p$$

$$\Rightarrow \quad P(A\cap B) = \frac{p}{3}$$

and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \quad \frac{5}{9} = p + \frac{1}{3} - \frac{p}{3}$$

$$\Rightarrow \quad \frac{5}{9} - \frac{1}{3} = \frac{2p}{3}$$

$$\Rightarrow \quad \frac{5-3}{9} = \frac{2p}{3}$$

$$\Rightarrow \quad p = \frac{2}{9} \times \frac{3}{2} = \frac{1}{3}$$

2 Marks Questions

Q 1. A bag contains 5 white, 7 red and 4 black balls. If four balls are drawn one by one with replacement, what is the probability that none is white ?

[ISC, 2020]

Sol. Number of White balls= 5

Red balls= 7

Black balls= 4

Total balls= 5 + 7 + 4 = 16

Probability of the ball is not White

$$= \frac{11}{16}$$

Probability of the ball is not White when 4 balls are drawn one by one with replacement

$$= \left(\frac{11}{16}\right)^4$$

$$= \frac{14641}{65536}$$

$$= 0.2234$$

[ISC Marking Scheme, 2020]

Q 2. Let A and B be two events such that $P(A) = \frac{1}{2}$, $P(B) = p$ and $P(A \cup B) = \frac{3}{5}$ find 'p' if A and B are independent events. **[ISC, 2020]**

Sol. $P(A) = \frac{1}{2}$, $P(B) = p$ and $P(A \cup B) = \frac{3}{5}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{3}{5} = \frac{1}{2} + p - P(A \cap B)$$

$$\frac{3}{5} - \frac{1}{2} = p - P(A)\, P(B)$$

Since, [$P(A \cap B) = P(A)P(B)$ as A and B are independent events]

$$\frac{6-5}{10} = p - \frac{1}{2}p$$

$$\frac{1}{10} = \frac{p}{2}$$

$$\therefore \quad p = \frac{1}{2}$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Some students are confused in case of independent events i.e., if A and B are independent events, then $P(A \cap B) = P(A).P(B)$ or $P(A \cap B) = P(A) + P(B)$.

Answering Tip

- Always remember in case of independent events $P(A \cap B) = P(A).P(B)$ and $P(A \cup B) = P(A) + P(B)$.

AI Q. 3. Two balls are drawn from an urn containing 3 white, 5 red and 2 black balls, one by one without replacement. What is the probability that at least one ball is red? **[ISC, 2019]**

Sol. $3W$, $5R$, $2B$ Event A: not drawing a red ball in 1st instance.

Event B: not drawing a red ball in 2nd instance.

$P(A) : \frac{5}{10}$, or $P(B) = \frac{4}{9}$

Probability of not drawing red ball in 1st and 2nd instances:

$$= P(A).P(B) = \frac{5}{10}.\frac{4}{9}$$

$\therefore$ Probability of drawing at least one red ball is

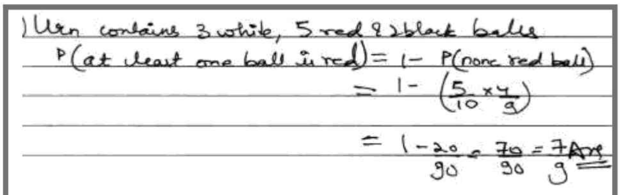
$$= 1 - P(A).P(B)$$

$$= 1 - \frac{5}{10}.\frac{4}{9}$$

$$= 1 - \frac{2}{9} = \frac{7}{9}$$

[ISC Marking Scheme, 2019]

OR



) Urn contains 3 white, 5 red & 2 black balls

P(at least one ball is red) = 1 − P(none red ball)

$= 1 - \left(\frac{5}{10} \times \frac{4}{9}\right)$

$= 1 - \frac{20}{90} = \frac{70}{90} = \frac{7}{9}$ Ans

[Topper's Answer, 2019]

Examiner's Comment

- Most of the candidates could find the probability of both red balls but failed to find the probability of one red ball and another non-red one.

Answering Tip

- Do enough practice of based on the probability concept when more than one ball is drawn at a time/one by one, with or without replacement.

Q. 4. If events A and B are independent, such that $P(A) = \frac{3}{5}, P(B) = \frac{2}{3}$, find $P(A \cup B)$. **[ISC, 2019]**

Sol. $P(A \cup B) = P(A) + (B) - P(A \cap B)$

$P(A \cap B) = P(A).P(B)$

$= \frac{3}{5} \cdot \frac{2}{3} = \frac{2}{5}$

$P(A \cup B) = \frac{3}{5} + \frac{2}{3} - \frac{2}{5} = \frac{13}{15}$

[ISC Marking Scheme, 2019]

Examiner's Comment

- Several candidates failed to find the probability of mutually exclusive events and independent events.

Answering Tip

- Emphasise on mutually exclusive events and independent events. A good number of problems need to be practiced by the students in order to get complete clarity on mutually exclusive events and independent events.

AI Q. 5. If A and B are events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$, then find :

(i) $P(A/B)$ **(ii) $P(B/A)$** **[ISC, 2018]**

Sol. Given, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$

(i) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{4}}{\frac{1}{3}} = \dfrac{3}{4}$

(ii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \dfrac{2}{4} = \dfrac{1}{2}$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Most of the candidates attempted this question correctly but a few got confused while presenting the solution and result in proper.

Answering Tip

- Understand the concept of probability by doing plenty of examples.

Q. 6. In a race, the probabilities of A and B winning the race are $\frac{1}{3}$ and $\frac{1}{6}$ respectively. Find the probability of neither of them winning the race. **[ISC, 2018]**

Sol. Given, $P(A) = \frac{1}{3} \Rightarrow P(\overline{A}) = 1 - \frac{1}{3} = \frac{2}{3}$

and $P(B) = \frac{1}{6} \Rightarrow P(\overline{B}) = 1 - \frac{1}{6} = \frac{5}{6}$

Probability to neither of them winning the race

$= P(\overline{A}).P(\overline{B})$

$= \frac{2}{3} \cdot \frac{5}{6} = \frac{5}{9}$

[ISC Marking Scheme, 2018]

Examiner's Comment

- A large number of candidates could not find the probability of neither of them winning the race correctly.

Answering Tip

- Concept of an event, complement of an event and the idea of finding the probability of neither the events occurring at a times are need to be understand by practicing plenty of questions.

Q. 7 A family has two children. What is the probability that both children are boys, given that at least one of them is a boy ? **[ISC Specimen Paper, 2018]**

Sol. Let, $S = \{(b, b), (b, g), (g, b), (g, g)\}$

$n(s) = 4$

Let E : Event that atleast one is a boy

$\therefore$ $E = \{(b, b), (b, g), (g, b)\} \Rightarrow n(E) = 3$

Let F : Event that both are boys

Let $F = \{(b, b)\} \Rightarrow n(F) = 1$

$\therefore P\left(\frac{F}{E}\right) = \frac{P(F \cap E)}{P(E)} = \frac{1}{3}$

AI Q. 8. Given that events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = k$. Find k if :

(i) A and B are mutually exclusive.

(ii) A and B are independent.

[ISC Specimen Paper, 2018]

Sol. Given, $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = k$

(i) A and B are mutually exclusive

$$\Rightarrow \quad A \cap B = \phi$$

$$\Rightarrow \quad P(A \cap B) = 0$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B)$$

$$\frac{3}{5} = \frac{1}{2} + k$$

$$\Rightarrow \quad = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$$

(ii) A and B are independent

$$\Rightarrow \quad P(A \cap B) = P(A).P(B)$$

$$= \frac{1}{2} \times k = \frac{k}{2}$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{3}{5} = \frac{1}{2} + k - \frac{k}{2}$$

$$\Rightarrow \quad \frac{3}{5} - \frac{1}{2} = \frac{k}{2} \Rightarrow \frac{k}{2} = \frac{1}{10}$$

$$\Rightarrow \quad k = \frac{2}{10} = \frac{1}{5}.$$

Q. 9. A pair of dice is thrown. What is the probability of getting an even number on the first die or a total of 8.

[ISC Specimen Paper, 2016]

Sol. Sample space for a pair of dice is given b y

$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

Let E be the event of getting an even number on the first die or a total of 8

$\therefore E = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6), (3, 5), (5, 3)\}$

Therefore, $n(S) = 36$ and $n(E) = 20$

$$\therefore \text{ Required Probability, } P(E) = \frac{n(E)}{n(S)} = \frac{20}{36} = \frac{5}{9}$$

[ISC Marking Scheme, 2016]

Detailed Solution :

We know that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ...(i)

Here, $P(A)$ is the probability of getting an even number on first die. *i.e.,*

$$P(A) = \frac{18}{36}$$

$P(B)$ is the probability of getting a total of 8 *i.e.,*

$$P(B) = \frac{5}{36}$$

$P(A \cap B)$ is the probability of getting even no. on first die and a total of 8 *i.e.,*

$$P(A \cap B) = \frac{3}{36}$$

$\therefore$ From (i), Required probability,

$$P(A \cup B) = \frac{18}{36} + \frac{5}{36} - \frac{1}{12}$$

$$= \frac{20}{36}.$$

Examiner's Comment

- A number of candidates made mistakes in getting all formable cases. Several candidates had difficulty in calculating the values of $P(A)$, $P(B)$ and $P(A \cap B)$

Answering Tip

- Emphasis should be given on mutually exclusive events and independent events. Plenty of practice is required to understand the application of
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Q. 10. A bag contains 20 balls numbered from 1 to 20. One ball is drawn at random from the bag. What is the probability that the ball drawn is marked with a number which is multiple of 3 or 4 ?

[ISC, 2014]

Sol. P(No. of multiple of 3 or 4)

$$= \frac{6}{20} + \frac{5}{20} - \frac{1}{20}$$

$$= \frac{10}{20} = \frac{1}{2}$$

[ISC Marking Scheme, 2014]

Detailed Answer :

Let Event A be the ball drawn is marked with a number which is multiple of 3.

i.e., $A = \{3, 6, 9, 12, 15, 18\}$ $\therefore n(A) = 6$

Let event B be the ball drawn is marked with a number which is multiple of 4.

i.e., $B = \{4, 8, 12, 16, 20\}$ $\therefore n(B) = 5$

$\therefore$ A ball drawn is marked with a number which is multiple of 3 and 4 both

i.e., $A \cap B = \{12\}$ $\therefore n(A \cap B) = 1$

Now, $$P(A) = \frac{n(A)}{n(S)} = \frac{6}{20}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{20}$$

and $$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{20}$$

(Here, $n(S) = 20$)

Required Probability,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{20} + \frac{5}{20} - \frac{1}{20} = \frac{10}{20} = \frac{1}{2}$$

AI Q. 11. Kamal and Monika appear for an interview for two vacancies. The probability of Kamal's selection is $\frac{1}{3}$ and that of Monika's selection is $\frac{1}{5}$. Find the probability that only one of them will be selected.

[ISC, 2007]

Sol. Let K : Kamal's selection for an interview

and M : Monika's selection for an interview

Given : $P(K) = \frac{1}{3}$ and $P(M) = \frac{1}{5}$

Now, $P(\bar{K}) = 1 - \frac{1}{3} = \frac{2}{3}$ and $P(\bar{M}) = 1 - \frac{1}{5} = \frac{4}{5}$

P(only one will be selected)

$$= P(K)\cdot P(\bar{M}) + P(M)\cdot P(\bar{K})$$

($\because$ K and M are independent events)

$$= \frac{1}{3}\cdot\frac{4}{5} + \frac{1}{5}\cdot\frac{2}{3} = \frac{4}{15} + \frac{2}{15} = \frac{6}{15} = \frac{2}{5}$$

4 Marks Questions

AI Q. 1. Bag A contains 4 white balls and 3 black balls, while Bag B contains 3 white balls and 5 black balls. Two balls are drawn from Bag A and placed in Bag B. Then, what is the probability of drawing a white ball from Bag B ?

[ISC, 2019] [ISC Specimen Paper, 2021]

Sol. Case 1

P(2 WB are transferred from Bag A to Bag B then 1 WB is drawn from Bag B)

$$= \frac{^4C_2}{^7C_2} \times \frac{^5C_1}{^{10}C_1} = \frac{10}{70}$$

Case 2

P(2 BB are transferred from Bag A to Bag B, then 1 WB is drawn from Bag B)

$$= \frac{^3C_2}{^7C_2} \times \frac{^3C_1}{^{10}C_1} = \frac{3}{70}$$

Case 3

P(1 WB an 1 BB are transferred from Bag A to the Bag B and 1 WB is drawn from Bag B)

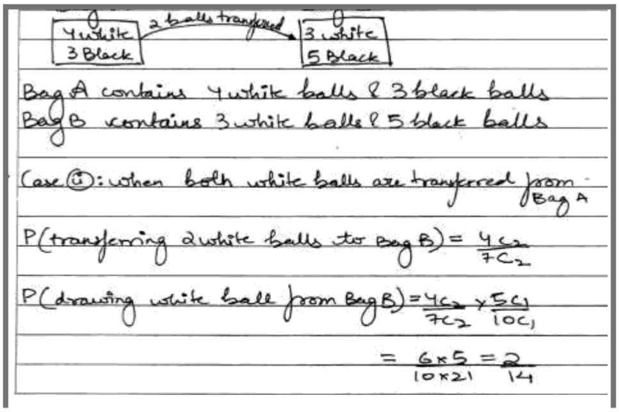
$$= \frac{^4C_1 \times {}^3C_1}{^7C_2} \times \frac{^4C_1}{^{10}C_1} = \frac{16}{70}$$

$\therefore$ P(Choosing WB from Bag B)

$$= \frac{10}{70} + \frac{3}{70} + \frac{16}{70} = \frac{29}{70}$$

[ISC Marking Scheme, 2019]

OR



4 white
3 Black

2 balls tranfered

3 white
5 Black

Bag A contains 4 white balls & 3 black balls

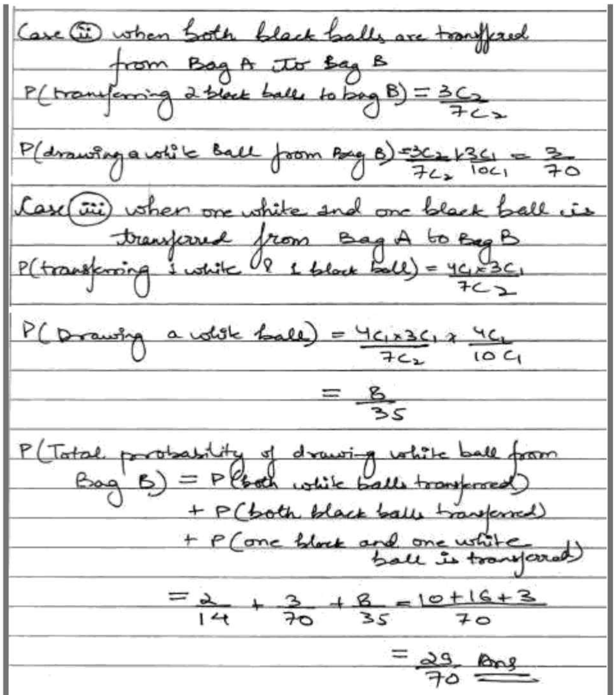
Bag B contains 3 white balls & 5 black balls

Case (i): when both white balls are tranferred from Bag A

P(transferring 2 white balls to Bag B) = $\frac{4C_2}{7C_2}$

P(drawing white ball from Bag B) = $\frac{4C_2}{7C_2} \times \frac{5C_1}{10C_1}$

$= \frac{6\times5}{10\times21} = \frac{2}{14}$



Case (ii) when both black balls are transffered from Bag A to Bag B

P(transferring 2 black balls to bag B) = $\frac{3C_2}{7C_2}$

P(drawing a white Ball from Bag B) = $\frac{3C_2}{7C_2} \times \frac{3C_1}{10C_1} = \frac{3}{70}$

Case (iii) when one white and one black ball is transferred from Bag A to Bag B

P(transferring 1 white & 1 black ball) = $\frac{4C_1 \times 3C_1}{7C_2}$

P(Drawing a white ball) = $\frac{4C_1 \times 3C_1}{7C_2} \times \frac{4C_1}{10C_1}$

$= \frac{8}{35}$

P(Total probability of drawing white ball from Bag B) = P(both white balls transferred)
+ P(both black balls transferred)
+ P(one black and one white ball is transfered)

$= \frac{2}{14} + \frac{3}{70} + \frac{8}{35} = \frac{10+16+3}{70}$

$= \frac{29}{70}$ Ans

[Topper's Answer, 2019]

Examiner's Comment

- A large number of candidates could not identify all possible cases while transferring two balls from Bag A to Bag B and drawing one white ball from Bag B. Candidates also made errors while writing the summation of product of probabilities of all the cases.

Answering Tip

- Understand concepts related to the theory of probability and practice problems on reasoning out all possible cases in a given random experiment.

Q. 2. Three persons A, B and C shoot to hit a target. Their probabilities of hitting the target are $\frac{5}{6}$, $\frac{4}{5}$ and $\frac{3}{4}$ respectively. Find the probability that :

(i) Exactly two persons hit the target.

(ii) At least one person hits the target.

[ISC, 2020]

Sol. Probability of hitting the target

$P(A) = 5/6, P(B) = 4/5, P(C) = 3/4$ (Given)

(i) Exactly two persons hit the target

$P(A) \times P(B) \times P(\overline{C}) + P(A) \times P(\overline{B}) \times P(C)$
$+ \mathrm{P}(\overline{\mathrm{A}}) \times \mathrm{P}(\mathrm{B}) \times \mathrm{P}(\mathrm{C})$

$$= \frac{5}{6} \times \frac{4}{5} \times \left(1 - \frac{3}{4}\right) + \frac{5}{6} \times \left(1 - \frac{4}{5}\right)\left(\frac{3}{4}\right) + \left(1 - \frac{5}{6}\right) \times \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)$$

$$= \frac{5}{6} \times \frac{4}{5} \times \frac{1}{4} + \frac{5}{6} \times \frac{1}{5} \times \frac{3}{4} + \frac{1}{6} \times \frac{4}{5} \times \frac{3}{4}$$

$$= \frac{20}{120} + \frac{15}{120} + \frac{12}{120}$$

$$= \frac{47}{120} = 0.3917 \text{ (Approx.)}$$

$\therefore$ Probability of exactly two persons hit the target $= \frac{47}{120}$

(ii) At least one person hits the target

$$= 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C})$$

$$= 1 - \left(1 - \frac{5}{6}\right)\left(1 - \frac{4}{5}\right)\left(1 - \frac{3}{4}\right)$$

$$= 1 - \left(\frac{1}{6}\right)\left(\frac{1}{5}\right)\left(\frac{1}{4}\right)$$

$$= 1 - \frac{1}{120} = \frac{119}{120}$$

$\therefore$ Probability of at least one person hits the target

$$= \frac{119}{120}$$

$= 0.9917$ Approx. **[ISC Marking Scheme, 2020]**

Examiner's Comment

- In some cases, the probability of hitting the target were not found correctly. In part (ii), several candidates could not understand the meaning of 'at least' while some did not apply 'AND' theorem.

Answering Tip

- Law of probability should be practised with plenty of examples and illustrations.
- Terms such as 'at least','at most','exact','none' should be understood and problems based on them practised.

Q. 3. ***A* speaks truth in 60% of the cases, while *B* in 40% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact ?** **[ISC Specimen Paper, 2018]**

Sol. Let *A* speaks truth is event *A* and *B* speaks truth is event *B*.

Given, $P(A) = 60\% = \frac{60}{100}$

and $P(B) = 40\% = \frac{40}{100}$

$\therefore$ $P(\overline{A}) = 1 - \frac{60}{100} = \frac{40}{100}$

and $P(\overline{B}) = 1 - \frac{40}{100} = \frac{60}{100}$

Now, required probability

$$= P(A).P(\overline{B}) + P(\overline{A}).P(B)$$

$$= \frac{60}{100} \times \frac{60}{100} + \frac{40}{100} \times \frac{40}{100}$$

$$= \frac{3600}{10000} + \frac{1600}{10000}$$

$$= \frac{36}{100} + \frac{16}{100}$$

$$= \frac{52}{100}$$

Hence, required probability in percentage

$= \frac{52}{100} \times 100 = 52\%$ **[ISC Marking Scheme, 2018]**

Examiner's Comment

- Most of the candidates attempted this question correctly but left the result as a fraction / decimal fraction. A few candidates made error while applying the concept of $p.q_1 + q.p_1$.

Answering Tip

- Students are advised to read the question paper and each question carefully and note what is to be answered.

AI **Q. 4. A problem is given to three students whose chances of solving it are $\frac{1}{4}, \frac{1}{5}$ and $\frac{1}{3}$ respectively. Find the probability that the problem is solved.** **[ISC, 2017]**

Sol. Let *A* : Problem is solved by student *A*
B : Problem is solved by student *B*
C : Problem is solved by student *C*

$\therefore$ $P(A) = \frac{1}{4}$, $\mathrm{P(B)} = \frac{1}{5}$ and $P(C) = \frac{1}{3}$

$\therefore$ $P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(\overline{B}) = 1 - P(B) = 1 - \frac{1}{5} = \frac{4}{5}$

and $P(\overline{C}) = 1 - P(C) = 1 - \frac{1}{3} = \frac{2}{3}$

Probability of none of them solve the problem

$$= P(\overline{A}).P(\overline{B}).P(\overline{C})$$

$$= \frac{3}{4} \times \frac{4}{5} \times \frac{2}{3} = \frac{2}{5}$$

$\therefore$ Probability that the problem is being solved = 1 – Probability of none of them solve the problem

$$= 1-\frac{2}{5}=\frac{3}{5}$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Most candidates calculated complement of given probabilities but could not complete the solution due to lack of understanding of the concept of probability.

Answering Tip

- Do plenty of practice to understand the application of $P(A \cup B \cup C) = 1 - P(\bar{A}).P(\bar{B}).P(\bar{C})$.

Q. 5. A card is drawn from a well shuffled pack of playing cards. What is the probability that it is either a spade or an ace or both ? [ISC, 2015]

Sol. $P(E)= \frac{13}{52}+\frac{4}{52}-\frac{1}{52}=\frac{16}{52}=\frac{4}{13}$

[ISC Marking Scheme, 2015]

Detailed Answer :

Let A be the event of getting a spade

B be the event of getting an ace

Here, $n(S) = 52$, $n(A) = 13$, $n(B) = 4$, $n(A \cap B) = 1$

$\therefore$ Probability of getting a spade card,

$$P(A) = \frac{n(A)}{n(S)} = \frac{13}{52}$$

Probability of getting an ace card,

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{52}$$

Probability of getting a spade ace,

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{52}$$

Using formula, $P(A \cup B) = P(A) + PB) - P(A \cap B)$ we get

Required Probability

$$P(A \cup B) = \frac{13}{52}+\frac{4}{52}-\frac{1}{52} = \frac{16}{52}=\frac{4}{13}.$$

Examiner's Comment

- Many candidates found this topic difficult. Some did not understand the meaning of 'either or' term in the question.

Answering Tip

- Theorem 'either or' and theorem 'and' need to be understand by the students. Number of outcomes and number of favourable outcomes need to be found carefully.

Q. 6. A bag contains 8 red and 5 white balls. Two successive draws of 3 balls are made at random from the bag without replacements. Find the probability that the first draw yields 3 white balls and the second draw 3 red balls. [ISC, 2012]

Sol. Let E : Event that 3 balls in the first draw are all white.

F: Event that 3 balls in the second draw are all red.

3 balls can be drawn out of 13 in ${}^{13}C_3$ ways and 3 white balls can be drawn out of 5 in ${}^{5}C_3$ ways.

$$P(E) = \frac{{}^{5}C_3}{{}^{13}C_3} = \frac{5!}{3!2!}\times\frac{3!10!}{13!} = \frac{5}{143}$$

Since, 3 balls are not replaced before the second draw, we are left with 8 red and 2 white balls.

Now, 3 balls can be drawn in ${}^{10}C_3$ ways and 3 red balls can be drawn in ${}^{8}C_3$ ways.

$$P(F/E) = \frac{{}^{8}C_3}{{}^{10}C_3} = \frac{8!}{3!5!}\times\frac{3!7!}{10!} = \frac{7}{15}$$

$$\therefore \ P(E \cap F) = P(E)\cdot P(F/E) = \frac{5}{143}\cdot\frac{7}{15} = \frac{7}{429}$$

AI Q. 7. Two horses are considered for a race. The probability of selection of the first horse is $\frac{1}{4}$ and that of the second is $\frac{1}{3}$. What is the probability that :

(i) both of them will be selected.

(ii) only one of them will be selected.

(iii) none of them will be selected. [ISC, 2009]

Sol. Given, $P(A) = P(\text{I horse selected}) = \frac{1}{4}$

$$P(B) = P(\text{II horse selected}) = \frac{1}{3}$$

$$P(\bar{A}) = 1 - P(A) = 1-\frac{1}{4}=\frac{3}{4}$$

and $$P(\bar{B}) = 1 - P(B) = 1-\frac{1}{3}=\frac{2}{3}$$

(i) $P(\text{Both of them selected}) = P(AB) = P(A)P(B)$

($\because$ A and B are independent events)

$$=\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$$

(ii) $P(\text{only one selected}) = P(A\bar{B}) + P(\bar{A}B)$

$$= P(A)P(\bar{B}) + P(\bar{A})P(B)$$

$$=\frac{1}{4}\times\frac{2}{3}+\frac{3}{4}\times\frac{1}{3}=\frac{5}{12}$$

($\because$ A and B are independent events $\Rightarrow A$ and $\bar{B}, \bar{A}$ and B are independent).

(iii) $P(\text{None of them selected}) = P(\bar{A}\bar{B}) = P(\bar{A})P(\bar{B})$

$$=\frac{3}{4}\times\frac{2}{3}=\frac{1}{2}$$

Q. 8. ***A* and *B* throw two dice each. If *A* gets a sum of 9 on his two dice, then find the probability of *B* getting a higher sum.**

Sol. *A* and *B* throw two dice each.

Sample space for two dice, $n(S) = 36$

Let *A* getting sum 9 on throwing a pair of dice,

$$A = \{(3, 6), (6, 3), (4, 5), (5, 4)\}$$

$$P(A) = \frac{4}{36} = \frac{1}{9}$$

B getting sum of more than 9 on throwing a pair of dice

$$B = \{(4, 6), (6, 4), (5, 5), (6, 5), (5, 6), (6, 6)\}$$

$$P(B) = \frac{6}{36} = \frac{1}{6}$$

Required probability (when *A* gets sum 9 and *B* gets the sum higher than 9)

$$= P(A)P(B) = \frac{1}{9} \times \frac{1}{6} = \frac{1}{54} \quad \text{(independent events)}$$

AI Q. 9. A candidate is selected for interview of management trainees in 3 companies. For the first company, there are 12 candidates, for the second there are 15 candidates and for the third, there are 10 candidates. Find the probability that he is selected by at least one of the companies.

Sol. Let C_1, C_2 and C_3 be the companies.

$P(C_1)$ = Probability of selection in $C_1 = \frac{1}{12}$

$P(C_2)$ = Probability of selection in $C_2 = \frac{1}{15}$

$P(C_3)$ = Probability of selection in $C_3 = \frac{1}{10}$

Probability (no selection in any company)

$$P(\overline{C_1C_2C_3}) = P(\bar{C}_1)P(\bar{C}_2)P(\bar{C}_3)$$

(C_1, C_2 and C_3 are independent events)

$$= \left(1 - \frac{1}{12}\right) \times \left(1 - \frac{1}{15}\right) \times \left(1 - \frac{1}{10}\right)$$

$$= \frac{11}{12} \times \frac{14}{15} \times \frac{9}{10} = \frac{231}{300} = \frac{77}{100}$$

Probability (selection in atleast one of the companies)

= 1 – Probability (no selection in any company)

$$= 1 - \frac{77}{100} = 1 - 0.77 = 0.23$$

Q. 10. ***P* speaks truth in 70% of the cases and *Q* in 80% of the cases. In what percent of cases are they likely to agree in stating the same fact?**

Sol. Let *E* and *F* denote the events '*P* speaks the truth' and '*Q* speaks the truth', respectively.

We have,

$$P(E) = \frac{70}{100}$$

$$P(\bar{E}) = 1 - P(E)$$

$$= 1 - \frac{70}{100} = \frac{30}{100}$$

and $$P(F) = \frac{80}{100}$$

$$P(\bar{F}) = 1 - P(F)$$

$$= 1 - \frac{80}{100} = \frac{20}{100}$$

So, required probability

$$= P(E)P(F) + P(\bar{E})P(\bar{F})$$

$$= \left(\frac{70}{100}\right)\left(\frac{80}{100}\right) + \left(\frac{30}{100}\right)\left(\frac{20}{100}\right)$$

$$= \frac{56}{100} + \frac{6}{100} = \frac{62}{100} = 62\%$$

Hence, they are likely to agree in 62% of the cases in stating the same fact.

Q. 11. The bag '*A*' contains 3 white and 2 black balls while the bag '*B*' contains 2 white and 5 black balls. One of the bags is chosen at random and a ball is drawn from it. What is the probability that the ball is white.

Sol. Probability (selecting bag *A*)

= Probability (selecting bag *B*)

$$= \frac{1}{2}$$

i.e., $P(A) = P(B) = \frac{1}{2}$

Let *W* : Drawing a white ball from any bag

Required probability

= Probability of selecting bag *A* × Probability of selecting a white ball from bag *A*

+ Probability of selecting bag *B* × Probability of selecting a white ball from bag *B*

= $P(A) \times P(W/A) + P(B) \times P(W/B)$

$$= \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{7} = \frac{3}{10} + \frac{1}{7}$$

$$= \frac{21 + 10}{70} = \frac{31}{70}$$

6 Marks Questions

Q. 1. ***A, B* and *C* throw a die one after the other in the same order till one of them gets a '6' and wins the game. Find their respective probability of winning, if *A* starts the game.**

[ISC Specimen Paper 2018]

Sol. Probability of winning (p) = $\frac{1}{6}$

Probability of losing (q) = $\frac{5}{6}$

A can win on 1st, 4th, 7th, ... trials

$$\therefore \quad P(A \text{ wins}) = p + qqqp + qqqqqqp + \ldots$$

$$= p + q^3p + q^6p + \ldots$$

$$= \frac{p}{1-q^3} = \frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^3}$$

$$= \frac{\frac{1}{6}}{6^3-5^3} \times 6^3$$

$$= \frac{6^2}{6^3-5^3} = \frac{36}{216-125}$$

$$= \frac{36}{91}$$

B can win on 2nd, 5th, 8th, ... trials.

$$\therefore \quad P(B \text{ wins}) = qp + qqqqp + qqqqqqqp + \ldots$$

$$= qp + q^4p + q^7p + \ldots$$

$$= \frac{qp}{1-q^3} = \frac{\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)}{1-\left(\frac{5}{6}\right)^3}$$

$$P(B) = \frac{5}{6^2} \times \frac{6^3}{91}$$

$$= \frac{30}{91}$$

Probability of C winning $= 1 - [P(A) + P(B)]$

$$= 1 - \left[\frac{36}{91} + \frac{30}{91}\right]$$

$$= 1 - \frac{66}{91}$$

$$P(C) = \frac{25}{91}.$$

AI Q. 2. In a class of 60 students, 30 opted for Mathematics, 32 opted fro Biology and 24 opted for both Mathematics and Biology. If one of these students is selected at random, find the probability that :

(i) The student opted for Mathematics or Biology.

(ii) The student has opted neither Mathematics nor Biology.

(iii) The student has opted mathematics but not Biology. [ISC, 2017]

Sol. Let A : event that students opted Mathematics

B : event that students opted Biology

$$\therefore P(A) = \frac{30}{60},\ P(B) = \frac{32}{60},\ P(A \cap B) = \frac{24}{60}$$

(i)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{30}{60} + \frac{32}{60} - \frac{24}{60}$$

$$= \frac{38}{60} = \frac{19}{30}$$

Hence, probability that student opted Mathematics or Biology is $\frac{19}{30}$.

(ii)

$$P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$$

$$= 1 - \frac{19}{30} = \frac{11}{30}$$

Hence, probability that student has neither opted Mathematics nor Biology is $\frac{11}{30}$.

(iii) Probability of student has opted Mathematics but not Biology $= P(A) - P(A \cap B)$

$$= \frac{30}{60} - \frac{24}{60} = \frac{6}{60} = \frac{1}{10}$$

[ISC Marking Scheme, 2017]

Q. 3. Bag A contain 1 white, 2 blue and 3 red balls. Bag B contains 3 white, 3 blue and 2 red balls. Bag C contains 2 white, 3 blue and 4 red balls. One bag is selected at random and then two balls are drawn from the selected bag. Find the probability that the balls drawn are white and red. [ISC, 2017]

Sol. Probability of selecting each bag $= \frac{1}{3}$

	A	B	C
White	1	3	2
Blue	2	3	3
Red	3	2	4
Prob of (1 W & 1R)	$\frac{1 \times {}^3C_1}{{}^6C_2}$	$\frac{{}^3C_1 \times {}^2C_1}{{}^8C_2}$	$\frac{{}^2C_1 \times {}^4C_1}{{}^9C_2}$

Required probability :

$$\frac{1}{3}\left[\frac{1 \times {}^3C_1}{{}^6C_2} + \frac{{}^3C_1 \times {}^2C_1}{{}^8C_2} + \frac{{}^2C_1 \times {}^4C_1}{{}^9C_2}\right]$$

$$= \frac{1}{3}\left[\frac{1 \times 3}{15} + \frac{3 \times 2}{28} + \frac{2 \times 4}{36}\right]$$

$$= \frac{1}{3}\left[\frac{1}{5} + \frac{3}{14} + \frac{2}{9}\right] = \frac{401}{1890}$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates did not write this part correctly. They mixed the probabilities of one white and one red. Many candidates wrote the result without multiplying the result with $\frac{1}{3}$.

Answering Tip

- Student should understand the problem and identify the cases that satisfy the situation.

Q. 4. A committee of 4 persons has to be chosen from 8 boys and 6 girls, consisting of at least one girl. Find the probability that the committee consists of more girls than boys. [ISC, 2016]

Sol. Number of ways the committee can be selected

$= {}^{14}C_4 - {}^{8}C_4$

$= 1001 - 70$

$= 931$

or $({}^{6}C_1.{}^{8}C_3 + {}^{6}C_2.{}^{8}C_2 + {}^{6}C_3.{}^{8}C_1) + {}^{6}C_4$

No. of committees consist of more girls than boys

$= {}^{6}C_4 + ({}^{6}C_3 \times {}^{8}C_1)$

$$P(E) = \frac{{}^{6}C_4 + ({}^{6}C_3 \times {}^{8}C_1)}{931}$$

$$= \frac{15+160}{931} = \frac{175}{931}$$

[ISC Marking Scheme, 2016]

Detailed Solution :

The committee of 4 persons can be formed in following ways :

1 Girl and 3 Boys

2 Girls and 2 Boys

3 Girls and 1 Boy

4 Girls and no Boy

No. of ways of selecting 1 girl and 3 boys = ${}^{6}C_1 \times {}^{8}C_3$

No. of ways of selecting 2 girls and 2 boys = ${}^{6}C_2 \times {}^{8}C_2$

No. of ways of selecting 3 girls and 1 boy = ${}^{6}C_3 \times {}^{8}C_1$

No. of ways of selecting 4 girls and no boy = ${}^{6}C_4$

∴ Total no. of ways of selecting 4 person out of 6 girls and 8 boys, consisting at least one girl.

$= ({}^{6}C_1 \times {}^{8}C_3) + ({}^{6}C_2 \times {}^{8}C_2) + ({}^{6}C_3 \times {}^{8}C_1) + {}^{6}C_4$

$= (6 \times 56) + (15 \times 28) + (20 \times 8) + 15 = 931$

No. of committee consists of more girls than boys

$= {}^{6}C_4 \times {}^{6}C_3 \times {}^{8}C_1$

$= 15 + 20 \times 8$

$= 15 + 160 = 175$

∴ Required Probability $= \frac{175}{931}$

Hence, the probability that the committee consists of more girls than boys in 175/931.

Q. 5. An urn contains 10 white and 3 black balls while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the second urn. Find the probability that the ball drawn from the second urn is a white ball. [ISC, 2016]

Sol.

I urn

W	10
B	3

II urn

W	3
B	5

P(transferring 2 white balls to urn II and drawing a white ball from urn II)

$$= \frac{{}^{10}C_2 \times {}^{5}C_1}{{}^{13}C_2 \times {}^{10}C_1} = \frac{45}{13 \times 12}$$

P(transferring 2 black balls to run II and drawing a white ball from urn II)

$$= \frac{{}^{3}C_2 \times {}^{3}C_1}{{}^{13}C_2 \times {}^{10}C_1} = \frac{9}{13 \times 12 \times 5}$$

P(transferring 1 white and 1 black to urn II and drawing a white ball from urn II)

$$= \frac{({}^{10}C_1 \times {}^{3}C_1) \times {}^{4}C_1}{{}^{13}C_2 \times {}^{10}C_1} = \frac{24}{13 \times 12}$$

$$\therefore \text{ Required probability} = \frac{1}{13 \times 12}\left[45 + 24 + \frac{9}{5}\right]$$

$$= \frac{59}{130}.$$

[ISC Marking Scheme, 2016]

Q. 6. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replaced into the urn. Otherwise, it is replaced with another ball of the same colour. The process is repeated. Find the probability that the third ball drawn is black. [ISC, 2015]

Sol. Case I : Here, it is assumed that, the black ball drawn is replaced along with another ball of the same colour.

$P(E) = P(WWB) + P(WBB) + P(BWB) + P(BBB)$

$$= \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2} + \frac{2}{4} \times \frac{2}{3} \times \frac{3}{4} + \frac{2}{4} \times \frac{2}{5} \times \frac{3}{4} + \frac{2}{4} \times \frac{3}{5} \times \frac{4}{6}$$

$$= \frac{1}{6} + \frac{1}{4} + \frac{3}{20} + \frac{1}{5}$$

$$= \frac{10+15+9+12}{60} = \frac{46}{60} = \frac{23}{30}$$

Case II :

Here, it is assumed that, the black ball drawn is with replacement.

Case II :

Here, it is assumed that, the black ball drawn is with replacement

$P(E) = P(WWB) + P(WBB) + P(BWB) + P(BBB)$

$$= \left(\frac{2}{4} \times \frac{1}{3} \times \frac{2}{2}\right) + \left(\frac{2}{4} \times \frac{2}{3} \times \frac{2}{3}\right) + \left(\frac{2}{4} \times \frac{2}{4} \times \frac{2}{3}\right) + \left(\frac{2}{4} \times \frac{2}{4} \times \frac{2}{4}\right)$$

$$= \frac{1}{6} + \frac{2}{9} + \frac{1}{6} + \frac{1}{8} = \frac{49}{72}$$

[ISC Marking Scheme, 2015]

Detailed Solution :

Case I : Assuming that the black ball drawn is replaced along with another ball of the same colour, we solve it as shown :

Here, there are four possibilities,

First Ball	Second Ball	Event
White	White	E_1
White	Black	E_2
Black	White	E_3
Black	Black	E_4

Let E denote the event of drawing a black ball in the third attempt

$P(E/E_1) = 1$, because there is no white ball to be selected.

$P(E/E_2) = 3/4$, because there are 3 black balls and 1 white ball.

$P(E/E_3) = 3/4$, because again there are 3 black balls and 1 white ball.

$P(E/E_4) = (4/6) = (2/3)$, because there are 4 black balls and 2 white balls

$$\therefore \quad P(E) = P(E_1)P(E/E_1) + P(E_2)P(E/E_2) + P(E_3)P(E/E_3) + P(E_4)P(E/E_4)$$

$$= \frac{1}{6}\times 1 + \frac{1}{3}\times\frac{3}{4} + \frac{1}{5}\times\frac{3}{4} + \frac{3}{10}\times\frac{2}{3}$$

$$= \frac{1}{6} + \frac{1}{4} + \frac{3}{20} + \frac{1}{5}$$

$$= \frac{23}{30}$$

Case II :

Here, we assume that the black ball drawn is with replacement.

Let E denote the event of drawing a black ball in the third attempt.

E_1 : getting two white balls in first two attempts.

E_2 : getting first white ball and second black ball.

E_3 : getting first black ball and second white ball.

E_4 : getting two black balls in first two attempts.

$$\therefore \quad P(E_1) = \frac{2}{4}\times\frac{1}{3} = \frac{1}{6}$$

$$P(E_2) = \frac{2}{4}\times\frac{2}{3} = \frac{1}{3}$$

$$P(E_3) = \frac{2}{4}\times\frac{2}{4} = \frac{1}{4}$$

$$P(E_4) = \frac{2}{4}\times\frac{2}{4} = \frac{1}{4}$$

$$\therefore \quad P(E) = P(E_1)P(E/E_1) + P(E_2)P(E/E_2) + P(E_3)P(E/E_3) + P(E_4)P(E/E_4)$$

$$= \frac{1}{6}\times 1 + \frac{1}{3}\times\frac{2}{3} + \frac{1}{4}\times\frac{2}{3} + \frac{1}{4}\times\frac{2}{4}$$

$$= \frac{1}{6} + \frac{2}{9} + \frac{1}{6} + \frac{1}{8} = \frac{49}{72}$$

Examiner's Comment

- Many candidates handled this problem well but some could not understand the underlying principle.

Answering Tip

- Write the correct interpretation of such problems.

AI **Q. 7. A bag contain 5 white and 4 black balls and another bag contains 7 white and 9 black balls. A ball is drawn from the first bag and two balls drawn from the second bag. What is the probability of drawing one white and two black balls ? [ISC 2014]**

Sol. Bag I : 5 white and 4 Black

Bag II : 7 white and 9 Black

Probability of drawing a white ball from Bag I = $\frac{5}{9}$

Probability of drawing a black ball from Bag I = $\frac{4}{9}$

Probability of drawing two back balls from Bag II

$$= \frac{^9C_2}{^{16}C_2}$$

Probability of drawing 1 black and 1 white ball from Bag II = $\frac{^7C_1 \times {^9C_1}}{^{16}C_2}$

$$\text{Required probability} = \frac{5}{9}\times\frac{^9C_2}{^{16}C_2} + \frac{4}{9}\times\frac{^7C_1\times{^9C_1}}{^{16}C_2}$$

$$= \frac{5}{9}\times\frac{9}{2\times 15} + \frac{4}{9}\times\frac{7\times 9}{8\times 15}$$

$$= \frac{1}{6} + \frac{7}{30} = \frac{12}{30} = \frac{2}{5}$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- The cases for withdrawal of balls from the bags in different colour combinations were not correctly identify by candidates. it was noticed that summation and product laws of probability were not understood by many candidates.

Answering Tip

- Separate the problems into different cases and solve them.

Q. 8. In a college, 70% students pass in Physics, 75% pass in Mathematics and 10% students fail in both. One student is chosen at random. What is the probability that :

(i) He passes in Physics and Mathematics.

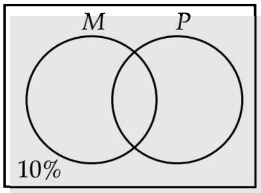
(ii) He passes in Mathematics given that he passes in Physics.

(iii) He passes in Physics given that he passes in Mathematics. [ISC, 2014]

Sol. $P(M) = 0.75$

$P(P) = 0.70$

$P(M \cup P) = 0.90$



(i) $P(M \cup P) = P(M) + P(P) - P(M \cap P)$

$0.90 = 0.75 + 0.70 - P(M \cap P)$

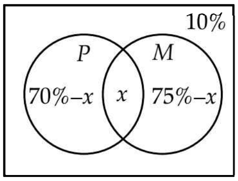
$\Rightarrow P(M \cap P) = 0.55$

(ii) $P(M/P) = \dfrac{P(M \cap P)}{P(P)} = \dfrac{0.55}{0.70} = \dfrac{11}{14}$

(iii) $P(P/M) = \dfrac{P(M \cap P)}{P(M)} = \dfrac{0.55}{0.75} = \dfrac{11}{15}$

[ISC Marking Scheme, 2014]

Detailed Solution :



Let $x\%$ students pass in both Physics and Maths.

Students who pass in Physics = 70%

$$\Rightarrow \qquad P(P) = \frac{70}{100}$$

Students who pass in Maths = 75%

$$\Rightarrow \qquad P(M) = \frac{75}{100}$$

Students who fail in both = 10%

$$70\% - x + x + 75\% - x = 90\%$$

$$x = 55\%$$

$$\Rightarrow \qquad P(M \cap P) = \frac{55}{100}$$

(i) P(Passes in Physics and Mathematics)

$$= \frac{55}{100} = \frac{11}{20}$$

(ii) $$P(M/P) = \frac{P(M \cap P)}{P(P)}$$

$$= \frac{55/100}{70/100} = \frac{55}{70} = \frac{11}{14}$$

(iii) $$P(P/M) = \frac{P(M \cap P)}{P(M)}$$

$$= \frac{55/100}{75/100} = \frac{55}{75} = \frac{11}{15}$$

Examiner's Comment

- Many candidates made error while finding $A \cap B$ and $A \cup B$. Errors were also made in using conditional probability formulae.

Answering Tip

- Probability theory needs to be understood through copious practice. Independent cases or mutually exclusive situations need plenty of illustration. Conditional probability problems needs to be learn with care.

AI **Q. 9. Aman and Bhuvan throw a pair of disc alternately. In order to win, they have to get a sum of 8. Find their respective probabilities of winning if Aman starts the game. [ISC, 2011]**

Sol. Success of Aman and Bhuvan is getting a total of 8 in a pair of disc.

$\therefore \quad S = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

Let p = Probability of getting a total of 8

and q = Probability of not getting a total of 8

$$\therefore \quad p = \frac{5}{36} \text{ and } q = \frac{31}{36}$$

If Aman starts the game, he can win it in the first throw, 3rd throw, 7th throw and so on.

Now, P(Aman's winning in the first throw)

$$= p = \frac{5}{36}$$

P(Aman winning in the third throw)

$$= qqp = \frac{31}{36} \times \frac{31}{36} \times \frac{5}{36} = \left(\frac{31}{36}\right)^2 \times \frac{5}{36}$$

P(Aman winning in 5th throw)

$$= qqqqp = \left(\frac{31}{36}\right)^4 \times \frac{5}{36} \text{ and so on}$$

Since, all cases are mutually exclusive, therefore the probability of Aman winning the game first

$$= p + qqp + qqqqp \ldots \infty$$

$$= p[1 + q^2 + q^4 + \ldots \infty]$$

$$= p\left[\frac{1}{1-q^2}\right] \qquad \left[\text{In a G.P., } S_\infty = \frac{a}{1-r}\right]$$

$$= \frac{5}{36}\left[\frac{1}{1-\left(\frac{31}{36}\right)^2}\right] = \frac{5}{36}\left[\frac{36 \times 36}{36 \times 36 - 31 \times 31}\right]$$

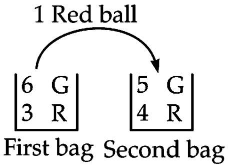
$$= \frac{5}{36} \times \frac{36 \times 36}{(5 \times 67)} = \frac{5 \times 36}{5 \times 67} = \frac{36}{67}$$

Since, Aman and Bhuvan are throwing the pair of dice alternatively, hence the events are mutually exclusive and therefore the sum of their probabilities of winning is 1.

$$\therefore \text{ Probability (Bhuvan's winning) } = 1 - \frac{36}{67} = \frac{31}{67}$$

Q. 10. There are two bags. One bag contains six green and three red balls. The second bag contains five green and four red balls. One ball is transferred from the first bag to the second bag. Then one ball is drawn from the second bag. Find the probability that it is a red ball. [ISC, 2010]

Sol. **Case I :** When a red ball is transferred from first bag to second bag.



Probability of drawing a red ball from first bag,

$$P_1 = \frac{3}{9} = \frac{1}{3}$$

Now, second bag contains 5 green and 5 red balls.

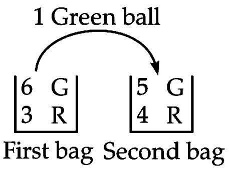
Probability of drawing a green ball from first second bag, $P_2 = \frac{5}{10} = \frac{1}{2}$

$\therefore$ Probability of both events occurring together,

$$P_1P_2 = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

(Multiplication theorem for independent event)

Case II : When a green ball is transferred from first bag to second bag.



Probability of drawing a green ball from first bag,

$$P_3 = \frac{6}{9} = \frac{2}{3}$$

Now, second bag contains 6 green and 4 red balls.

Probability of drawing a red ball from second bag,

$$P_4 = \frac{4}{10} = \frac{2}{5}$$

$\therefore$ Probability of both the events occurring together,

$$P_3P_4 = \frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$$

(Multiplication theorem for independent events)

The required probability

$$= P_1P_2 + P_3P_4$$

$$= \frac{1}{6} + \frac{4}{15} = \frac{5+8}{30} = \frac{13}{30}$$

Q. 11. An article manufactured by a company consists of two parts *A* and *B*. In the process of manufacture of part *A*, 9 out of 104 parts may be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part *B*. Calculate the probability that the article manufactured will not be defective. [ISC, 2009]

Sol. Article will not be defective if both the parts are non defective.

$$P(\text{Part } A \text{ to be defective}) = \frac{9}{104}$$

$$P(\text{Part } A \text{ not defective}) = 1 - \frac{9}{104} = \frac{95}{104}$$

$$P(\text{Part } B \text{ to be defective}) = \frac{5}{100} = \frac{1}{20}$$

$$P(\text{Part } B \text{ not defective}) = 1 - \frac{1}{20} = \frac{19}{20}$$

$$P(\text{None part defective}) = \frac{95}{104} \times \frac{19}{20} = \frac{361}{416}$$

Q. 12. There are 3 urns *A*, *B* and *C*. Urn *A* contains 4 red balls and 3 black balls. Urn *B* contains 5 red balls and 4 black balls. Urn *C* contains 4 red balls and 4 black balls. One ball is drawn from each of these urns. What is the probability that the 3 balls drawn consist of 2 red balls and 1 black ball? [ISC, 2008]

Sol. Given, Urn A : 4 red balls and 3 black balls
Urn B : 5 red balls and 4 black balls
Urn C : 4 red balls and 4 black balls.

The three possible cases are,
$(R, R, B), (B, R, R), (R, B, R)$

$\therefore$ Required probability

$$= P(RRB \text{ or } BRR \text{ or } RBR)$$

$$= P(RRB) + P(BRR) + P(RBR)$$

$$= \left(\frac{4}{7} \times \frac{5}{9} \times \frac{4}{8}\right) + \left(\frac{3}{7} \times \frac{5}{9} \times \frac{4}{8}\right) + \left(\frac{4}{7} \times \frac{4}{9} \times \frac{4}{8}\right)$$

$$= \frac{80 + 60 + 64}{504} = \frac{204}{504} = \frac{17}{42}$$

TOPIC-2
Random Variable and its Probability Distributions, Bernoulli Trials and Binomial Distribution

Revision Notes

- **Random Variable :**
  A random variable is a real valued function defined over the sample space of an experiment. In other words, a random variable is a real-valued function whose domain is the sample space of a random experiment. A random variable is usually denoted by uppercase letter *X*, *Y*, *Z* etc.
- **Probability Distribution of a Random Variable :**
  If the values of a random variable together with the corresponding probabilities are given, then this description is called a probability distribution of the random variable.

Scan to know more about this topic

Random variable and it's probability distributions

- **Bernoulli Trials :**

  Trials of a random experiment are called Bernoulli trials, if they satisfy the following four conditions :

  (a) The trials should be finite in numbers.

  (b) The trials should be independent of each other.

  (c) Each of the trial yields exactly two outcomes, *i.e.*, success or failure.

  (d) The probability of success or the failure remains the same in each of the trial.

  If an experiment is repeated n times under the similar conditions, we says that n trials of the experiment have been made.

Scan to know more about this topic

Bernoulli Trial and Binomial Distribution

- **Binomial Distribution :**

  Let E be an event. Let p = probability of success in one trial (*i.e.*, occurrence of event E in one trial) and, $q = 1 - p$ = probability of failure in one trial (*i.e.*, non-occurrence of event E in one trial).

  Let X = number of successes (*i.e.*, number of times event E occurs in n trials)

  Then, probability of X successes in n trials is given by the relation :

  $$P(X = r) = P(r) = {}^nC_r p^r q^{n-r}$$

  Where $r = 0, 1, 2, 3, \ldots n$; p = probability of success in one trial, $q = 1 - p$ = probability of failure in one trial and

  $${}^nC_r = \frac{n!}{r!(n-2\pi)!}$$

Important Fact

- The result $P(X = r) = P(r) = {}^nC_r p^r q^{n-r}$ can be used only when :
  - (i) the probability of success in each trial is the same.
  - (ii) each trial must surely result in either a success or a failure.

Important Facts

- $P(X = r)$ or $P(r)$ is also called the probability of occurrence of event E exactly r times in n trials.
- ${}^nC_r p^r q^{n-r}$ is the $(r + 1)$th term in binomial expansion of $(q + p)^n$
- A Binomial Distribution with n Bernoulli trials and probability of success in each trial as p is denoted by $B(n, p)$. Here, n and p are known as the parameters of Binomial Distribution.
- The expression $P(X = r)$ or $\boldsymbol{P(r)}$ is called the probability function of Binomial Distribution.

- **Mean, Variance and Standard Deviation of Binomial Distribution :**
  - Mean $= \sum_{r=0}^{n} r.P(r) = np$
  - Variance $= \sum_{r=0}^{n} r^2.P(r) - (\text{Mean})^2 = npq$
  - Standard Deviation (S.D.) $= \sqrt{npq}$

Know the Terms

- **Discrete random variable :** It is a random variable which can take only finite or countable number of values.
- **Continuous random variable :** A variable can take any value between two given limits is called a continuous random variable.

Know the Formulae

- Mean of Expectation of a random variable $X = \mu = \sum_{i=1}^{n} x_i P_i$
- Variance $= (\sigma)^2 = \sum_{i=1}^{n} P_i x_i^2 - \mu^2$
- Standard Deviation $= \sigma = \sqrt{\text{Variance}}$
- Binomial distribution :

  $$P(X = r) = {}^nC_r p^r q^{n-r}$$

  where, $r = 0, 1, 2, \ldots n$, p = success, $q = 1 - p$ = failure, n = no. of trials

1 Mark Questions

Q. 1. For the following probability distribution :

X	1	2	3	4
P(X)	1/10	1/5	3/10	2/5

$E(X^2)$ is equal to

(a) 3 (b) 5 (c) 7 (d) 10

Sol. Correct option : (d)

Explanation :

$$E(X^2)=\sum X^2\ P(X)$$

$$=1.\frac{1}{10}+4.\frac{1}{5}+9.\frac{3}{10}+16.\frac{2}{5}$$

$$=\frac{1}{10}+\frac{4}{5}+\frac{27}{10}+\frac{32}{5}$$

$$=\frac{1+8+27+64}{10}=10$$

Q. 2 For the following probability distribution :

X	−4	−3	−2	−1	0
P(X)	0.1	0.2	0.3	0.2	0.2

***E(X)* is equal to**

(a) 0 (b) −1 (c) −2 (d) −1.8

Sol. Correct option : (d)

Explanation :

$$E(X)=\sum X\ P(X)$$

$$=-4\times(0.1)+(-3\times 0.2)+(-2\times 0.3)+(-1\times 0.2)+(0\times 0.2)$$

$$=-0.4-0.6-0.6-0.2+0=-1.8$$

AI Q. 3 A binomial distribution has mean 5 and variance 4. Find the number of trials.

[ISC Specimen Paper, 2021]

Sol. $npq = 4, np = 5$

$$\therefore \quad \frac{npq}{np} = \frac{4}{5}$$

$$\Rightarrow \quad q = \frac{4}{5}$$

$$\therefore \quad P = 1-q = 1-\frac{4}{5} = \frac{1}{5}$$

$$\therefore \quad n\times\frac{1}{5} = 5$$

$$\Rightarrow \quad n = 25$$

Q. 4. A discrete random variable *X* has the probability distribution given as below :

X	0.5	1	1.5	2
P(X)	k	k^2	$2k^2$	k

Then find the value of *k*.

Sol. We know that, $\sum_{i=1}^{n} P_i = 1$, where $P_i \geq 0$

$$\Rightarrow P_1+P_2+P_3+P_4=1$$

$$\Rightarrow k+k^2+2k^2+k=1$$

$$\Rightarrow 3k^2+2k-1=0$$

$$\Rightarrow (3k-1)(k+1)=0$$

$$\Rightarrow k=\frac{1}{3}\ (\text{as } k \text{ is} \geq 0)$$

Q. 5. Three balls are drawn from a bag containing 2 red and 5 black balls, if the random variable *x* represents the number of red balls drawn, then how many value the *x* can take.

Sol. As there are two red balls, the maximum number of red balls can be 2.

2 Marks Questions

Q 1. Find the probability distribution of *X*, the number of heads in a simultaneous toss of two coins.

Sol. Let *X* be the number of heads.

Possible values of X are 0, 1, 2

$P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{4}$. The probability distribution of *X* is :

X	0	1	2
P(X)	1/4	1/2	1/4

AI Q. 2. Find the mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

(a) 1 (b) 2

(c) 5 (d) 8/3

Sol. Let *X* be the random variable representing a number on the die.

The total number of observations is 6. Therefore,

$$P(X=1)=\frac{3}{6}=\frac{1}{2}$$

$$P(X=2)=\frac{2}{6}=\frac{1}{3}$$

$$P(X=5)=\frac{1}{6}$$

Therefore, the probability distribution is as follows.

X	1	2	5
P(X)	1/2	1/3	1/6

$$\text{Mean} = E(X)$$

$$=\sum p_i x_i$$

$$=\frac{1}{2}\times 1+\frac{1}{3}\times 2+\frac{1}{6}\times 5$$

$$=\frac{1}{2}+\frac{2}{3}+\frac{5}{6}$$

$$=\frac{3+4+5}{6}=\frac{12}{6}=2$$

4 Marks Questions

Q. 1. Determine the binomial distribution where mean is 9 and standard deviation is $\frac{3}{2}$. Also, find the probability of obtaining at most one success.

[ISC Specimen Paper, 2021] [ISC, 2019]

Sol. Given that :

$$np = 9\,;\ \sqrt{npq} = \frac{3}{2}$$

$$\therefore \quad npq = \frac{9}{4}$$

$$\therefore \quad 9q = \frac{9}{4}$$

$$\therefore \quad q = \frac{1}{4}$$

$$\therefore \quad \text{P} = \frac{3}{4}$$

Now, $n \times \frac{3}{4} = 9$

$$\Rightarrow \quad n = 12$$

$\therefore$ Binomial distribution is $\left(\frac{1}{4}+\frac{3}{4}\right)^{1/12}$

Now, Let X be the random variable which counts the no. of successes

P(at most one success) $= P(X \leq 1)$

$$= P(X = 0) + P(X = 1)$$

$$= {}^{12}C_0\left(\frac{1}{4}\right)^{12} + {}^{12}C_1\left(\frac{1}{4}\right)^{11}\frac{3}{4}$$

$$= 0.000002$$

[ISC Marking Scheme, 2019]

Examiner's Comment

- Many candidates used incorrect formula for Standard Deviation and calculated incorrect values of n, p, q. Some candidates could not understand the meaning of at most one success.

Answering Tip

- Understand the concept and all the formulae of Binomial Probability and representing as $B(n,p)$ of $(q+p)n$. Do adequate practice of various type of problems.

Q. 2. Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the value 'x' has the following form, where 'k' is some unknown constant.

$$p\,(X = x) = \begin{cases} 0.1, & \text{if } x = 0 \\ kx, & \text{if } x = 1 \text{ or } 2 \\ k(5-x), & \text{if } x = 3 \text{ or } 4 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find is value of 'k'.

(ii) What is the probability that you study :

(a) atleast two hours ?

(b) exactly two hours ?

(c) at most 2 hours ? [ISC Specimen Paper, 2018]

Sol.

$$P(X = x) = \begin{cases} 0.1, & \text{if } x = 0 \\ kx, & \text{if } x = 1 \text{ or } 2 \\ k(5-x), & \text{if } x = 3 \text{ or } 4 \\ 0, & \text{otherwise} \end{cases}$$

Given

x	0	1	2	3	4
$P(X = x)$	0.1	k	$2k$	$2k$	k

(i) $\Sigma P(x) = 1$

$\Rightarrow \quad 0.1 + k + 2k + 2k + k = 1$

$\Rightarrow \quad 6k = 0.9$

$\Rightarrow \quad k = \frac{0.9}{6} = 0.15$

(ii) (a) P(at least two hours) :

$$P(x \geq 2) = P(x = 2) + P(x = 3) + P(x = 4)$$

$$\Rightarrow \quad P(x \geq 2) = 2k + 2k + k$$

$$= 5k = 5 \times 0.15$$

$$= 0.75$$

(b) P(exactly two hours)

$$= P(x = 2)$$

$$= 2k = 2 \times 0.15$$

$$= 0.3$$

(c) P(atmost two hours) $= P(x \leq 2)$

$$= P(x = 0) + P(x = 1) + P(x = 2)$$

$$= 0.1 + k + 2k$$

$$= 0.1 + (3 \times 0.15)$$

$$= 0.1 + 0.45$$

$$= 0.55$$

AI Q. 3. Five bad eggs are mixed with 10 good ones. If 3 eggs are drawn one by one with replacement, then find the probability distribution of the number of good eggs drawn. [ISC, 2010]

Sol. Given that, total eggs = 15

$$p = \frac{10}{15} = \frac{2}{3} \quad \text{(10 good eggs)}$$

$$q = \frac{5}{15} = \frac{1}{3}, n = 3 \quad \text{(5 bad eggs)}$$

Let X denotes the number of good eggs in 3 draws. Then, X can take values 0, 1, 2 and 3.

$$\therefore \quad P(X = 0) = {}^3C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$P(X = 1) = {}^3C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 = 3 \times \frac{2}{3} \times \frac{1}{9} = \frac{2}{9}$$

$$P(X=2)={}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1=3\times\frac{4}{9}\times\frac{1}{3}=\frac{4}{9}$$

$$P(X=3)={}^3C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0=\frac{8}{27}$$

Hence, probability distribution of X is :

X	0	1	2	3
***P*(*X*)**	1/27	2/9	4/9	8/27

Q. 4. The mean and variance of a binomial distribution are 4 and 2 respectively. Find the probability of at least 6 successes. [ISC, 2009]

Sol. Given, mean = np = 4 and variance = npq = 2

$\therefore \quad \frac{npq}{np}=\frac{2}{4}$

$\Rightarrow \quad q=\frac{1}{2}$

and $\quad p=1-q$

$\quad =1-\frac{1}{2}=\frac{1}{2}$

Since, $\quad$ Mean $=n\times\frac{1}{2}=4 \Rightarrow n=8$

$\therefore$ Required Probability

$$={}^8C_6\left(\frac{1}{2}\right)^8+{}^8C_7\left(\frac{1}{2}\right)^8+{}^8C_8\left(\frac{1}{2}\right)^8$$

$$=\left(\frac{1}{2}\right)^8[28+8+1]=\frac{37}{256}$$

Q. 5. The probability that a teacher will give an unannounced test during any class meeting is $\frac{1}{5}$. If a student is absent twice, find the probability that the student will miss at least one test. [ISC, 2008]

Sol. Given, Probability that a teacher gives an unannounced test during any class $=\frac{1}{5}$

Let $\quad p=\frac{1}{5}$

$\therefore \quad q=1-p$

$\Rightarrow \quad q=1-\frac{1}{5}=\frac{4}{5}$

Here, n = 2

P(student misses atleast one test) = $P(x\geq 1)$

$= P(x = 1) + P(x = 2)$

$$={}^2C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)+{}^2C_2\left(\frac{1}{5}\right)^2$$

$$=2\times\frac{4}{25}+\frac{1}{25}=\frac{8}{25}+\frac{1}{25}=\frac{9}{25}$$

Q. 6. Assuming that on an average one telephone out of 10 is busy, seven telephone numbers are randomly selected and called. Find the probability that three of them will be busy. [ISC, 2007]

Sol. Probability of the telephone being busy is $\frac{1}{10}$.

i.e., $\quad p=\frac{1}{10},\ q=\frac{9}{10},\ n=7$

$\therefore$ Required probability

$= P(x = 3)$

$$={}^7C_3\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^4=\frac{35\times(9)^4}{10^7}=0.0229$$

AI **Q. 7. In a binomial distribution, the sum of its mean and the variance is 1.8. Find the probability of two successes if the event was conducted 5 times. [ISC, 2007]**

Sol. Mean = np and Variance = nqp, n = 5

Given, $\quad np + npq = 1.8$

$\therefore \quad p+pq=\frac{1.8}{5}=\frac{9}{25} \qquad$ ...(i)

But $\quad p+q=1 \Rightarrow p=1-q$

$\therefore \quad (1-q)+(1-q)q=\frac{9}{25}$

$\Rightarrow \quad (1-q)(1+q)=\frac{9}{25}$

$\Rightarrow \quad 1-q^2=\frac{9}{25} \Rightarrow q^2=\frac{16}{25}$

$\Rightarrow \quad q=\frac{4}{5}=0.8$

$\therefore \quad p=0.2, q=0.8$

Now, probability(2 successes)

$={}^5C_2(0.2)^2(0.8)^3$

$\therefore \quad P(x=2)=10\times 0.04\times 0.512=0.2048$

Q. 8. How many minimum number of times must a man toss a fair coin, so that the probability of having at least one head is more than 80% ?

Sol. Let the coin be tossed n times

$\therefore P$(getting atleast one head) $>\frac{80}{100}$

$\therefore \quad 1-P(0)>\frac{8}{10}$

$\Rightarrow \quad P(0)<1-\frac{8}{10}=\frac{2}{10}=\frac{1}{5}$

$\therefore \quad {}^nC_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^n<\frac{1}{5}$

or $\quad \frac{1}{2^n}<\frac{1}{5}$ or $2^n>5 \qquad$ ...(i)

The minimum value of n satisfying the eqn. (i) is 3.

$\Rightarrow \quad n$ = 3 (*i.e.,* 3 times)

Q. 9. A dice is tossed three times. Getting a '3' or a '5' is considered a success. Find the probability of atleast 2 success.

Sol. Given,

The total number of tossing dice = 3

So, n = 3

The probability (getting (3 or 5)) $p=\frac{1}{3}$

The probability (not getting (3 or 5))

$q=1-p=1-\frac{1}{3}=\frac{2}{3} \qquad [\because p+q=1]$

$\therefore$ Required probability, at least 2 successes

$= P(X = 2) + P(X = 3)$

$$= {}^3C_2(p)^2(q)^1 + {}^3C_3(p)^3q^0 = 3p^2q + p^3$$

$$= 3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^3$$

$$= 3 \times \frac{1}{9} \times \frac{2}{3} + \frac{1}{27} = \frac{7}{27}$$

AI Q. 10. Four dice are thrown simultaneously. If the occurrence of an odd number in a single dice is considered a success, then find the probability of at most 2 successes. [ISC 2006]

Sol. Given, the total number of dice = 4

So, $n = 4$

∴ The probability of getting an odd number

$$= \frac{3}{6} = \frac{1}{2}$$

So, $p = \frac{1}{2}$

Probability of not getting an odd number

$$q = 1 - p = 1 - \frac{1}{2}$$

$$= \frac{2-1}{2} = \frac{1}{2} \qquad [\because p + r = 1]$$

By using Binomial distribution,

$$P(x = r) = {}^nC_r p^r q^{n-r}$$

∴ Required probability = P(at most 2 successes)

$$= P(x=0) + P(x=1) + P(x=2)$$

$$= {}^4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 + {}^4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$$

$$= \left(\frac{1}{2}\right)^4 + 4\times\left(\frac{1}{2}\right)^4 + 6\times\left(\frac{1}{2}\right)^4$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16}$$

$$= \frac{1+4+6}{16} = \frac{11}{16}$$

6 Marks Questions

Q. 1. The probability that a bulb produced in a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulb :

(i) None will fuse after 150 days of use.

(ii) Not more than one will fuse after 150 days of use.

(iii) More than one will fuse after 150 days of use.

(iv) At least one will fuse after 150 days of use.

[ISC, 2020]

Sol. $x = 5, p = 0.05$

∴ $d = 1 - p = 0.95$

(i) None will fuse after 150 days of use

i.e. $r = 0$

$$P(r) = {}^nC_r\, p^r\, d^{n-r}$$

$$P(0) = {}^5C_0\,(0.05)^0(0.95)^{5-0}$$

$$= (0.95)^5$$

$$P(0) = (0.95)^5$$

Probability of bulb is none is fused

= 0.77378 (approx.)

(ii) Not more than one will fuse after 150 days of use

$r = 0, 1$

$$P(0) + P(1) = (0.95)^5 + {}^5C_1(0.05)^1(0.95)^4$$

$$= (0.95)^5 + 5 \times 0.05(0.95)^4$$

$$= (0.95)^4(0.95 + 0.25)$$

$$= (0.95)^4(1.20)$$

= 0.97740 (approx.)

Probability of bulb is not more than one will fuse after 150 days of use

= 0.97740

(iii) More then one will fuse after 150 days of use

= 1{P(0) + P(1)}

= 1 – 0.97740

= 0.0226

Probability of bulb more than one will fuse after 150 days of use

= 0.0226

(iv) At least one will fuse after 150 days of use

= 1 – P(0)

= 1 – 0.77378

= 0.2262 (approx.)

∴ Probability of bulb at least one will be fuse after 150 days of use

= 0.2262 (approx.)

[ISC Marking Scheme, 2020]

Examiner's Comment

- Most candidates made errors in computing the values of p and q and applying the binomial probability formula $P(X = r) = {}^nC_r\, p^r\, q^{n-r}$. Most of the candidates were not clear in writing the correct Binomial Distribution.

Answering Tip

- Extensive drill on concepts of mean and variance of binomial distribution is required.

Q. 2. From a lot of 6 items containing 2 defective items, a sample of 4 items one drawn at random. Let the random variable X denote the number of defective items in the sample. If the sample is drawn without replacement, find :

(i) The probability distribution of X

(ii) Mean of X

(iii) Variance of X [ISC, 2018]

Sol. Random variable, X : 0, 1, 2, values (defective items)

$$P(X = 0) = \frac{^4C_4}{^6C_4} = \frac{1}{15}$$

$$P(X = 1) = \frac{^2C_1 \times {^4C_3}}{^6C_4} = \frac{8}{15}$$

$$P(X = 2) = \frac{^2C_2 \times {^4C_2}}{^6C_4} = \frac{6}{15}$$

Probability Distribution :

Defective Items (X_i)	0	1	2	
Probability P_i	$\frac{1}{15}$	$\frac{8}{15}$	$\frac{6}{15}$	
$P_iX_i^2$	0	$\frac{8}{15}$	$\frac{24}{15}$	$\sum P_iX_i^2 = \frac{32}{15}$

$$\therefore \quad \text{Mean} = \sum P_iX_i$$

$$= \frac{1}{15}\times 0 + \frac{8}{15}\times 1 + \frac{6}{15}\times 2$$

$$= \frac{8}{15} + \frac{12}{15} = \frac{20}{15} = \frac{4}{3}$$

$$\text{Variance} = \sum P_iX_i^2 - \left(\sum P_iX_i\right)^2$$

$$= \frac{32}{15} - \left(\frac{20}{15}\right)^2$$

$$= \frac{32}{15} - \frac{400}{225}$$

$$= \frac{480-400}{225} = \frac{80}{225} = \frac{16}{45}$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Most of the candidates used the concept of Binomial probability distribution to solve this problem.

Answering Tip

- Students understand the concepts of probability distribution, Binomial probability distribution and their conditions. Sufficient practice on different types of probability distribution problems should be needed.

AI Q. 3. An urn contains 25 balls of which 10 balls are red and the remaining green. A ball is drawn at random from the urn, the colour is noted and the ball is replaced. If 6 balls are drawn in this way, find the probability that :

(i) All the balls are red.

(ii) Not more than 2 balls are green.

(iii) Number of red balls and green balls are equal. [ISC, 2017]

Sol. Number of red balls : 10, number of green balls: 15

$$P(R) = \frac{10}{25} = \frac{2}{5};\ P(G) = \frac{15}{25} = \frac{3}{5}$$

$$p = \frac{2}{5},\ q = \frac{3}{5}$$

Number of ways of drawing a red ball :

$$X = 0, 1, 2, 3, 4, 5, 6$$

$$P(X = r) = {^nC_r}.p^r.q^{n-r}$$

(i) All are red

$$P(X = 6) = {^6C_6}\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^0 = \left(\frac{2}{5}\right)^6$$

(ii) Not more than 2 are green :

$$\Rightarrow \quad = P(X = 4) + P(X = 5) + P(X = 6)$$

$$P(X \geq 4) = {^6C_4}\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^2 + {^6C_5}\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^1 + {^6C_6}\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^0$$

(iii) Equal number of balls *i.e.*, no. of red balls and green balls are equal

$$P(X = 3) = {^6C_3}\left(\frac{2}{5}\right)^3\cdot\left(\frac{3}{5}\right)^3$$

[ISC Marking Scheme, 2017]

Q. 4. The difference between mean and variance of a binomial distribution is 1 and the difference of their squares is 11. Find the distribution. **[ISC, 2016]**

Sol. $np - npq = 1,\ np(1-q) = 1$ ...(i)

$(np)^2 - (npq)^2 = 11,\ (np)^2(1-q^2) = 11$ ...(ii)

Dividing (ii) by (i), we get $(1+q)/(1-q) = 11$

$$1 + q = 11 - 11q$$
$$12q = 10$$
$$q = 5/6,\ p = 1/6$$

We get, $n = 36$

The distribution is given by $\left(\frac{1}{6}+\frac{5}{6}\right)^{36}$

or $X \sim B\left(36, \frac{1}{6}\right)$ **[ISC Marking Scheme, 2016]**

Detailed Solution :

Let the binomial distribution be $(q + p)^n$.

We know that, mean = np and variance = npq

Now, given $np - npq = 1$

$\Rightarrow$ $np(1-q) = 1$ ...(i)

and $(np)^2 - (npq)^2 = 11$

$\Rightarrow$ $(np)^2(1-q)^2 = 11$ ...(ii)

Dividing equation (ii) by the square of equation (i), we get

$$\frac{(np)^2(1-q^2)}{(np)^2(1-q)^2} = \frac{11}{1}$$

$$\frac{(1+q)(1-q)}{(1-q)(1-q)} = 11$$

$\Rightarrow$ $1 + q = 11 - 11q$

$\Rightarrow$ $q = \frac{10}{12} = \frac{5}{6}$

$$p = 1 - q = 1 - \frac{5}{6} = \frac{1}{6}$$

Putting the values of p and q in (i), we get

$$n\frac{1}{6}\left(1-\frac{5}{6}\right) = 1$$

$\Rightarrow$ $n = 36$

Hence, the binomial distribution is $\left(\frac{5}{6}+\frac{1}{6}\right)^{36}$.

Examiner's Comment

- Several candidates made errors in solving the equations $np - npq = 1$, $(np)^2 - (npq)^2 = 11$. Most of the candidates were not clear in writing the correct Binomial Distribution.

Answering Tip

- Revision must be done of concepts, mean and variance of binomial probability distribution. While solving for n, p, and q it must be noted that $p + q = 1$. More practice in solving equations in two/three unknowns is essential.

Q. 5. Five dice are thrown simultaneously. If the occurrence of an odd number in a single dice is considered a success, find the probability of maximum three successes. **[ISC, 2015]**

Sol. $n = 5,\ p = 1/2,\ q = 1/2$

$$p(X \le 3) = 1 - p(X = 4, 5)$$
$$= 1 - {}^5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) - {}^5C_5\left(\frac{1}{2}\right)^5$$
$$= 1 - \frac{5}{32} - \frac{1}{32} = \frac{26}{32} = \frac{13}{16} \text{ or } 0.8125$$

[ISC Marking Scheme, 2015]

Detailed Solution :

Let A be the event occurring an odd number and it is success.

$$P(A) = \frac{1}{2},\ P(A') = \frac{1}{2},\ i.e., p = \frac{1}{2} \text{ and } q = \frac{1}{2}$$

$\therefore$ P(at most 3 success)

or P(maximum of 3 success)

or $P(X \le 3) = \sum_{r=0}^{3} {}^nC_r p^r q^{n-r}$, where n is 5

$$= {}^5C_0\left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4$$
$$+ {}^5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$$
$$= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32}$$
$$= \frac{26}{32} = \frac{13}{16} = 0.8125$$

Examiner's Comment

- Many candidates were unable to understand the problem correctly. The concept of $P(X \le 3)$ was not clear to many candidates. Probability distribution theory was incorrectly applied by some candidates.

Answering Tip

- Revise the Binomial theorem thoroughly and understand the each term of binomial expansion. Practise the situation of maximum and minimum successes.

AI **Q. 6. On dialling certain telephone numbers, assume that on an average, one telephone number out of five is busy. Ten telephone numbers are randomly selected and dialled. Find the probability that at least three of them will be busy. [ISC, 2014]**

Sol. $p=\frac{1}{5},\ q=\frac{4}{5},\ n=10$

$$P(X\geq 3)=1-P(X=0,1,2)$$

$$=1-\left[{}^{10}C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{10}+{}^{10}C_1\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^9+{}^{10}C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^8\right]$$

$$=1-\left[\left(\frac{4}{5}\right)^{10}+10\times\frac{1}{5}\times\left(\frac{4}{5}\right)^9+45\times\frac{1}{25}\times\left(\frac{4}{5}\right)^8\right]$$

$$=1-\left(\frac{4}{5}\right)^8\left[\frac{16}{25}+10\times\frac{4}{25}+\frac{45}{25}\right]$$

$$=1-0.678=0.322$$

[ISC Marking Scheme, 2014]

Detailed Solution :

Probability of the telephone being busy is $\frac{1}{5}$ *i.e.*, $q=1\times p=1-\frac{1}{4}=\frac{4}{5}$ and n = 10.

Probability that at least three of them busy =

1 – [Prob that no telephone being busy + Prob that one telephone being + Prob that two telephone being busy]

i.e., $p(X > 3) = 1 - [p(X = 0) + p(X = 1) + p(X = 7)]$

$$=\left[{}^{10}C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{10}+{}^{10}C_1\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^9+{}^{10}C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^8\right]$$

$$=1-\left[\left(\frac{4}{5}\right)^{10}+10\times\frac{1}{5}\times\left(\frac{4}{5}\right)^9+45\times\frac{1}{5}\times\left(\frac{4}{5}\right)^8\right]$$

$$=1-\left(\frac{4}{5}\right)^8\left[\frac{16}{25}+10\times\frac{4}{25}+\frac{45}{25}\right]$$

$$=1-\left(\frac{4}{5}\right)^8\left(\frac{101}{25}\right)$$

$$= 1 - 0{\cdot}678$$

$$= 0.322$$

Examiner's Comment

- Probability distribution theory was incorrectly applied by some candidates owing to carelessness or lack of understanding. In the expansion of $(q + p)^n$, some took the value of n as 3 instead of 10 by mistake.

Answering Tip

- Students must understand the meaning of 'at least 3' and 'at most 3' situations.

Q. 7. If the sum and the product of the mean and variance of a Binomial Distribution are 1.8 and 0.8 respectively, find the probability distribution and the probability of at least one success. [ISC, 2013]

Sol. We have $np + npq = 1.8$

$\Rightarrow\quad np(1 + q) = 1.8 \quad \ldots\text{(i)}$

and $\quad np\cdot npq = 0.8$

$\Rightarrow\quad n^2p^2q = 0.8 \quad \ldots\text{(ii)}$

Dividing the square of (i) by (ii), we get

$$\frac{n^2p^2(1+q)^2}{n^2p^2q}=\frac{1.8\times1.8}{0.8}$$

$$\Rightarrow\quad \frac{(1+q)^2}{q}=\frac{3.24}{0.8}$$

$$\Rightarrow\quad \frac{(1+q)^2}{q}=\frac{324}{80}$$

$$\Rightarrow\quad \frac{(1+q)^2}{q}=\frac{81}{20}$$

$\Rightarrow\ 20+2\times20q + 20q^2 = 81q$

$\Rightarrow\ 20q^2 - 41q + 20 = 0$

$\Rightarrow\ 20q^2 - 25q - 16q + 20 = 0$

$\Rightarrow\ 5q(4q-5) - 4(4q - 5) = 0$

$\Rightarrow\ (4q - 5)(5q - 4) = 0$

$\Rightarrow\quad q=\frac{5}{4}$ or $q=\frac{4}{5}$ but $q\neq\frac{5}{4}>1$

$\therefore\quad q=\frac{4}{5}$

$\Rightarrow\quad p=\frac{1}{5}$

From (i), $\quad n\times\frac{1}{5}\left(1+\frac{4}{5}\right)=1.8$

$$n\times\frac{1}{5}\times\frac{9}{5}=1.8$$

$\Rightarrow\quad 9n = 25 \times 1.8$

$\Rightarrow\quad n=\frac{25\times1.8}{9}$

$\Rightarrow\quad n=\frac{45}{9}\quad \therefore\ n = 5$

Hence, the binomial distribution is

$$(p+q)^n=\left(\frac{1}{5}+\frac{4}{5}\right)^5$$

Probability(getting atleast 1 success)

$$= P(r \geq 1)$$

$$= P(1) + P(2) + P(3) + P(4) + P(5)$$

$$= {}^5C_1 \cdot \frac{1}{5}\left(\frac{4}{5}\right)^4 + {}^5C_2 \cdot \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3$$

$$+ {}^5C_3 \cdot \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2 + {}^5C_4 \cdot \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)$$

$$+ {}^5C_5 \cdot \left(\frac{1}{5}\right)^5$$

$$= 5 \cdot \frac{1}{5} \times \frac{256}{625} + 10 \cdot \frac{1}{25} \cdot \frac{64}{125}$$

$$+ 10 \cdot \frac{1}{125} \cdot \frac{16}{25} + 5 \cdot \frac{1}{625} \cdot \frac{4}{5} + 1 \cdot \frac{1}{3125}$$

$$= \frac{256}{625} + \frac{128}{625} + \frac{32}{625} + \frac{4}{625} + \frac{1}{3125}$$

$$= \frac{1280 + 640 + 160 + 20 + 1}{3125}$$

$$= \frac{2101}{3125} = 0.67232$$

Q. 8. The probability that a bulb produced by a factory will fuse in 100 days of use is 0.05. Find the probability that out of 5 such bulbs, after 100 days of use :

(i) None fuse.

(ii) Not more than one fuses.

(iii) More than one fuses.

(iv) At least one fuses [ISC, 2012]

Sol. Given, $p = 0.05 = \frac{1}{20}, q = \frac{19}{20}$

(i) $P(\text{None fuse}) = {}^5C_0 p^0 q^{5-0} = \left(\frac{1}{20}\right)^0 \left(\frac{19}{20}\right)^5 = \left(\frac{19}{20}\right)^5$

(ii) P(Not more than one fuses)

$$= \sum_{r=0}^{1} {}^nC_r p^r q^{n-r}$$

$$= {}^5C_0 \left(\frac{1}{20}\right)^0 \left(\frac{19}{20}\right)^5 + {}^5C_1 \left(\frac{1}{20}\right)\left(\frac{19}{20}\right)^4$$

$$= \left(\frac{19}{20}\right)^4 \left(\frac{19}{20} + \frac{5}{20}\right)$$

$$= \left(\frac{19}{20}\right)^4 \left(\frac{24}{20}\right) = \frac{6}{5}\left(\frac{19}{20}\right)^4$$

(iii) P(More than one fuses)

$$= 1 - P(\text{Not more than one fuse})$$

$$= 1 - \frac{6}{5}\left(\frac{19}{20}\right)^4$$

(iv) P(At least one fuses)

$$= 1 - P(\text{None fuses})$$

$$= 1 - \left(\frac{19}{20}\right)^5.$$

AI Q. 9. A box contains 4 red and 5 black marbles. Find the probability distribution of the red marbles in a random draw of three marbles. Also, find the mean and standard deviation of the distribution. [ISC, 2011]

Sol. Let X denote the number of red marbles drawn from the box. Since, there are 4 red marbles therefore, X can take the values 0, 1, 2 and 3.

Now, $P(X = 0)$

= Probability of getting no red marbles

= Probability of 3 marbles drawn are black

$$= \frac{{}^5C_3}{{}^9C_3} = \frac{10}{84} = \frac{5}{42}$$

$P(X = 1)$ = Probability of getting 1 red marble

$$= \frac{{}^4C_1 \times {}^5C_2}{{}^9C_3} = \frac{4 \times 10}{84} = \frac{10}{21}$$

$P(X = 2)$ = Probability of getting 2 red marbles

$$= \frac{{}^4C_2 \times {}^5C_1}{{}^9C_3} = \frac{6 \times 5}{84} = \frac{5}{14}$$

$P(X = 3)$ = Probability of getting 3 red marbles

$$= \frac{{}^4C_3}{{}^9C_3} = \frac{4}{84} = \frac{1}{21}$$

∴ The probability distribution of X is given as :

X	0	1	2	3
$P(X)$	5/42	10/21	5/14	1/21

Again,

X_i	$P_i(X)$	$P_i \times X_i$	X_i^2	$P_i \times X_i^2$
0	5/42	0	0	0
1	10/21	10/21	1	10/21
2	5/14	5/7	4	10/7
3	1/21	1/7	9	3/7
		$\Sigma P_i X_i = 4/3$		$\Sigma P_i \times X_i^2 = 7/3$

Now, Mean, $\mu = \Sigma P_i X_i = \frac{4}{3}$

Standard Deviation, $\sigma = \sqrt{\Sigma P_i X_i^2 - \mu^2}$

$$= \sqrt{\frac{7}{3} - \frac{16}{9}} = \sqrt{\frac{21 - 16}{9}}$$

$$= \frac{\sqrt{5}}{3} = 0.745$$

Q. 10. Eight coins are thrown simultaneously.

(i) Show that the probability of getting at least 6 heads is $\frac{37}{256}$.

(ii) What is the probability of getting at least 3 heads? **[ISC, 2008]**

Sol. Given, total number of coins, $n = 8$

Here, probability of getting head $= p = \frac{1}{2}$

$$\therefore \quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

(i) $p = \frac{1}{2}, q = \frac{1}{2}, n = 8$

Probability(at least 6 heads)

$$= {}^8C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2 + {}^8C_7\left(\frac{1}{2}\right)^8 + {}^8C_8\left(\frac{1}{2}\right)^8$$

$$= \left[\frac{8\times7}{1\times2} + 8 + 1\right]\left(\frac{1}{2}\right)^8$$

$$= 37 \times \frac{1}{256} = \frac{37}{256}$$

(ii) Probability(at least three heads)

$$= 1 - \left[{}^8C_0\left(\frac{1}{2}\right)^8 + {}^8C_1\left(\frac{1}{2}\right)^8 + {}^8C_2\left(\frac{1}{2}\right)^8\right]$$

$$= 1 - \frac{1}{2^8}[1 + 8 + 28]$$

$$= 1 - \frac{37}{256} = \frac{256 - 37}{256} = \frac{219}{256}$$

Q. 11. Two cards are drawn successively with replacement from a well-shuffled pack of 52 cards. Find the probability distribution of the number of kings and hence find the mean of the distribution.

Sol. Let X is the random variable denoting the number of kings.

$\therefore$ X can take value 0, 1 or 2.

$\because$ Drawing of a card is done with replacement, the two draws form independent experiments.

$$P_{(X=0)} = P_{(\text{no king \& no king})}$$

$$= P_{(\text{no king})} \times P_{(\text{no king})}$$

$$= \frac{48}{52} \times \frac{48}{52} = \frac{144}{169}$$

$$P_{(X=1)} = P_{(\text{king \& no king})} + P_{(\text{no king \& king})}$$

$$= \frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52} = \frac{24}{169}$$

$$P_{(X=2)} = P_{(\text{king})} \text{ and } P_{(\text{king})} \times P_{(\text{king})}$$

$$= \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$$

$\therefore$ Required probability distribution is :

X	0	1	2
$P(X)$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

$$\text{Mean } M = \sum_{i=0}^{2} x_i P(x_i)$$

$$= 0 \times \frac{144}{169} + 1 \times \frac{24}{169} + 2 \times \frac{1}{169} = \frac{26}{169} = \frac{2}{13}$$

Q. 12. Out of a group of 30 honest people, 20 always speak the truth. Two persons are selected at random from the group. Find the probability distribution of the number of selected persons who speak the truth. Also, find the mean of the distribution.

Sol. Total no. of honest people = 30

No. of people speaking truth always = 20

Let X denotes the number of persons speaking the truth. So, $X = 0, 1, 2$.

X	0	1	2
$P(X)$	$\frac{{}^{10}C_2}{{}^{30}C_2} = \frac{9}{87}$	$\frac{{}^{10}C_1 \times {}^{20}C_1}{{}^{30}C_2} = \frac{40}{87}$	$\frac{{}^{20}C_2}{{}^{30}C_2} = \frac{38}{87}$

$$\text{Mean} = \sum_{i=0}^{2} X_i P(X_i)$$

$$= 0 \times \frac{9}{87} + 1 \times \frac{40}{87} + 2 \times \frac{38}{87} = \frac{116}{87}$$

TOPIC-3
Bayes' Theorem

Revision Notes

Scan to know more about this topic

Baye's Theorem

➢ **Bayes' Theorem :**

If $E_1, E_2, E_3, ..., E_n$ are n non-empty events constituting a partition of sample space S, *i.e.*, $E_1, E_2, E_3, ..., E_n$ are pair wise disjoint and $E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = S$ and A is any event of non-zero probability then,

$$P(E/A)=\frac{P(E_i)\cdot P(A/E_i)}{\sum_{j=1}^{n}P(E_j)P(A/E_j)},i=1,2,3,\ldots,n$$

e.g.,
$$P(E_1/A)=\frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)+P(E_3)\cdot P(A/E_3)}$$

- Bayes' theorem is also known as the formula for the probability of causes.
- If $E_1, E_2, E_3, \ldots, E_n$ form a partition of S and A be any event, then

  $P(A)=P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)+\ldots+P(E_n)\cdot P(A/E_n)$ $[\because\ P(E_i\cap A)=P(E_i)\cdot P(A/E_i)]$
- The probabilities $P(E_1), P(E_2), \ldots, P(E_n)$ which are known before the experiment takes place are called **prior probabilities** and $P(A/E_n)$ are called **posterior probabilities**.

4 Marks Questions

Q. 1. A company has two plants, which manufactures scooters. Plant I manufactures 80% of the scooters while plant II manufactures 20% of the scooters. At plant I, 85 out of 100 scooters are rated as being of standard quality, while at plant II only 65 out of 100 scooters are rated as being of standard quality. If a scooter is of standard quality, what is the probability that it came from plant I?

Sol. Let A = Scooters manufactured by plant I

B = Scooters manufactured by plant II

Let S represents the event that scooter manufactured by plants is of standard quality.

Given, plant I manufactured = 80% scooters

Plant II manufactured = 20% scooters

So, the probability of manufactured scooters by plants I and II are

$$P(A)=\frac{80}{100}=0.8 \text{ and } P(B)=\frac{20}{100}=0.2$$

Now, the probability that the scooter of standard quality manufactured by plants I and II are

$$P(S/A)=\frac{85}{100}=0.85$$

$$P(S/B)=\frac{65}{100}=0.65$$

The probability of selecting scooters with standard quality manufactures from plant I,

$$P(A/S)=\frac{P(A)\cdot P(S/A)}{P(A)\cdot P(S/A)+P(B)\cdot P(S/B)}$$

[by using Bayes' theorem]

$$=\frac{0.8\times0.85}{0.8\times0.85+0.2\times0.65}$$

$$=\frac{0.68}{0.68+0.13}=\frac{0.68}{0.81}=\frac{68}{81}$$

AI **Q. 2. A man is known to speak the truth 3 out of 5 times. He throws a die and reports that it is 1. Find the probability that it is actually 1.**

Sol. E_1 = Event that 1 occur;

E_2 = Event that 1 does not occur;

A = Event that the man reports that 1 occur.

$$P(E_1)=\frac{1}{6};P(E_2)=\frac{5}{6};$$

$$P(A/E_1)=\frac{3}{5};\ P(A/E_2)=\frac{2}{5};$$

$$P(E_1/A)=\frac{\frac{1}{6}\cdot\frac{3}{5}}{\frac{1}{6}\cdot\frac{3}{5}+\frac{5}{6}\cdot\frac{2}{5}}=\frac{3}{13}$$

Q. 3. In answering questions on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who guesses the answer will be correct with probability $\frac{1}{3}$, what is the probability that the student knows the answer given that the answered it correctly?

Sol. Let, E_1 : Student knows the answer

E_2 : Student guesses the answer

A : Answers correctly

Given, $P(E_1)=\frac{3}{5},P(E_2)=\frac{2}{5}$

and $P(A/E_1)=1,P(A/E_2)=\frac{1}{3}$

By Bayes' theorem,

$$P(E_1/A)=\frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)}$$

$$=\frac{\frac{3}{5}\cdot1}{\frac{3}{5}\cdot1+\frac{2}{5}\cdot\frac{1}{3}}=\frac{9}{11}$$

6 Marks Questions

AI **Q. 1. Given three identical Boxes *A*, *B* and *C*, Box *A* contains 2 gold and 1 silver coin, Box *B* contains 1 gold and 2 silver coins and Box *C* contains 3 silver coins. *A* person chooses a Box at random and takes out a coin. If the coin drawn is of silver, find the probability that it has been drawn from the Box which has the remaining two coins also of silver.**

[ISC, Specimen Paper, 2021] [ISC, 2019]

Sol. $A \rightarrow$ Box A is chosen
$B \rightarrow$ Box B is chosen
$C \rightarrow$ Box C is chosen
$S \rightarrow$ Coin drawn is silver coin

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

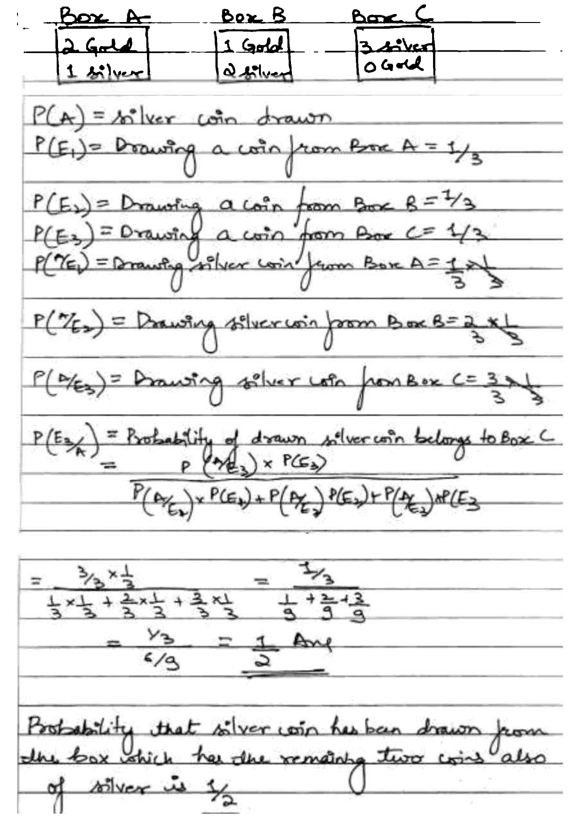
$$P(S|A) = \frac{1}{3};\ P(S|B) = \frac{2}{3},\ P(S|C) = \frac{3}{3}$$

$$P(C|S) = \frac{P(C).P(S/C)}{P(A).P(S/A) + P(B).P(S/B) + P(C).P(S/C)}$$

$$= \frac{\frac{3}{3} \times \frac{1}{3}}{\frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{3}{3}} = \frac{1}{2}$$

[ISC Marking Scheme, 2019]

OR



Box A	Box B	Box C
2 Gold 1 silver	1 Gold 2 silver	3 silver 0 Gold

P(A) = silver coin drawn
$P(E_1)$ = Drawing a coin from Box A = $1/3$
$P(E_2)$ = Drawing a coin from Box B = $1/3$
$P(E_3)$ = Drawing a coin from Box C = $1/3$
$P(A/E_1)$ = Drawing silver coin from Box A = $\frac{1}{3} \times \frac{1}{3}$
$P(A/E_2)$ = Drawing silver coin from Box B = $\frac{2}{3} \times \frac{1}{3}$
$P(A/E_3)$ = Drawing silver coin from Box C = $\frac{3}{3} \times \frac{1}{3}$
$P(E_3/A)$ = Probability of drawn silver coin belongs to Box C

$$= \frac{P(A/E_3) \times P(E_3)}{P(A/E_1) \times P(E_1) + P(A/E_2)\, P(E_2) + P(A/E_3) \times P(E_3}$$

$$= \frac{3/3 \times \frac{1}{3}}{\frac{1}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} + \frac{3}{3} \times \frac{1}{3}} = \frac{1/3}{\frac{1}{9} + \frac{2}{9} + \frac{3}{9}}$$

$$= \frac{1/3}{6/9} = \frac{1}{2} \text{ Ans}$$

Probability that silver coin has been drawn from the box which has the remaining two coins also of silver is $1/2$

[Topper's Answer 2019]

Examiner's Comment

- Most of the candidates attempted this question well. However, some candidates could not identify the Box C, from where a silver coin was drawn. A few candidates, instead of using Bayes' theorem, applied the concept of total probability.

Answering Tip

- Understand Bayes' theorem as one of the concepts of conditional probability with illustrations and tree diagram.

Q. 2. A fair die is rolled. If face 1 turns up, a ball is drawn from Bag *A*. If face 2 or 3 turns up, a ball is drawn from Bag *B*. If face 4 or 5 or 6 turns up, a ball is drawn from Bag *C*. Bag *A* contains 3 red and 2 white balls, Bag *B* contains 3 red and 4 white balls and Bag *C* contains 4 red and 5 white balls. The die is rolled, a Bag is picked up and a ball is drawn. If the drawn ball is red, what is the probability that it is drawn from Bag *B*?

[ISC, 2017]

Sol. E_1 : Event that bag *A* is picked
E_2 : Event that bag *B* is picked
E_3 : Event that bag *C* is picked

$$P(E_1) = \frac{1}{6},\ P(E_2) = \frac{2}{6},\ P(E_3) = \frac{3}{6}$$

$$P(R/E_1) = \frac{3}{5},\ P(R/E_2) = \frac{3}{7},\ P(R/E_3) = \frac{4}{9}$$

By applying Bayes' theorem :

$$P(E_2/R) = \frac{P(E_2).P(R/E_2)}{P(E_1)P((R/E_1) + P(E_2).P(R/E_2) + P(E_3).P(R/E_3)}$$

$$= \frac{\frac{2}{6} \cdot \frac{3}{7}}{\frac{1}{6} \cdot \frac{3}{5} + \frac{2}{6} \cdot \frac{3}{7} + \frac{3}{6} \cdot \frac{4}{9}} = \frac{90}{293}$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Some candidates made errors in computing the probabilities of selecting Bag *A*, Bag *B* and Bag *C* as 1/3 instead of 1/6, 2/6 and 3/6 respectively and were unable to get the result. Some candidates did not apply the theorem correctly.

Answering Tip

- Remember that probabilities are ratios not numbers. Clear the concepts of conditional probability to understand advance the concepts of conditional probability to understand advance concept of Bayes' Theorem.

AI **Q. 3. In an automobile factory, certain parts are to be fixed into the chassis in a section before it moves into another section. On a given day, one of three people *A*, *B* and *C* carries out this task. *A* has 45% chance, *B* has 35% chance and *C* has 20% chance of doing the task. The probability that *A*, *B* and *C* will take more than the allotted time is $\frac{1}{6}, \frac{1}{10}$ and $\frac{1}{20}$ respectively. If it is found that the time taken is more than the allotted time, what is the probability that *A* has done the task? [ISC, 2016]**

Sol. Let E_1, E_2, E_3 denote the events of carrying out the task by *A*, *B* and *C* respectively.

Let *H* denote the event of taking more time.

Then, $P(E_1) = \frac{45}{100} = 0.45$, $P(E_2) = \frac{35}{100} = 0.35$,

$$P(E_3) = \frac{20}{100} = 0.20$$

Given, $P(H/E_1) = \frac{1}{6}$, $P(H/E_2) = \frac{1}{10}$,

$$P(H/E_3) = \frac{1}{20}$$

$$P\left(\frac{E_1}{H}\right) = \frac{P(E_1)\cdot P(H/E_1)}{P(E_1)\cdot P\left(\frac{H}{E_1}\right) + P(E_2)\cdot P\left(\frac{H}{E_2}\right) + P(E_3)\cdot P\left(\frac{H}{E_3}\right)}$$

$$= \frac{0.075}{0.075 + 0.035 + 0.01} = 0.625 = \frac{5}{8}$$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Many candidates used Bayes' theorem correctly but took the probabilities of *A*, *B*, *C* as 45, 35, 20, instead of percentages. Some candidates did not implement the theorem correctly.

Answering Tip

- Be clear that probabilities are ratios and not numbers. Learn conditional probability concept which helps the student to understand the advanced concept Bayes' theorem.

Q. 4. Box I contains two white and three black balls, box II contains four white and one black balls and box III contains three white and four black balls. A dice having three red, two yellow and one green face, is thrown to select the box. If red face turns up, we pick up box I, if a yellow face turns up we pick up box II, otherwise, we pick up box III. Then, we draw a ball from the selected box. If the ball drawn is white, what is the probability that the dice had turned up with a red face? [ISC, 2015]

Sol. $P(A) = 3/6$, $P(B) = 2/6$, $P(C) = 1/6$

Let *D* be the probability of drawing a white ball.

$P(D/A) = 2/5$, $P(D/B) = 4/5$, $P(D/C) = 3/7$

$P(A/D)$

$$= \frac{P(A) \times P(D/A)}{P(A)P(D/A) + P(B)P(D/B) + P(C)P(D/C)}$$

$$= \frac{3/6 \times 2/5}{(3/6 \times 2/5 + 2/6 \times 4/5 + 1/6 \times 3/7)}$$

$= (6/30) \times (210/113)$

$= 42/113 = 0.37$ **[ISC Marking Scheme, 2015]**

Detailed Solution :

Let *A* : Getting red face on dice

B : Getting yellow face on dice

C : Getting green face on dice

D : Getting of drawing a white ball.

Bag I	Bag II	Bag III
2 white	4 white	3 white
3 Black	1 Black	4 Black
Total Balls = 5	Total Balls = 5	Total Balls = 7

$\therefore P(A) = \frac{3}{6}$, $P(B) = \frac{2}{6}$, $P(C) = \frac{1}{6}$,

and $P\left(\frac{D}{A}\right) = \frac{2}{5}$, $P\left(\frac{D}{B}\right) = \frac{4}{5}$, $P\left(\frac{C}{D}\right) = \frac{1}{6}$

Using Bayes' Theorem,
Required Probability,

$$P\left(\frac{A}{D}\right) = \frac{P(A).P\left(\frac{D}{A}\right)}{P(A)P.\left(\frac{D}{A}\right) + P(B).P\left(\frac{D}{B}\right) + P(C).P\left(\frac{D}{C}\right)}$$

$$= \frac{\frac{3}{6} \times \frac{2}{5}}{\left(\frac{3}{6} \times \frac{2}{5}\right) + \left(\frac{2}{6} \times \frac{4}{5}\right) + \left(\frac{1}{6} \times \frac{3}{7}\right)}$$

$$= \frac{\frac{1}{5}}{\frac{1}{5} + \frac{4}{15} + \frac{1}{14}}$$

$$= \frac{\frac{1}{5}}{\frac{42 + 56 + 15}{210}}$$

$$= \frac{\frac{1}{5}}{\frac{113}{210}} = \frac{210}{5 \times 113}$$

$$= \frac{42}{113}$$

Examiner's Comment

- The concept of Bayes' theorem was clear to most candidates but some candidates found incorrect probability. While some candidates found conditional probability for the happening of an event incorrectly, even probability of a specific known event was found wrongly by a few candidates.

Answering Tip

- Learn Bayes' theorem with proper explanation and illustration. Pay heed to the laws of total probability. Do Adequate practice of Bayes' theorem.

AI **Q. 5. In a bolt factory, three machines *A*, *B* and *C* manufacture 25%, 35% and 40% of the total production respectively. Of their respective outputs, 5%, 4% and 2% are defective. A bolt is drawn at random from the total production and it is found to be defective. Find the probability that it was manufactured by machine *C*. [ISC, 2014]**

Sol. $P(A) = 1/4, P(B) = 7/20, P(C) = 2/5$

Let D be the probability of drawing a defective bolt.

$P(D/A) = 1/20, P(D/B) = 1/25, P(D/C) = 1/50$

$P(C/D)$

$$= \frac{P(C)\times P(D/C)}{P(A)P(D/A)+P(B)P(D/B)+P(C)P(D/C)}$$

$$= \frac{2/5\times 1/50}{(1/4\times 1/20+7/20\times 1/25+2/5\times 1/50)}$$

$= 16/69$ **[ISC Marking Scheme, 2014]**

Detailed Solution :

Let, event A : bolt is manufactured by machine A

event B : bolt is manufactured by machine B

event C : bolt is manufactured by machine C.

D : probability of drawing a defective bolt.

$$\therefore \quad P(A) = 25\% = \frac{25}{100} = \frac{1}{4}$$

$$P(B) = 35\% = \frac{35}{100} = \frac{7}{20}$$

$$P(C) = 40\% = \frac{40}{100} = \frac{2}{5}$$

$$\text{and} \quad P\left(\frac{D}{A}\right) = 5\% = \frac{5}{100} = \frac{1}{20}$$

$$P\left(\frac{D}{B}\right) = 4\% = \frac{4}{100} = \frac{1}{25}$$

$$P\left(\frac{D}{C}\right) = 2\% = \frac{2}{100} = \frac{1}{50}$$

Using Bayes' Theorem,

Required Probability,

$$P\left(\frac{C}{D}\right) = \frac{P(C).P\left(\frac{D}{C}\right)}{P(A).P\left(\frac{D}{A}\right)+P(B).P\left(\frac{D}{B}\right)+P(C).P\left(\frac{D}{C}\right)}$$

$$= \frac{\frac{2}{5}\times\frac{1}{50}}{\left(\frac{1}{4}\times\frac{1}{20}\right)+\left(\frac{7}{20}\times\frac{1}{25}\right)+\left(\frac{2}{5}\times\frac{1}{50}\right)}$$

$$= \frac{\frac{1}{125}}{\frac{1}{80}+\frac{7}{500}+\frac{1}{125}}$$

$$= \frac{\frac{1}{125}}{\frac{25+28+16}{2000}}$$

$$= \frac{\frac{1}{125}}{\frac{69}{2000}}$$

$$= \frac{1}{125}\times\frac{2000}{69}$$

$$= \frac{16}{69}$$

Examiner's Comment

- While using Bayes' theorem, some candidates used actual numbers given instead of probabilities of the events. Probability of an event and the conditional probability for the happening of an event were not understood and also incorrectly substituted in several cases.

Answering Tip

- Remember that in Bayes' Theorem the required probability is the ratio of conditional probability of specific known event to the sum of the conditional probabilities of all events under consideration.

Q. 6. For *A*, *B* and *C*, the chances of being selected as the manager of a firm are 4 : 1 : 2 respectively. The probabilities for them to introduce a radical change in the marketing strategy are 0.3, 0.8 and 0.5 respectively. If a change takes place; find the probability that it is due to the appointment of *B*. [ISC, 2013]

Sol. Let E_1, E_2, E_3 and A be the events as defined below :

E_1 = A is selected as Manager

E_2 = B is selected as Manager

E_3 = C is selected as Manager

and A = radical change occurs in marketing strategy

$$P(E_1) = \frac{4}{4+1+2} = \frac{4}{7}, \quad P(E_2) = \frac{1}{4+1+2} = \frac{1}{7},$$

$$P(E_3) = \frac{2}{4+1+2} = \frac{2}{7}$$

$$\text{Given,} \quad P\left(\frac{A}{E_1}\right) = 0.3$$

$$P\left(\frac{A}{E_2}\right)=0.8$$

$$P\left(\frac{A}{E_3}\right)=0.5$$

We want to find $P(E_2/A)$.

By Bayes' theorem,

$$P\left(\frac{E_2}{A}\right)=\frac{P(E_2)\cdot P\left(\frac{A}{E_2}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right)+P(E_2)\cdot P\left(\frac{A}{E_2}\right)+P(E_3)\cdot P\left(\frac{A}{E_3}\right)}$$

$$=\frac{\frac{1}{7}\times 0.8}{\frac{4}{7}\times 0.3+\frac{1}{7}\times 0.8+\frac{2}{7}\times 0.5}$$

$$=\frac{0.8}{1.2+0.8+1}=\frac{0.8}{3}=\frac{8}{30}=\frac{4}{15}$$

Q. 7. In a class of 75 students, 15 are above average, 45 are average and the rest below average achievers. The probability that an above average achieving student fails is 0.005, that an average achieving student fails is 0.05 and the probability of a below average achieving students failing is 0.15. If a student is known to have passed, what is the probability that he is a below average achiever? [ISC, 2012]

Sol. Let, E_1 : student is above average

E_2 : student is average

E_3 : student is below average

A : student is known to have passed

$$P(E_1)=\frac{15}{75}=\frac{1}{5},\quad P(E_2)=\frac{45}{75}=\frac{3}{5},$$

$$P(E_3)=\frac{75-(15+45)}{75}=\frac{1}{5}$$

$$P\left(\frac{A}{E_1}\right)=1-0.005=0.995$$

$$P\left(\frac{A}{E_2}\right)=1-0.05=0.95$$

$$P\left(\frac{A}{E_3}\right)=1-0.15=0.85$$

Using Bayes' theorem,

$$P(E_3/A)=\frac{P(E_3)\cdot P(A/E_3)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)+P(E_3)\cdot P(A/E_3)}$$

$$=\frac{\frac{1}{5}\times\frac{85}{100}}{\frac{1}{5}\times\frac{995}{1000}+\frac{3}{5}\times\frac{95}{100}+\frac{1}{5}\times\frac{85}{100}}$$

$$=\frac{\frac{17}{100}}{\frac{199}{1000}+\frac{57}{100}+\frac{17}{100}}=\frac{\frac{17}{100}}{\frac{199+570+170}{1000}}$$

$$=\frac{\frac{17}{100}}{\frac{939}{1000}}=\frac{17\times 1000}{100\times 939}=\frac{170}{939}$$

Q. 8. Bag *A* contains 2 white, 1 black and 3 red balls, bag *B* contains 3 white, 2 black and 4 red balls and bag *C* contains 4 white, 3 black and 2 red balls. One Bag is chosen at random and 2 balls are drawn at random from that bag. If the randomly drawn balls happen to be red and black, what is the probability that both balls come from Bag *B*? [ISC, 2011]

Sol.

Bag *A*		Bag *B*		Bag *C*	
2	W	3	W	4	W
1	B	2	B	3	B
3	R	4	R	2	R

Let E_1, E_2 and E_3 be the events of choosing bag *A*, bag *B* and bag *C* respectively.

$$\therefore\quad P(E_1)=P(E_2)=P(E_3)=\frac{1}{3}$$

Let E be the event of drawing 1 red and 1 black ball from the bags, then $P(E/E_1)$ = Probability(1 red and 1 black balls are drawn from bag *A*)

$$=\frac{{}^3C_1\times{}^1C_1}{{}^6C_2}=\frac{3}{15}=\frac{1}{5}$$

Similarly, $P(E/E_2)=\dfrac{{}^4C_1\times{}^2C_1}{{}^9C_2}=\dfrac{4\times 2}{36}=\dfrac{2}{9}$

and $P(E/E_3)=\dfrac{{}^2C_1\times{}^3C_1}{{}^9C_2}=\dfrac{2\times 3}{36}=\dfrac{1}{6}$

Required probability = Probability (2 balls (1 red, 1 black) are drawn from bag B) is given by

$$P(E_2/E)=\frac{P(E_2)\cdot P(E/E_2)}{P(E_1)\cdot P(E/E_1)+P(E_2)\cdot P(E/E_2)+P(E_3)\cdot P(E/E_3)}$$

$$=\frac{\frac{1}{3}\times\frac{2}{9}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{2}{9}+\frac{1}{3}\times\frac{1}{6}}$$

$$=\frac{\frac{1}{3}\times\frac{2}{9}}{\frac{1}{3}\left[\frac{1}{5}+\frac{2}{9}+\frac{1}{6}\right]}=\frac{\frac{2}{9}}{\frac{18+20+15}{90}}$$

$$=\frac{2\times 90}{9\times 53}=\frac{20}{53}$$

AI Q. 9. A factory has three machines *A*, *B* and *C* producing 1500, 2500 and 3000 bulbs per day, respectively. Machine *A* produces 1.5% defective bulbs, machine *B* produces 2% defective bulbs and machine *C* produces 2.5% defective bulbs. At the end of the day, a bulb is drawn at random and is found to be defective. What is the probability that the defective bulb has been produced by machine *B*? [ISC, 2010]

Sol. Given that, total daily production of bulbs

= 1500 + 2500 + 3000 = 7000 bulbs.

Let A, B and C be the event if drawing a bulb produced by machines A, B and C respectively.

$$\therefore \quad P(A)=\frac{1500}{7000}=\frac{3}{14}$$

$$P(B)=\frac{2500}{7000}=\frac{5}{14}$$

and $$P(C)=\frac{3000}{7000}=\frac{3}{7}$$

Let F be the event of producing a defective bulb.

$$P(F/A)=1.5\%=\frac{1.5}{100}=\frac{3}{200}$$

$$P(F/B)=2\%=\frac{2}{100}=\frac{1}{50}$$

$$P(F/C)=2.5\%=\frac{2.5}{100}=\frac{1}{40}$$

By Bayes' theorem,

$$P(B/F)=\frac{P(B)\cdot P(F/B)}{P(A)\cdot P(F/A)+P(B)\cdot P(F/B)+P(C)\cdot P(F/C)}$$

$$=\frac{\frac{5}{14}\times\frac{1}{50}}{\frac{3}{14}\times\frac{3}{200}+\frac{5}{14}\times\frac{1}{50}+\frac{3}{7}\times\frac{1}{40}}$$

$$=\frac{\frac{1}{140}}{\frac{9}{2800}+\frac{1}{140}+\frac{3}{280}}=\frac{20}{59}$$

Q.10. An insurance company insured 4000 doctors, 8000 teachers and 12000 engineers. The probabilities of a doctor, a teacher and an engineer dying before the age of 58 years are 0.01, 0.03 and 0.05 respectively. If one of the insured person dies before the age of 58 years, find the probability that he is a doctor. [ISC, 2009]

Sol. Let E_1, E_2 and E_3 be the events that the person is a doctor, teacher and engineer respectively.

$$P(E_1)=\frac{4000}{4000+8000+12000}=\frac{4000}{24000}=\frac{1}{6}$$

Similarly, $P(E_2)=\frac{1}{3}$, $P(E_3)=\frac{1}{2}$

and $$P\left(\frac{A}{E_1}\right)=0.01,\ P\left(\frac{A}{E_2}\right)=0.03,\ P\left(\frac{A}{E_3}\right)=0.05$$

By Bayes' theorem,

$$P\left(\frac{E_1}{A}\right)=\frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right)+P(E_2)\cdot P\left(\frac{A}{E_2}\right)+P(E_3)\cdot P\left(\frac{A}{E_3}\right)}$$

$$=\frac{0.01\times\frac{1}{6}}{\frac{1}{6}\times0.01+\frac{1}{3}\times0.03+\frac{1}{2}\times0.05}$$

$$=\frac{\frac{0.01}{6}}{\frac{0.01+0.06+0.15}{6}}=\frac{0.01}{0.22}=\frac{1}{22}$$

Q. 11. A class consists of 50 students out of which there are 10 girls. In the class 2 girls and 5 boys are rank holders in an examination. If a student is selected at random from the class and is found to be a rank holder, what is the probability that the student selected is a girl? [ISC, 2008]

Sol. Given, Total number of students = 50

No. of girls = 10

No. of boys = 40

No. of rank holders = 2 girls + 5 boys = 7 students

Let R : student achieve rank in the examination

and G : selecting a girl,

B : selecting a boy,

$$\therefore \quad P(G)=\frac{10}{50}=\frac{1}{5},\quad P(B)=\frac{40}{50}=\frac{4}{5}$$

Also, $P(R/G)=\frac{2}{10}=\frac{1}{5}$ and $P(R/B)=\frac{5}{40}=\frac{1}{8}$

By Bayes' theorem,

$$P(G/R)=\frac{P(R/G)\cdot P(G)}{P(R/G)\cdot P(G)+P(R/B)\cdot P(B)}$$

$$=\frac{\frac{1}{5}\times\frac{1}{5}}{\frac{1}{5}\times\frac{1}{5}+\frac{4}{5}\times\frac{1}{8}}$$

$$=\frac{\frac{1}{25}}{\frac{1}{25}+\frac{1}{10}}=\frac{\frac{1}{25}}{\frac{2+5}{50}}=\frac{2}{7}$$

AI **Q. 12. An insurance company insured 1500 scooter drivers, 2500 car drivers and 4500 truck drivers. The probability of a scooter, a car and a truck meeting with an accident is 0.01, 0.02 and 0.04 respectively. If one of the insured persons meets with an accident, find the probability that he is a scooter driver. [ISC, 2007]**

Sol. Let E_1, E_2 and E_3 be events of insured driver of scooter, car and truck, respectively.

Then, $$P(E_1)=\frac{1500}{8500}=\frac{15}{85}=\frac{3}{17}$$

$$P(E_2)=\frac{2500}{8500}=\frac{5}{17}$$

and $$P(E_3)=\frac{4500}{8500}=\frac{9}{17}$$

Now, probability that the drivers meets with an accident is scooter driver.

$$P(A/E_1) = 0.01$$

Probability that the driver meets with an accident is a car driver $P(A/E_2) = 0.02$

Probability that the driver meets with an accident is a truck drivers,

$$P(A/E_3) = 0.04$$

The probability that scooter driver meets with an accident,

$$P\left(\frac{A}{E_1}\right)=\frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)+P(E_3)\cdot P(A/E_3)}$$

[by using Bayes' theorem]

$$=\frac{\frac{3}{17}\times 0.01}{\frac{3}{17}\times 0.01+\frac{5}{17}\times 0.02+\frac{9}{17}\times 0.04}$$

$$=\frac{\frac{3}{17}\times 0.01}{\frac{1}{17}(3\times 0.01+5\times 0.02+9\times 0.04)}$$

$$=\frac{3\times 0.01}{0.03+0.10+0.36}$$

$$=\frac{0.03}{0.10+0.39}=\frac{0.03}{0.49}=\frac{3}{49}$$

Q. 13. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn at random and are found to be hearts. Find the probability of the missing card to be a heart.

Sol. Let C_1, C_2, C_3, C_4 be the events that the lost card is of heart, spade, diamond or club respectively.

Obviously, $P(C_1)=P(C_2)=P(C_3)=P(C_4)$

$$=\frac{13}{52}=\frac{1}{4}$$

Let S be the event of drawing two cards of heart from the remaining 51 cards. We wish to find $P\left(\frac{C_1}{S}\right)$

Now, $P\left(\frac{S}{C_1}\right)$ is the probability of drawing two heart cards from 51 cards given that one heart card is lost

$$P\left(\frac{S}{C_1}\right)=\frac{{}^{12}C_2}{{}^{51}C_2}=\frac{12\times 11}{1\times 2}\times\frac{1\times 2}{51\times 50}=\frac{22}{425}$$

and $$P\left(\frac{S}{C_2}\right)=P\left(\frac{S}{C_3}\right)=P\left(\frac{S}{C_4}\right)=\frac{{}^{13}C_2}{{}^{51}C_2}$$

$$=\frac{13\times 12}{1\times 2}\times\frac{1\times 2}{51\times 50}=\frac{26}{425}$$

By Bayes' theorem,

$$P\left(\frac{C_1}{S}\right)=\frac{P(C_1)\cdot P\left(\frac{S}{C_1}\right)}{\Sigma P(C_1)\cdot P\left(\frac{S}{C_1}\right)}$$

$$=\frac{\frac{1}{4}\times\frac{22}{425}}{\frac{1}{4}\times\frac{22}{425}+\frac{1}{4}\times\frac{26}{425}+\frac{1}{4}\times\frac{26}{425}+\frac{1}{4}\times\frac{26}{425}}$$

$$=\frac{22}{22+26+26+26}=\frac{11}{50}$$

Q. 14. Assume that the chances of a patient having a heart attack are 40%. Assume that a meditation and yoga course reduces the risk of heart attack by 30% and the prescription of certain drugs and certain restrictions reduces the risk by 25%. At a time, a patient chooses only one of the two options with equal probabilities. After going through one of the two options, the patient is selected at random who is suffering from a heart attack. Find the probability that the patient followed a course of meditation and yoga.

Sol. Let X be the event that the patient follows a course of meditation and yoga.

Let Y be the event that the patient takes certain drugs, then

Given, $P(X)=\frac{1}{2}, P(Y)=\frac{1}{2}$

Let E be the event that patient suffers a heart attack.

$$P(E/X)=\left(40-\frac{30}{100}\times 40\right)\%$$

$$=\frac{40}{100}\left(1-\frac{30}{100}\right)$$

$$=\frac{40}{100}\times\frac{70}{100}$$

$$P(E/Y)=\left(40-\frac{25}{100}\times 40\right)\%$$

$$=\frac{40}{100}\left(1-\frac{25}{100}\right)$$

$$=\frac{40}{100}\times\frac{75}{100}$$

∴ By Bayes' Theorem, we get

$$P\left(\frac{X}{E}\right)=\frac{\left(\frac{70}{100}\times\frac{40}{100}\right)\times\frac{1}{2}}{\left(\frac{70}{100}\times\frac{40}{100}\right)\times\frac{1}{2}+\left(\frac{75}{100}\times\frac{40}{100}\right)\times\frac{1}{2}}$$

$$=\frac{14}{29}$$

Q. 15. A girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin two times and notes the number of heads obtained. If she obtained exactly two heads, what is the probability that she threw 1, 2, 3 or 4 with the die?

Sol. When a die is thrown, sample space

= {1, 2, 3, 4, 5, 6}

Let E_1 = she gets 5 or 6

E_2 = she gets 1, 2, 3 or 4

Then, $P(E_1)=\frac{2}{6}=\frac{1}{3}$ and $P(E_2)=\frac{4}{6}=\frac{2}{3}$

When she gets 5 or 6, she throws a coin three times.

Sample space

= {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Out of these 8 outcomes, exactly two heads are obtained in the 3 outcomes *i.e.*, HHT, HTH, THH.

If A denotes the event of getting exactly two heads, then

$$P(A/E_1)=\frac{3}{8}$$

Again, when she gets 1, 2, 3 or 4, she throws a coin two times.

Sample space = {HH, HT, TH, TT}

$$\therefore \quad P(A/E_2)=\frac{1}{4}$$

Thus, by Bayes' theorem, we have probability that she gets exactly two heads when she threw 1, 2, 3 or 4 with the die

$$P(E_2/A)=\frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$=\frac{\frac{2}{3}\times\frac{1}{4}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{4}}=\frac{\frac{1}{6}}{\frac{1}{8}+\frac{1}{6}}$$

$$=\frac{1}{6}\times\frac{24}{3+4}=\frac{4}{7}$$

❍❍

SELF ASSESSMENT TEST

UNIT–IV

Maximum Marks : 25 **Timing : 1 Hour**

Q. 1. If $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$, then find $P(A \cup B)$.

Q. 2. The probability distribution of the discrete variable X is given as :

X	2	3	4	5
$P(X)$	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

Then find the value of k.

Q. 3. Two dice are thrown. If it is known that the sum of numbers on the dice was less than 6, then find the probability of getting a sum 3.

Q. 4. The random variable X has a probability distribution $P(X)$ of the following form, where 'k' is some number.

$$P(X = x) = \begin{cases} k, \text{ if } x = 0 \\ 2k, \text{ if } x = 1 \\ 3k, \text{ if } x = 2 \\ 0, \text{ otherwise} \end{cases}$$

Q. 5. The probability of A, B and C solving a problem are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$, respectively. If all try and solve the problem simultaneously, find the probability that only one of them will solve it.

Q. 6. A coin is tossed 5 times. What is the probability of getting at least 3 heads?

Q. 7. Three persons A, B and C shoot to hit a target. If in trials, A hits the target 4 times in 5 shots, B hits 3 times in 4 shots and C hits 2 times in 3 trials. Find the probability that :

(i) Exactly two person hit the target.

(ii) At least two persons hit the target.

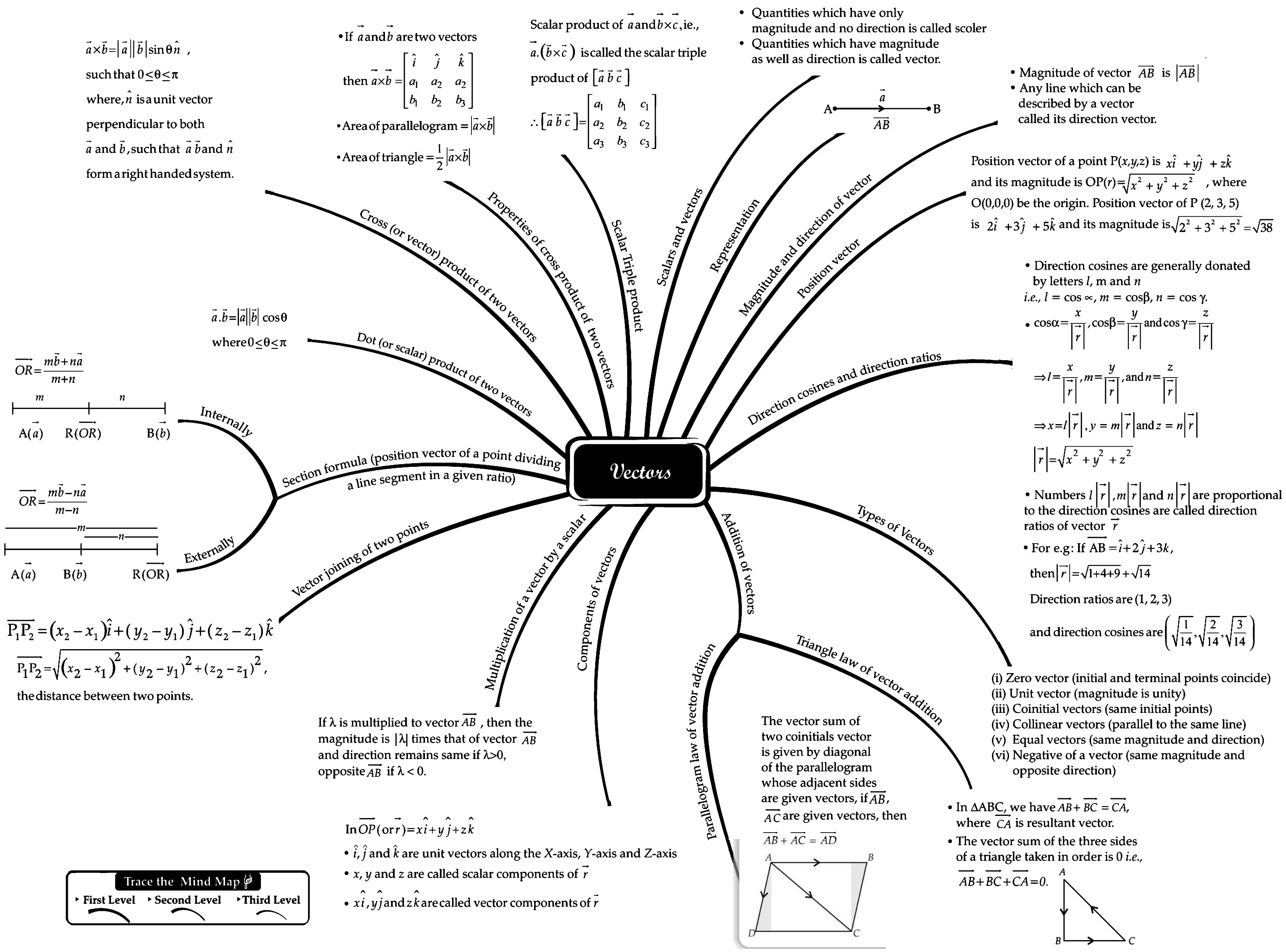
(iii) None hit the target.

Q. 8. An insurance company insured 6000 scooter drivers, 3000 car drivers and 9000 truck drivers. The probability of an accident involving a scooter, a car and a truck is 0.02, 0.06 and 0.30, respectively. One of the insured persons meets with an accident. Find the probability that he is a car driver.

OSWAAL PREPARATION TOOLS

Solutions for Self-assessment Test

Visit : *http://qr.w69b.com/g/pi8gdC0qk* **OR Scan the Code**



Vectors

Cross (or vector) product of two vectors

$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$,

such that $0\le\theta\le\pi$

where, $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}\ \vec{b}$ and $\hat{n}$ form a right handed system.

Properties of cross product of two vectors

- If $\vec{a}$ and $\vec{b}$ are two vectors then $\vec{a}\times\vec{b}=\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_2\\ b_1 & b_2 & b_3\end{bmatrix}$
- Area of parallelogram $=|\vec{a}\times\vec{b}|$
- Area of triangle $=\frac{1}{2}|\vec{a}\times\vec{b}|$

Scalar Triple product

Scalar product of $\vec{a}$ and $\vec{b}\times\vec{c}$, ie., $\vec{a}.(\vec{b}\times\vec{c})$ is called the scalar triple product of $[\vec{a}\ \vec{b}\ \vec{c}]$

$\therefore[\vec{a}\ \vec{b}\ \vec{c}]=\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}$

Scalars and vectors

- Quantities which have only magnitude and no direction is called scoler
- Quantities which have magnitude as well as direction is called vector.

Representation

Magnitude and direction of vector

- Magnitude of vector $\overline{AB}$ is $|\overline{AB}|$
- Any line which can be described by a vector called its direction vector.

Position vector

Position vector of a point P(*x*,*y*,*z*) is $x\hat{i}+y\hat{j}+z\hat{k}$ and its magnitude is $OP(r)=\sqrt{x^2+y^2+z^2}$, where O(0,0,0) be the origin. Position vector of P (2, 3, 5) is $2\hat{i}+3\hat{j}+5\hat{k}$ and its magnitude is $\sqrt{2^2+3^2+5^2}=\sqrt{38}$

Direction cosines and direction ratios

- Direction cosines are generally donated by letters *l*, m and *n*

i.e., $l=\cos\propto$, $m=\cos\beta$, $n=\cos\gamma$.

- $\cos\alpha=\frac{x}{|\vec{r}|}$, $\cos\beta=\frac{y}{|\vec{r}|}$ and $\cos\gamma=\frac{z}{|\vec{r}|}$

$\Rightarrow l=\frac{x}{|\vec{r}|}, m=\frac{y}{|\vec{r}|}$, and $n=\frac{z}{|\vec{r}|}$

$\Rightarrow x=l|\vec{r}|, y=m|\vec{r}|$ and $z=n|\vec{r}|$

$|\vec{r}|=\sqrt{x^2+y^2+z^2}$

- Numbers $l|\vec{r}|, m|\vec{r}|$ and $n|\vec{r}|$ are proportional to the direction cosines are called direction ratios of vector $\vec{r}$
- For e.g: If $\overrightarrow{AB}=\hat{i}+2\hat{j}+3k$,

then $|\vec{r}|=\sqrt{1+4+9}+\sqrt{14}$

Direction ratios are (1, 2, 3)

and direction cosines are $\left(\sqrt{\frac{1}{14}},\sqrt{\frac{2}{14}},\sqrt{\frac{3}{14}}\right)$

Types of Vectors

(i) Zero vector (initial and terminal points coincide)
(ii) Unit vector (magnitude is unity)
(iii) Coinitial vectors (same initial points)
(iv) Collinear vectors (parallel to the same line)
(v) Equal vectors (same magnitude and direction)
(vi) Negative of a vector (same magnitude and opposite direction)

Addition of vectors

Triangle law of vector addition

- In ΔABC, we have $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CA}$, where $\overrightarrow{CA}$ is resultant vector.
- The vector sum of the three sides of a triangle taken in order is 0 *i.e.*, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0.$

Parallelogram law of vector addition

The vector sum of two coinitials vector is given by diagonal of the parallelogram whose adjacent sides are given vectors, if $\overrightarrow{AB}$, $\overrightarrow{AC}$ are given vectors, then

$\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AD}$

Components of vectors

In $\overrightarrow{OP}(\text{or }\vec{r})=x\hat{i}+y\hat{j}+z\hat{k}$

- $\hat{i},\hat{j}$ and $\hat{k}$ are unit vectors along the *X*-axis, *Y*-axis and *Z*-axis
- *x*, *y* and *z* are called scalar components of $\vec{r}$
- $x\hat{i}, y\hat{j}$ and $z\hat{k}$ are called vector components of $\vec{r}$

Multiplication of a vector by a scalar

If λ is multiplied to vector $\overline{AB}$, then the magnitude is |λ| times that of vector $\overline{AB}$ and direction remains same if λ>0, opposite $\overline{AB}$ if λ < 0.

Vector joining of two points

$\overline{P_1P_2}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$

$\overline{P_1P_2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$,

the distance between two points.

Section formula (position vector of a point dividing a line segment in a given ratio)

Internally

$\overrightarrow{OR}=\frac{m\vec{b}+n\vec{a}}{m+n}$

Externally

$\overrightarrow{OR}=\frac{m\vec{b}-n\vec{a}}{m-n}$

Dot (or scalar) product of two vectors

$\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

where $0\le\theta\le\pi$

Trace the Mind Map
- First Level
- Second Level
- Third Level

UNIT-V
Vectors

CHAPTER
10
VECTORS

SYLLABUS

- *Vectors and Scalars, Magnitude and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, Components of a vector, addition of vectors, Multiplication of a vector by a scalar, Position vector of a point dividing a line segment in a given ratio. Definition, Geometrical Interpretation, properties and application of scalar (dot) product of vectors, vector (cross) product of vectors, scalar triple product of vectors.*
  - *As directed line segments.*
  - *Magnitude and direction of a vector.*
  - *Types : equal vectors, unit vectors, zero vector.*
  - *Position vector.*
  - *Components of a vector.*
  - *Vectors in two and three dimension.*
  - *$\hat{i}, \hat{j}, \hat{k}$ as unit vector along the x, y and the z axes; expressing a vector in terms of the unit vectors.*
  - *Operations : Sum and Difference of vectors; Scalar multiplication of a vector.*
  - *Section formula.*
  - *Triangle inequalities.*
  - *Scalar (dot) product of vectors and its geometrical significance*
  - *Cross product — its properties - area of a triangle, area of parallelogram, collinear vectors.*
  - *Scalar triple product — volume of a parallelepiped, coplanarity.*

Note : Proofs of geometrical theorems by using vector algebra are excluded.

Trend Analysis

List of Concepts	2016	2017	2018	2019	2020
Basic Algebra of Vectors and Dot Product		1 Q (6 marks)	1 Q (2 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (2 marks)
Cross Product	1 Q (6 marks)		1 Q (4 marks)		1 Q (4 marks)
Scalar Triple Product	1 Q (6 marks)	1 Q (6 marks)	1 Q (4 marks)	OR 1 Q (4 marks)	OR 1 Q (4 marks)

TOPIC-1
Basic Algebra of Vectors and Dot Product

TOPIC - 1
Basic Algebra of Vectors and Dot Product .... **P. 279**
TOPIC - 2
Cross Product .... **P. 289**
TOPIC - 3
Scalar Triple Product .... **P. 295**

Revision Notes

- **Vector and their Representation**

  A quantity having magnitude as well as the direction is called a vector. It is denoted by a directed line segment as $\vec{AB}$ or $\vec{a}$. Its magnitude (or modulus) is $|\vec{AB}|$ or $|\vec{a}|$ otherwise, simply AB or a.

A is called the initial point and B is called the terminal point.

- **Types of vectors :**

- **Zero or Null vector :** It is the vector whose initial and terminal points are coincident. It is denoted by $\vec{0}$. Its magnitude is 0 (zero).

  Note : Any non-zero vector is called a **proper vector.**

- **Negative of a vectors :** The vector which has the same magnitude as the $\vec{r}$ but opposite direction. It is denoted by $-\vec{r}$. Hence if, $\overrightarrow{AB}=\vec{r} \Rightarrow \overrightarrow{BA}=-\vec{r}$ *i.e.,* $\overrightarrow{AB}=-\overrightarrow{BA}$, $\overrightarrow{PQ}=-\overrightarrow{QP}$ etc.

Scan to know more about this topic

Types of vectors

- **Unit vector :** It is a vector with the unit magnitude, the unit vector $\vec{r}$ in the direction of vector $\vec{r}$ is given by $\hat{r}=\frac{\vec{r}}{|\vec{r}|}$, such that $|\hat{r}|=1$. So, if $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then its unit vector is :

$$\hat{r}=\frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i}+\frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j}+\frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

Scan to know more about this topic

Basics of vectors

- **Equal vectors :** Two vectors are said to be equal, if they have the same magnitude as well as direction, regardless of the position of their initial points,

Thus ,
$$\vec{a}=\vec{b} \Leftrightarrow \begin{cases} |\vec{a}|=|\vec{b}| \\ \vec{a} \text{ and } \vec{b} \text{ have same direction} \end{cases}$$

Also, if $\vec{a}=\vec{b} \Rightarrow a_1\hat{i}+a_2\hat{j}+a_3\hat{k}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k} \Rightarrow a_1=b_1,\ a_2=b_2,\ a_3=b_3$.

Scan to know more about this topic

Dot product of vectors

- **Collinear or Parallel vectors :** Two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel, if there exists a non-zero scalar λ such that $\vec{a}=\lambda\vec{b}$

  - It is important to note that the respective coefficients of $\hat{i},\hat{j},\hat{k}$ in $\vec{a}$ and $\vec{b}$ are proportional provided they are parallel or collinear to each other.
  - The d.r's of parallel vectors are same (or are in proportion).
  - The vectors $\vec{a}$ and $\vec{b}$ will have same or opposite direction as λ is positive or negative respectively.

- **Position Vector :**

The position vector of a point say $P(x,y,z)$ is $\overrightarrow{OP}=\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and the magnitude is $|\vec{r}|=\sqrt{x^2+y^2+z^2}$.

The vector $\overrightarrow{OP}=\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ is said to be in its **component form.** Here, *x, y, z* are called the scalar components or rectangular components of $\vec{r}$ and $x\hat{i},\ y\hat{j},\ z\hat{k}$ are the vector components of $\vec{r}$ along *x, y, z*-axis, respectively.

- Also, $\overrightarrow{AB}=$ (Position vector of B) – (Position vector of A).

For example, let $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$. Then, $\overrightarrow{AB}=(x_2\hat{i}+y_2\hat{j}+z_2\hat{k})-(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})$

Here, $\hat{i},\hat{j}$ and $\hat{k}$ are the unit vectors along the axes *OX, OY* and *OZ* respectively.

- **Direction Ratio and Direction Cosine :**

If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then coefficient of $\hat{i},\hat{j},\hat{k}$ in $\vec{r}$ *i.e., x, y, z* are called the direction ratio (abbreviated as. d.r's) of vector $\vec{r}$. These are denoted by *a, b, c* (*i.e., a = x, b=y, c=z;* in a manner we can say that scalar components of vector $\vec{r}$ and its d.r.'s both are the same).

Also, the coefficients of $\hat{i},\hat{j},\hat{k}$ in $\vec{r}$ (which is the unit vector of $\vec{r}$) *i.e.,* $\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}$ are called direction cosines (which is abbreviated as d.c.'s) of vector $\vec{r}$.

- These direction cosines are denoted by *l, m, n* such that $l=\cos\alpha, m=\cos\beta, n=\cos\gamma$ and $l^2+m^2+n^2=1$ $\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.
- It can be easily concluded that $\vec{r}=lr\hat{i}+mr\hat{j}+nr\hat{k}=r(\cos\alpha\hat{i}+\cos\beta\hat{j}+\cos\gamma\hat{k})$. [Here $r=|\vec{r}\,|$]

➢ **Addition of vectors :**

(a) Triangular law: If two adjacent sides (say sides *AB* and *BC*) of a triangle *ABC* are represented by $\vec{a}$ and $\vec{b}$ taken in same order, then the third side of the triangle taken in the reverse order gives the sum of vectors $\vec{a}$ and $\vec{b}$ *i.e.*, $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC} \Rightarrow \overrightarrow{AC}=\vec{a}+\vec{b}$.

Also, since $\overrightarrow{AC}=-\overrightarrow{CA} \Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{AA}=\vec{0}$.

(b) Parallelogram law : If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and the direction by the two adjacent sides (say *AB* and *AD*) of a parallelogram *ABCD*, then their sum is given by that diagonal of parallelogram which is co-initial with $\vec{a}$ and $\vec{b}$ *i.e.*, $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{OB}$.

➢ **Properties of Vector Addition :**

(a) Commutative property : $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

Consider $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ be any two given vectors,

Then $\vec{a}+\vec{b}=(a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}=\vec{b}+\vec{a}$

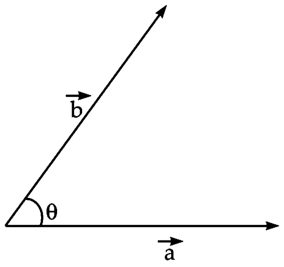
- **Associative property :** $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$
- **Additive identity :** $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$
- **Additive inverse :** $(-\vec{a})+\vec{a}=\vec{0}\ +(-\vec{a})=\vec{0}$

➢ **Multiplication of a Vector by a Scalar :** Let $\vec{a}$ be any vector and *k* be any scalar. Then the product $k\vec{a}$ is defined as a vector whose magnitude is $|k|$ times that of $\vec{a}$ and the direction is

(i) Same as that of $\vec{a}$ if *k* is positive and

(ii) Opposite that of $\vec{a}$, if *k* is negative

➢ **Scalar Product or Dot Product :** The dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined by $\vec{a}\cdot\vec{b}=|\vec{a}\,||\,\vec{b}\,|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0\le\theta\le\pi$



Consider $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$,

then $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$.

➢ **Properties of Dot product :**

- $\hat{i}\cdot\hat{i}=|\hat{i}|\,|\hat{i}|\cos 0=1 \Rightarrow \hat{i}\cdot\hat{i}=1=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}$
- $\hat{i}\cdot\hat{j}=|\hat{i}|\,|\hat{j}|\cos\frac{\pi}{2}=0 \Rightarrow \hat{i}\cdot\hat{j}=0=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}$
- $\vec{a}\cdot\vec{b}\in R$, where *R* is real number *i.e.*, any scalar.

- $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (Commutative property of dot product).
- $\vec{a}\cdot\vec{b}=0 \Leftrightarrow 0\,\vec{a}\perp\vec{b}$ **or** $|\vec{a}|=0$ or $|\vec{b}|=0$
- If $\theta=0$ then $\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|$. Also, $\vec{a}\cdot\vec{a}=|\vec{a}|^2=a^2$ as θ in this case is 0.
- Moreover if $\theta=\pi$, then $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$.
- $\vec{a}\cdot\left(\vec{b}+\vec{c}\right)=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (Distributive property of dot product).
- $\vec{a}\cdot\left(-\vec{b}\right)=-\left(\vec{a}\cdot\vec{b}\right)=\left(-\vec{a}\right)\cdot\vec{b}$

➤ **Projection of Vectors :**

- Projection of a vector $\vec{a}$ on the other vector say $\vec{b}$ is given as $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)$.
- Projection of a vector $\vec{b}$ on the other vector say $\vec{a}$ is given as $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)$.

Know the Formulae

➤ **Section Formula**

The position vector of a point say *P* dividing a line segment joining the points *A* and *B* whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio *m* : *n*.

(a) internally, $\overrightarrow{OP}=\dfrac{m\vec{b}+n\vec{a}}{m+n}$ **(b)** externally, $\overrightarrow{OP}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$

- Also, if point *P* is the mid-point of line segment *AB*, then $\overrightarrow{OP}=\dfrac{\vec{a}+\vec{b}}{2}$

➤ **Angle between two vectors $\vec{a}$ and $\vec{b}$ can be found by the expression given below :**

$$\cos\theta=\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right) \quad \text{or} \quad \theta=\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

1 Mark Questions

Q. 1. If the vectors $a\hat{i}+3\hat{j}-2\hat{k}$ and $3\hat{i}-4\hat{j}+b\hat{k}$ are collinear, then (*a*, *b*) =

(a) $\left(\frac{9}{4},\frac{8}{3}\right)$ **(b)** $\left(-\frac{9}{4},\frac{8}{3}\right)$

(c) $\left(\frac{9}{4},-\frac{8}{3}\right)$ **(d)** $\left(-\frac{9}{4},-\frac{8}{3}\right)$

[ISC Specimen Paper, 2021]

Sol. Correct Option : (b)

Explanation :

Given that :

$a\hat{i}+3\hat{j}-2\hat{k}$ and $3\hat{i}-4\hat{j}+b\hat{k}$ are collinear

$$\therefore \quad \frac{a}{3}=\frac{3}{-4}=\frac{-2}{b}$$

$$\therefore \quad a=-\frac{9}{4} \text{ and } b=\frac{8}{3}$$

$$\therefore \quad (a>b)=\left(-\frac{9}{4},\frac{8}{3}\right)$$

AI Q. 2. If $\vec{a}$ is a non-zero vector of magnitude '*a*' and λ a non-zero scalar, then $\lambda\vec{a}$ is unit vector if

(a) $\lambda=1$ **(b)** $\lambda=-1$

(c) $a=|\lambda|$ **(d)** $a=1/|\lambda|$

Sol. Correct option : (d)

Explanation :

Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}|=1$.

Now,

$$|\lambda\vec{a}|=1$$

$$\Rightarrow |\lambda||\vec{a}| = 1$$

$$\Rightarrow \quad |\vec{a}| = \frac{1}{|\lambda|} \qquad [\lambda \neq 0]$$

$$\Rightarrow \quad a = \frac{1}{|\lambda|} \qquad [|\vec{a}| = a]$$

So that, vector $\lambda\vec{a}$ is a unit vector if $a = \frac{1}{|\lambda|}$.

Q. 3. The vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 is

(a) $\hat{i} - 2\hat{j} + 2\hat{k}$ **(b)** $\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$

(c) $3(\hat{i} - 2\hat{j} + 2\hat{k})$ **(d)** $9(\hat{i} - 2\hat{j} + 2\hat{k})$

Sol. Correct option : (c)

Explanation :

Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Any vector in the direction of a vector $\vec{a}$ is given by $\frac{\vec{a}}{|\vec{a}|}$.

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$\therefore$ Vector in the direction of $\vec{a}$ with magnitude 9.

$$= 9\,\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$$= 3(\hat{i} - 2\hat{j} + 2\hat{k})$$

Q. 4. The position vector of the point which divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$ in the ratio 3 : 1 is :

(a) $\frac{3\vec{a} - 2\vec{b}}{2}$ **(b)** $\frac{7\vec{a} - 8\vec{b}}{4}$

(c) $\frac{3\vec{a}}{4}$ **(d)** $\frac{5\vec{a}}{4}$

Sol. Correct option : (d)

Explanation :

Let the position vector of the R divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$.

$\therefore$ Position vector, $R = \frac{3(\vec{a} + \vec{b}) + 1(2\vec{a} - 3\vec{b})}{3 + 1}$

Since, the position vector of a point R dividing the line segments joining the points P and Q, whose position vectors are $\vec{p}$ and $\vec{q}$ in the ration $m : n$ internally, is given by $\frac{m\vec{q} + n\vec{p}}{m + n}$.

$\therefore R = \frac{5\vec{a}}{4}$.

AI **Q. 5. Find the angle between the vectors $\vec{a} = 6\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - 9\hat{j} + 6\hat{k}$**

[ISC Specimen Paper, 2021]

Sol.
$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$$

$$\cos\theta = \frac{12 - 18 + 18}{\sqrt{6^2 + 2^2 + 3^2}.\sqrt{2^2 + (-a)^2 + 6^2}}$$

$$\therefore \quad \cos\theta = \frac{12}{77}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{12}{77}\right)$$

Q. 6. Write a unit vector in the direction of the sum of vectors $\vec{a} = 2\vec{i} + 2\vec{j} - 5\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} - 7\vec{k}$.

Sol. Let
$$\vec{r} = \vec{a} + \vec{b} = 4\vec{i} + 3\vec{j} - 12\vec{k}$$

$$|\vec{r}| = \sqrt{16 + 9 + 144} = \sqrt{169} = 13$$

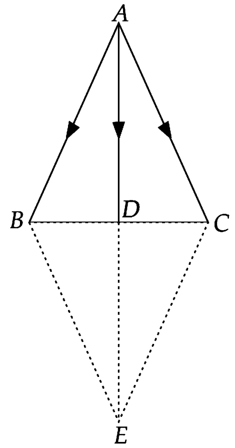
So, unit vector

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{13}$$

$$= \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k}$$

AI **Q. 7. The two vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the two sides *AB* and *AC*, respectively of ΔABC. Find the length of the median through *A*.**

Sol. $\vec{AB} = \hat{j} + \hat{k}$ and $\vec{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$



Now *ABEC* represent a parallelogram with *AE* as the diagonal.

$$\vec{AE} = \vec{AB} + \vec{AC}$$

$$= (\hat{j} + \hat{k}) + (3\hat{i} - \hat{j} + 4\hat{k}) = 3\hat{i} + 5\hat{k}$$

Now,
$$|\vec{AE}| = \sqrt{(3)^2 + (5)^2} = \sqrt{9 + 25} = \sqrt{34}$$

$$\therefore \quad |\vec{AD}| = \frac{1}{2}\sqrt{34} \text{ units}$$

Q. 8. Write the value of $\vec{p}$ for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel vectors.

Sol.
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \quad \text{[For parallel vectors]}$$

or
$$\frac{3}{1} = \frac{2}{-2p} = \frac{9}{3}$$

or
$$p = -\frac{1}{3}$$

2 Marks Questions

AI Q. 1. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$.
[ISC Specimen Paper, 2021] [ISC, 2019]

Sol. $\vec{a} \perp \vec{b}$

$$\Rightarrow \quad \vec{a}.\vec{b} = 0 \ \& \ |\vec{a}| = 5$$

Now, $|\vec{a}+\vec{b}| = 13$

$$\therefore \quad |\vec{a}+\vec{b}|^2 = 169$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}.\vec{b}) = 169$$

$$\Rightarrow \quad 5^2 + |\vec{b}|^2 + 2.0 = 169$$

$$\therefore \quad |\vec{b}| = 12$$

[ISC Marking Scheme, 2019]

Examiner's Comment

- Majority of the candidates performed well. However, a few candidates applied incorrect formula for perpendicular vectors.

Answering Tip

- Understand the concepts of dot product and cross product of two vectors and their properties.

AI Q. 2. Find λ if the scalar projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.
[ISC Specimen Paper, 2021]

Sol. A.T.Q, $\dfrac{2\lambda + 6 + 12}{\sqrt{2^2 + 6^2 + 3^2}} = 4$

$$\Rightarrow \quad 2\lambda + 18 = 28$$

$$\therefore \quad \lambda = 5$$

Examiner's Comment

- Most of the candidates calculated dot product instead of applying projection formula.

Answering Tip

- Clarify the concept of scalar projection of vector thoroughly.

Q 3. Write a vector of magnitude of 18 units in the direction of the vector $\hat{i} - 2\hat{j} - 2\hat{k}$.
[ISC, 2020]

Sol. $\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$

$$|\vec{a}| = \left|\sqrt{1^2 + (-2)^2 + (-2)^2}\right|$$

$$= \left|\sqrt{1+4+4}\right| = 3$$

So, its Unit Vector, $\vec{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\hat{i} - 2\hat{j} - 2\hat{k}}{3}$

$$18\vec{a} = 18 \times \frac{\hat{i} - 2\hat{j} - 2\hat{k}}{3}$$

$$= 6(\hat{i} - 2\hat{j} - 2\hat{k})$$

Required vector $= 6\hat{i} - 12\hat{j} - 12\hat{k}$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Most of candidates know how to find the vector in the direction of the given vector of required magnitude but some of them only find the unit vector of the given vector.

Answering Tip

- Understand the difference between the unit vector and the vector in the direction of the given vector of required magnitude.

Q. 4. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$ making an angle of $\dfrac{\pi}{4}$ with x-axis, $\dfrac{\pi}{2}$ with y-axis and an acute angle θ with z-axis.

Sol. Let $\hat{a}$ be the unit vector in the direction of vector $\vec{a}$. Since vector makes an angle of $\dfrac{\pi}{4}$ with x-axis, $\dfrac{\pi}{2}$ with y-axis and an acute angle θ with z-axis .

$$\therefore \quad \cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{2} + \cos^2\theta = 1$$

[Using $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$]

$$\Rightarrow \quad \frac{1}{2} + 0 + \cos^2\theta = 1$$

$$\Rightarrow \quad \cos^2\theta = \frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\therefore \quad \vec{a} = 5\sqrt{2}\hat{a}$$

$$= 5\sqrt{2}\left(\cos\frac{\pi}{4}\hat{i} + \cos\frac{\pi}{2}\hat{j} + \cos\frac{\pi}{4}\hat{k}\right)$$

$$= 5\hat{i} + 5\hat{k}$$

Q. 5. *P* and *Q* are two points with position vectors $3\vec{a}-2\vec{b}$ and $\vec{a}+\vec{b}$ respectively. Write the position vector of a point *R* which divides the line segment *PQ* externally in the ratio 2 : 1.

Sol. Consider two points *P* and *Q* with position vectors $\vec{OP}=3\vec{a}-2\vec{b}$ and $\vec{OQ}=\vec{a}+\vec{b}$ then position vector of the point *R* dividing the line joining of *P* and *Q* externally in the ratio 2 : 1 is

$$\vec{OR}=\frac{2(\vec{a}+\vec{b})-(3\vec{a}-2\vec{b})}{2-1}$$

$$=4\vec{b}-\vec{a}$$

AI **Q. 6.** Write the value of λ so that the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other ?

Sol. Given, $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$

and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$

For perpendicular,

$$\vec{a}\cdot\vec{b}=0$$

$$\Rightarrow\quad 2\times1+\lambda(-2)+1\times3=0$$

$$\Rightarrow\quad 2\lambda=5$$

$$\Rightarrow\quad \lambda=\frac{5}{2}$$

Q. 7. Find $|\vec{x}|$, if for a unit vector $\vec{a}$ and $(\vec{x}+\vec{a})\cdot(\vec{x}-\vec{a})=15$

Sol. $|\vec{a}|=1$

Given, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=15$

$$\Rightarrow\quad \vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}+\vec{a}\cdot\vec{a}=15$$

$$|\vec{x}|^2-|a|^2=15$$

$$\Rightarrow\quad |\vec{x}|^2-1=15$$

$$|\vec{x}|^2=16\Rightarrow|\vec{x}|=\pm4$$

As magnitude of a vector is non-negative.

So $|\vec{x}|=4$

4 Marks Questions

AI **Q. 1.** If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, find the value of x such that the vectors $\vec{\alpha}=(x-2)\vec{a}+\vec{b}$ and $\vec{\beta}=(3+2x)\vec{a}-2\vec{b}$ are collinear. **[ISC, 2019]**

Sol. Since $\vec{\alpha}$ and $\vec{\beta}$ are collinear

$$\vec{\alpha}=k\vec{\beta}$$

$$(x-2)\vec{a}+\vec{b}=k[3+2x)\vec{a}-2\vec{b}]$$

$$x-2=k[3+2x]$$

$$1=-2k,\ k=-\frac{1}{2}$$

$$x-2=-\frac{1}{2}(3+2x)$$

$$2x-4=-3-2x$$

$$4x=1$$

$$x=\frac{1}{4}$$

[ISC Marking Scheme, 2019]

Examiner's Comment

- Many candidates had no idea of the concept of the collinear vectors and coplanar vectors. Some candidates made errors while comparing the components of collinear vectors.

Answering Tip

- Understand the concepts like collinear vectors, coplanar vectors and linear combination of vectors, etc.
- Do adequate practice of problems based on this concept.

Q. 2. If $\vec{a}$ and $\vec{b}$ are two vector such that $|\vec{a}+\vec{b}|=|\vec{a}|$, then show that vector $|2\vec{a}+\vec{b}|$ is perpendicular to $\vec{b}$.

Sol. Given that, $|\vec{a}+\vec{b}|=|\vec{a}|$

$$\therefore\quad |\vec{a}+\vec{b}|^2=|\vec{a}|^2$$

(Squaring both sides)

$$\Rightarrow\quad |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{a}|^2$$

$$\Rightarrow\quad |\vec{b}|^2+2\vec{a}\cdot\vec{b}=0 \qquad \text{... (1)}$$

To show that $(2\vec{a}+\vec{b})\cdot(\vec{b})=0$

Consider $(2\vec{a}+\vec{b})\cdot(\vec{b})=2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}$

$$=2\vec{a}\cdot\vec{b}+|\vec{b}|^2$$

$$=0 \qquad \text{(From (i))}$$

$$\therefore\quad 2\vec{a}+\vec{b}\ \text{is}\perp\text{to}\ \vec{b}$$

Q. 3. The position vectors of the vertices of a triangle are given as $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$. Prove that it is a right angled triangle.

Sol. Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$

$\vec{b}=\hat{i}-3\hat{j}-5\hat{k}$

and $\vec{c}=3\hat{i}-4\hat{j}-4\hat{k}$

be the position vectors of A, B and C.

Then, $\overrightarrow{AB} = \vec{b} - \vec{a}$

$= \hat{i} - 3\hat{j} - 5\hat{k} - (2\hat{i} - \hat{j} + \hat{k})$

$= -\hat{i} - 2\hat{j} - 6\hat{k}$

$\overrightarrow{BC} = \vec{c} - \vec{b}$

$= 3\hat{i} - 4\hat{j} - 4\hat{k} - (\hat{i} - 3\hat{j} - 5\hat{k})$

$= 2\hat{i} - \hat{j} + \hat{k}$

and $\overrightarrow{AC} = \vec{c} - \vec{a}$

$= 3\hat{i} - 4\hat{j} - 4\hat{k} - (2\hat{i} - \hat{j} + \hat{k})$

$= \hat{i} - 3\hat{j} - 5\hat{k}$

Now, $\overrightarrow{AB} + \overrightarrow{BC} = -\hat{i} - 2\hat{j} - 6\hat{k} + 2\hat{i} - \hat{j} + \hat{k}$

$= \hat{i} - 3\hat{j} - 5\hat{k}$

$= \overrightarrow{AC}$

$\therefore$ $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AC}$ form a triangle.

Now, $\overrightarrow{AB} \cdot \overrightarrow{BC} = (-\hat{i} - 2\hat{j} - 6\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})$

$= (-1)(2) + (-2)(-1) + (-6)(1)$

$= -2 + 2 - 6 = -6 \neq 0$

Consider $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k})$

$= (2)(-1) + (-1)(3) + (1)(5)$

$= -2 - 3 + 5$

$= 0$

$\Rightarrow$ $\overrightarrow{BC} \cdot \overrightarrow{CA} = 0$

$\Rightarrow$ $\overrightarrow{BC} \perp \overrightarrow{CA}$

$\Rightarrow$ ΔABC is a right angled triangle.

AI **Q. 4. The vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular. Given that $|\vec{a}| = |\vec{b}|$. Then, find the values of x and y.**

Sol. Given that $\vec{a}$ and $\vec{b}$ are mutually perpendicular.

$\Rightarrow$ $\vec{a} \cdot \vec{b} = 0$

$\Rightarrow$ $(3\hat{i} + x\hat{j} - \hat{k}) \cdot (2\hat{i} + \hat{j} + y\hat{k}) = 0$

$\Rightarrow$ $(3)(2) + (x)(1) + (-1)(y) = 0$

$\Rightarrow$ $6 + x - y = 0$

$\Rightarrow$ $x - y = -6$ ...(i)

Also, given that

$|\vec{a}| = |\vec{b}|$

$\Rightarrow$ $|\vec{a}|^2 = |\vec{b}|^2$

$\Rightarrow$ $9 + x^2 + 1 = 4 + 1 + y^2$

$\Rightarrow$ $x^2 - y^2 = 5 - 10$

$\Rightarrow$ $x^2 - y^2 = -5$ ...(ii)

From (i), $y = x + 6$

Substitute value of y in (ii),

$x^2 - (x + 6)^2 = -5$

$\Rightarrow$ $x^2 - x^2 - 12x - 36 = -5$

$\Rightarrow$ $-12(x + 3) = -5$

$\Rightarrow$ $x = \frac{5}{12} - 3$

$\Rightarrow$ $x = \frac{-31}{12}$

Substitute in $y = x + 6$

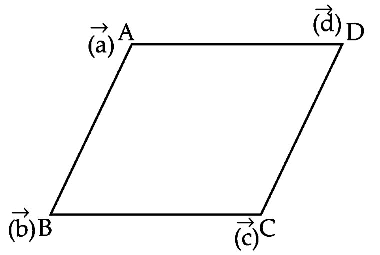
We get, $y = \frac{-31}{12} + 6$

$\Rightarrow$ $y = \frac{41}{12}$

$\therefore$ $x = \frac{-31}{12}$ and $y = \frac{41}{12}$

Q. 5. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of four points A, B, C and D respectively and if $\vec{b} - \vec{a} = \vec{c} - \vec{d}$, show that $ABCD$ is a parallelogram.

Sol. Given that $\vec{b} - \vec{a} = \vec{c} - \vec{d}$



$\Rightarrow$ $\overrightarrow{AB} = \overrightarrow{DC}$

Also, $|\overrightarrow{AB}| = |\overrightarrow{DC}|$

(Taking magnitudes on both sides)

$\therefore$ $\overrightarrow{AB} \,||\, \overrightarrow{DC}$ and $|\overrightarrow{AB}| = |\overrightarrow{DC}|$

Opposite sides are equal and parallel

$\therefore$ $ABCD$ is a parallelogram.

Q. 6. If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of λ so that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular vectors.

Sol. Given, $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$

and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$

$$\vec{a}+\vec{b}=(\hat{i}-\hat{j}+7\hat{k})+(5\hat{i}-\hat{j}+\lambda\hat{k})$$

$$=6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}$$

and $$\vec{a}-\vec{b}=(\hat{i}-\hat{j}+7\hat{k})-(5\hat{i}-\hat{j}+\lambda\hat{k})$$

$$=-4\hat{i}+(7-\lambda)\hat{k}$$

Since, $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular vectors,

$\therefore$ $$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$$

$$\{6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}\}\cdot\{-4\hat{i}+(7-\lambda)\hat{k}\}=0$$

$\Rightarrow$ $$-24+(7+\lambda)(7-\lambda)=0$$

$\Rightarrow$ $$49-\lambda^2-24=0$$

$\Rightarrow$ $$\lambda^2=49-24=25$$

$\Rightarrow$ $$\lambda=\pm 5 \text{ units}$$

AI Q. 7. Vectors $\vec{a},\vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=0$ and $|\vec{a}|=3, |\vec{b}|=5$ and $|\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$.

Sol. $\therefore$ $\vec{a}+\vec{b}+\vec{c}=0,\quad \vec{a}+\vec{b}=-\vec{c}$

$$|\vec{a}+\vec{b}|^2=|-\vec{c}|^2=|\vec{c}|^2$$

$\Rightarrow$ $$|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{c}|^2$$

$\Rightarrow$ $$9+25+2|\vec{a}||\vec{b}|\cos\theta=49$$

Where θ being angle between $\vec{a}$ and $\vec{b}$.

$\therefore$ $$\cos\theta=\frac{15}{2\cdot 3\cdot 5}=\frac{1}{2}$$

$\Rightarrow$ $$\theta=\frac{\pi}{3}$$

Q. 8. If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}|=3, |\vec{b}|=4$ and $|\vec{c}|=5$ and each one of them is perpendicular to the sum of the other two, then find $|\vec{a}+\vec{b}+\vec{c}|$.

Sol. Since $\vec{a}\perp(\vec{b}+\vec{c})$,

$\therefore$ $$\vec{a}\cdot(\vec{b}+\vec{c})=0$$

$\Rightarrow$ $$\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$$

$\Rightarrow$ $$\vec{a}\cdot\vec{a}+(\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c})=\vec{a}\cdot\vec{a}+0$$

$\Rightarrow$ $$\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})=|\vec{a}|^2$$

$\Rightarrow$ $$\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})=3^2=9 \qquad \text{...(i)}$$

Similarly, $$\vec{b}\cdot(\vec{a}+\vec{b}+\vec{c})=|\vec{b}|^2=16 \qquad \text{...(ii)}$$

and $$\vec{c}\cdot(\vec{a}+\vec{b}+\vec{c})=|\vec{c}|^2=25 \qquad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})+\vec{b}\cdot(\vec{a}+\vec{b}+\vec{c})+\vec{c}\cdot(\vec{a}+\vec{b}+\vec{c})=50$$

$\Rightarrow$ $$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=50$$

$\Rightarrow$ $$|\vec{a}+\vec{b}+\vec{c}|^2=50$$

$\Rightarrow$ $$|\vec{a}+\vec{b}+\vec{c}|=\sqrt{50}=5\sqrt{2}$$

6 Marks Questions

AI Q. 1. If $\vec{a},\vec{b},\vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined with vectors $\vec{a},\vec{b}$ and $\vec{c}$.

[ISC, 2017]

Sol. $\vec{a},\vec{b}$ and $\vec{c}$ are mutually perpendicular.

$\therefore$ $$\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$$

$$|\vec{a}|=|\vec{b}|=|\vec{c}|=k \text{ (constant)}$$

$$|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$=\vec{a}.\vec{a}+\vec{b}.\vec{b}+\vec{c}.\vec{c}+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$$

$$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2=k^2+k^2+k^2=3k^2$$

$$|\vec{a}+\vec{b}+\vec{c}|=\sqrt{3}k$$

Let $(\vec{a}+\vec{b}+\vec{c})$ make angles α, β and γ with vectors $\vec{a},\vec{b}$ and $\vec{c}$ respectively, then

$$(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}=|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{a}|\cos\alpha$$

$$(\vec{a}+\vec{b}+\vec{c})\cdot\vec{b}=|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{b}|\cos\beta$$

$$(\vec{a}+\vec{b}+\vec{c})\cdot\vec{c}=|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{c}|\cos\gamma$$

$$\cos\alpha = \frac{\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}}{\sqrt{3}\,k.k} = \frac{k^2}{\sqrt{3}k^2} = \frac{1}{\sqrt{3}}$$

$$\cos\beta = \frac{\vec{b}.\vec{a}+\vec{b}.\vec{b}+\vec{b}.\vec{c}}{\sqrt{3}\,k.k} = \frac{k^2}{\sqrt{3}k^2} = \frac{1}{\sqrt{3}}$$

$$\cos\gamma = \frac{\vec{c}.\vec{a}+\vec{c}.\vec{b}+\vec{c}.\vec{c}}{\sqrt{3}\,k.k} = \frac{k^2}{\sqrt{3}k^2} = \frac{1}{\sqrt{3}}$$

$$\therefore \qquad \alpha = \beta = \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates used $\vec{a}\times(\vec{a}+\vec{b}+\vec{c})$ instead of $\vec{a}.(\vec{a}+\vec{b}+\vec{c})$ and could not simplify further. Many candidates were unable to express the formula for $\cos\alpha = \dfrac{\vec{a}.(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}|.|\vec{a}+\vec{b}+\vec{c}|}$

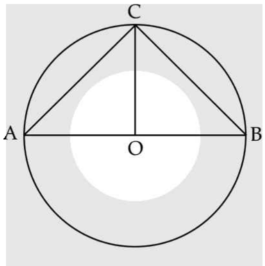
Answering Tip

- Students were advised to read the instructions given in the question carefully.

Q. 2. Using vectors, prove that angle in a semicircle is a right angle. [ISC, 2015]

Sol. Let O be the centre of the circle and AB be the diameter C is a point on the circumference. Take O as the origin and Let $\vec{OA}$ and $\vec{a}$ and $\vec{OC} = \vec{C}$

Therefore, $\quad \vec{OB} = -\vec{a}$



$$\vec{CA} = \vec{OA} - \vec{OC} = \vec{a} - \vec{c}$$

$$\vec{CB} = \vec{OB} - \vec{OC} = -\vec{a} - \vec{c}$$

$$\vec{CA}.\vec{CB} = (\vec{a}-\vec{c}).(-\vec{a}-\vec{c})$$

$$= -\vec{a}.\vec{a} - \vec{a}.\vec{c} + \vec{c}.\vec{a} + \vec{c}.\vec{c}$$

$$= -|\vec{a}|^2 + |\vec{c}|^2$$

$= -r^2 + r^2 = 0$, where r is radius

Therefore, angle ACB is a right angle.

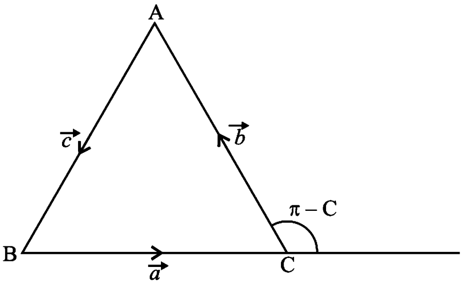
[ISC Marking Scheme, 2015]

Q. 3. In triangle *ABC*, using vectors, prove that $c^2 = a^2 + b^2 - 2ab\cos C$. [ISC, 2014]

Sol. In triangle ABC, $\vec{a}+\vec{b}+\vec{c} = \vec{0}$

$$\therefore \qquad \vec{a}+\vec{b} = -\vec{c}$$

$$|\vec{a}+\vec{b}|^2 = |-\vec{c}|^2$$



On squaring both sides, we get

$$(\vec{a}+\vec{b})^2 = (-\vec{c})^2$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos(\pi - C) = |\vec{c}|^2$$

$$\Rightarrow a^2 + b^2 - 2ab\cos C = c^2$$

$$\Rightarrow \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

Hence Proved.

[ISC Marking Scheme, 2014]

Examiner's Comment

- Diagram drawn by candidates did not show arrows to indicate directions of $\vec{a}, \vec{b}, \vec{c}$.

Answering Tip

- Vectors need to be correctly depicted. Dot and cross product needs thorough attention.

Q. 4. If $\vec{a}$ and $\vec{b}$ are unit vector and θ is the angle between them, then show that $|\vec{a}-\vec{b}| = 2\sin\frac{\theta}{2}$. [ISC, 2013]

Sol. We have $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$= 1\cdot 1\cdot\cos\theta = \cos\theta$$

Now, $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$

$$= 1 + 1 - 2\cos\theta = 2(1-\theta)$$

$$\Rightarrow \qquad |\vec{a}-\vec{b}|^2 = 2\cdot 2\sin^2\frac{\theta}{2}$$

$$\Rightarrow \qquad |\vec{a}-\vec{b}| = 2\sin\frac{\theta}{2}$$

Hence Proved.

TOPIC-2
Cross Product

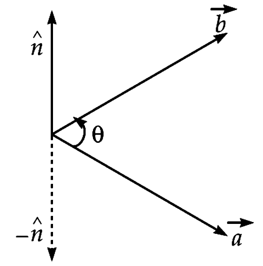
Scan to know more about this topic

Cross product of vectors

Revision Notes

- **The cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined by,**

  $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ (see Figure)



  Consider $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

  then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

  Note : Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

- **Properties of Cross Product :**
  - $\hat{i}\times\hat{i} = |\hat{i}||\hat{i}|\sin 0 = \vec{0} \Rightarrow \hat{i}\times\hat{i} = \vec{0} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k}$
  - $\hat{i}\times\hat{j} = |\hat{i}||\hat{j}|\sin\frac{\pi}{2}\cdot\hat{k} = \hat{k}$. Similarly, $\hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$
  - $\vec{a}\times\vec{b}$ is a vector $\vec{c}$ (say) then this vector $\vec{c}$ is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.
  - $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}\,||\,\vec{b}$ or, $\vec{a} = \vec{0}, \vec{b} = \vec{0}$
  - $\vec{a}\times\vec{a} = \vec{0}$
  - $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (Commutative property does not hold for cross product).
  - $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (Left distributive).
  - $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a}+\vec{c}\times\vec{a}$ (right distributive).

- **Triangle Inequality**

  For any two vectors $\vec{a}$ and $\vec{b}$ we always have $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$

  Proof : The given inequality holds trivially when either $\vec{a} = 0$ or $\vec{b} = 0$ *i.e.,* in such a case

  $$|\vec{a}+\vec{b}| = 0 = |\vec{a}|+|\vec{b}|$$

  So, let us check it for $|\vec{a}| \ne 0 \ne |\vec{b}|$.

  Then consider $\quad |\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}$

  $\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta$

  For $\cos\theta \le 1$, we have: $\quad 2|\vec{a}||\vec{b}|\cos\theta \le 2|\vec{a}||\vec{b}|$

  $\Rightarrow \quad |\vec{a}|^2|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta \le |\vec{a}|^2|\vec{b}|^2+2|\vec{a}||\vec{b}|$

  $\Rightarrow \quad |\vec{a}+\vec{b}|^2 \le \left(|\vec{a}|+|\vec{b}|\right)^2$

  $\Rightarrow \quad |\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$ **Hence Proved.**

  Note : For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}.\vec{b}| \le |\vec{a}||\vec{b}|$

Know the Formulae

- Angle between two vectors $\vec{a}$ and $\vec{b}$ in terms of cross-product can be found by the expression :

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \quad \textbf{or} \quad \theta = \sin^{-1}\left(\frac{\left(|\vec{a}\times\vec{b}|\right)}{\left(|\vec{a}||\vec{b}|\right)}\right)$$

- **Area of Triangle :** If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a Triangle, then the area of triangle can be obtained by evaluating $\frac{1}{2}|\vec{a}\times\vec{b}|$.
- **Area of Parallelogram**

(i) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then the area of parallelogram, can be obtained by evaluating $|\vec{a}\times\vec{b}|$.

(ii) The area of the parallelogram with diagonal $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$

1 Mark Questions

AI Q. 1. The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$ is

(a) 0 **(b) –1**

(c) 1 **(d) 3**

Sol. Correct option : (c)

Explanation :

$$\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$$
$$=\hat{i}.\hat{i}+\hat{j}.(-\hat{j})+\hat{k}.\hat{k}$$
$$=1-\hat{j}.\hat{j}+1$$
$$=1-1+1$$
$$=1$$

Q. 2. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$ when θ is equal to

(a) 0 **(b) $\frac{\pi}{4}$**

(c) $\frac{\pi}{2}$ **(d) π**

Sol. Correct option : (b)

Explanation :

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

$$|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$$
$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta=|\vec{a}||\vec{b}|\sin\theta$$
$$\Rightarrow \quad \cos\theta=\sin\theta \quad \left[\because |\vec{a}| \text{ and } |\vec{b}| \text{ are positive.}\right]$$
$$\Rightarrow \quad \tan\theta=1$$
$$\Rightarrow \quad \theta=\frac{\pi}{4}$$

So that, $|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$ when θ is equal to $\frac{\pi}{4}$.

Q. 3. Find the value of $(\hat{i}\times\hat{j}).\hat{k}+\hat{i}.\hat{j}$.

Sol. $$(\hat{i}\times\hat{j}).\hat{k}+\hat{i}.\hat{j}. = \hat{k}.\hat{k}+\hat{i}.\hat{j}$$
$$= 1 + 0 = 1$$

AI Q. 4. Find the value of $(\hat{k}\times\hat{j}).\hat{i}+\hat{j}.\hat{k}$.

Sol. $$(\hat{k}\times\hat{j}).\hat{i}+\hat{j}.\hat{k} = -\hat{i}.\hat{i}+0$$
$$= -\hat{i}.\hat{i} + 0$$
$$= -1 + 0 = -1$$

Q. 5. If $\vec{a}$ and $\vec{b}$ are two vectors of magnitude 3 and $\frac{2}{3}$ respectively such that $\vec{a}\times\vec{b}$ is a unit vector, write the angle between $\vec{a}$ & $\vec{b}$.

Sol. Since,

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.\hat{n}, \text{where } \hat{n}=1$$
$$|\vec{a}\times\vec{b}| = 1$$
$$|\vec{a}| = 3 \qquad \text{[Given]}$$
and $$|\vec{b}| = \frac{2}{3}$$
∴ $$1 = 3\times\frac{2}{3}\sin\theta$$
or $$\sin\theta = \frac{1}{2}$$
or $$\theta = \frac{\pi}{6}$$

Q. 6. If $|\vec{a}|=8, |\vec{b}|=3$ and $|\vec{a}\times\vec{b}|=12$, find the angle between $\vec{a}$ and $\vec{b}$.

Sol. We know

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

or $$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{12}{8\times 3} = \frac{1}{2}$$

$\therefore$ $$\theta = \frac{\pi}{6}$$

AI **Q. 7. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ having the same length $\sqrt{2}$ and their vector product is $-\hat{i}-\hat{j}+\hat{k}$.**

Sol. $$\sin\theta = \frac{|-\hat{i}-\hat{j}+\hat{k}|}{\sqrt{2}\cdot\sqrt{2}}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

or $$\theta = 60°$$

or $$\theta = \frac{\pi}{3}$$

Q. 8. If $(\vec{a}\times\vec{b})^2+(\vec{a}\cdot\vec{b})^2 = 225$ and $|\vec{a}| = 5$, then write the value of $|\vec{b}|$.

Sol. $$(\vec{a}\times\vec{b})^2+(\vec{a}\cdot\vec{b})^2 = 225$$

or $$|\vec{a}|^2|\vec{b}|^2(\sin^2\theta+\cos^2\theta) = 225$$

or $$(5)^2|\vec{b}|^2 = 225 \text{ or } |\vec{b}| = 3$$

2 Marks Questions

Q. 1. Find the area of the parallelogram whose adjacent sides are given by the vectors $\vec{a} = 3\hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$. [ISC Specimen Paper, 2018]

Sol. $\vec{a} = 3\hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= (1+1)\hat{i}-\hat{j}(3-1)+(-3-1)\hat{k}$$

$$\vec{a}\times\vec{b} = 2\hat{i}-2\hat{j}-4\hat{k}$$

Area of parallelogram $= |\vec{a}\times\vec{b}|$

$$= \sqrt{(2)^2+(-2)^2+(-4)^2}$$

$$= \sqrt{4+4+16}$$

$$= \sqrt{24} = 2\sqrt{6}$$

$= 4.89$ sq. units

AI **Q. 2. Find the unit vector perpendicular to the two vectors $\hat{i}+2\hat{j}-\hat{k}$ and $2\hat{i}+3\hat{j}+\hat{k}$. [ISC, 2009]**

Sol. Let $$\vec{a} = \hat{i}+2\hat{j}-\hat{k}$$

and $$\vec{b} = 2\hat{i}+3\hat{j}+\hat{k}$$

Now, unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$ is given by

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= (2+3)\hat{i}-(1+2)\hat{j}+(3-4)\hat{k}$$

$$= 5\hat{i}-3\hat{j}-\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(5)^2+(-3)^2+(-1)^2}$$

$$= \sqrt{25+9+1} = \sqrt{35}$$

$\therefore$ Unit vector $$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{5\hat{i}-3\hat{j}-\hat{k}}{\sqrt{35}}$$

$\therefore$ $$\hat{n} = \frac{5}{\sqrt{35}}\hat{i}-\frac{3}{\sqrt{35}}\hat{j}-\frac{1}{\sqrt{35}}\hat{k}$$

Q. 3. Find $\vec{a}\cdot\vec{b}$ if $|\vec{a}|=2, |\vec{b}|=5$ and $|\vec{a}\times\vec{b}|=8$. [ISC, 2008]

Sol. $$|a\times b| = |a||b|\sin\theta$$

$\therefore$ $$\sin\theta = \frac{8}{2\times 5} = \frac{4}{5}$$

$\therefore$ $$\cos\theta = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{1-\frac{16}{25}} = \frac{3}{5}$$

$\therefore$ $$a.b = |a||b|\cos\theta$$

$$= 2\times 5\times\frac{3}{5} = 6$$

AI **Q. 4. If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ find $(\vec{r}\times\hat{i})(\vec{r}\times\hat{j})+xy$.**

Sol. $(\vec{r}\times\hat{i}) = (x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i} \;= -y\hat{k}+z\hat{j}$

$(\vec{r}\times\vec{j}) = (x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j} \;= x\hat{k}-z\hat{i}$

$\therefore\; (\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j}) = (0\hat{i}+z\hat{j}-y\hat{k})\cdot(-z\hat{i}+0\hat{j}+x\hat{k})$

$= -xy$

$\therefore\; (\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy = 0$

Q. 5. For any vector $\vec{a}$, find the value of $(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2$

Sol. Let $\vec{a} = x\hat{i}+y\hat{j}+z\hat{k}$

$\therefore\quad \vec{a}^2 = x^2+y^2+z^2$

$$\therefore\quad \vec{a}\times\hat{i} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ 1 & 0 & 0\end{vmatrix}$$

$= \hat{i}[0]-\hat{j}[-z]+\hat{k}[-y]$

$= z\hat{j}-y\hat{k}$

$\therefore\; (\vec{a}\times\hat{i})^2 = (z\hat{j}-y\hat{k})(z\hat{j}-y\hat{k})$

$= y^2+z^2$

Similarly, $(\vec{a}\times\hat{j})^2 = x^2+z^2$ and $(\vec{a}\times\hat{k})^2 = x^2+z^2$

$(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2 = y^2+z^2+x^2+z^2+x^2+y^2$

$= 2(x^2+y^2+z^2) = 2\vec{a}^2$

AI **Q. 6. If $|\vec{a}| = 10, |\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 12$, then find the value of $|\vec{a}\times\vec{b}|$.**

Sol. Here, $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b}$ = 12 [Given]

$\therefore\quad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$12 = 10\times 2\cos\theta$

$\Rightarrow\quad \cos\theta = \frac{12}{20} = \frac{3}{5}$

$\Rightarrow\quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{25}}$

$\sin\theta = \pm\frac{4}{5}$

$\therefore\; |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}||\sin\theta|$

$= 10\times 2\times\frac{4}{5}$

$= 16$

Q. 7. For any vector $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+p\hat{j}+q\hat{k}) = \vec{0}$. Then find the values of p and q.

Sol. Given $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+p\hat{j}+q\hat{k}) = \vec{0}$

$$i.e.,\quad \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 27\\ 1 & p & q\end{vmatrix} = \vec{0}$$

$\Rightarrow (6q-27p)\hat{i}+(27-2q)\hat{j}+(2p-6)\hat{k} = \vec{0}$

$\Rightarrow 27 - 2q = 0$ and $2p - 6 = 0$

$\Rightarrow\quad q = \frac{27}{2}, p = 3$

4 Marks Questions

Q. 1. Using vectors, find the area of the triangle whose vertices are :

A(3, –1, 2), B(1, –1, –3) and C(4, –3, 1) [ISC, 2020]

Sol. A(3, – 1, 2); B(1, – 1, – 3) and C(4, – 3, 1)

$\vec{a} = 3\hat{i}-\hat{j}+2\hat{k}$; $\vec{b} = \hat{i}-\hat{j}-3\hat{k}$; $\vec{c} = 4\hat{i}-3\hat{j}+\hat{k}$

$\overrightarrow{AB} = \vec{b}-\vec{a}$

$= \hat{i}-\hat{j}-3\hat{k}-3\hat{i}+\hat{j}-2\hat{k}$

$\overrightarrow{AB} = -2\hat{i}-5\hat{k}$

$\overrightarrow{AC} = \vec{c}-\vec{a}$

$= 4\hat{i}-3\hat{j}+\hat{k}-3\hat{i}+\hat{j}-2\hat{k}$

$= \hat{i}-2\hat{j}-\hat{k}$

$\text{ar}\,\Delta ABC = \frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}$

$$= \frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & -5\\ 1 & -2 & -1\end{vmatrix}$$

$$= \frac{1}{2}\Big|\hat{i}\begin{vmatrix}0 & -5\\ -2 & 1\end{vmatrix} - \hat{j}\begin{vmatrix}-2 & -5\\ -1 & -1\end{vmatrix} + \hat{k}\begin{vmatrix}-2 & 0\\ 1 & -2\end{vmatrix}$$

$= \frac{1}{2}\left|-10\hat{i}-7\hat{j}+4\hat{k}\right|$

$\text{ar}\,\Delta ABC = \frac{1}{2}\left|\sqrt{(-10)^2+(-7)^2+(4)^2}\right|$

$= \frac{1}{2}\left|\sqrt{100+49+16}\right|$

$\text{ar}\,\Delta ABC = \frac{1}{2}\sqrt{165}$ unit²

[ISC Marking Scheme, 2020]

Examiner's Comment

- Few candidates find the area of the triangle using determinants and lose their marks.

Answering Tip

- Read the question carefully and find the area of the triangle using the approach as mentioned in the question.

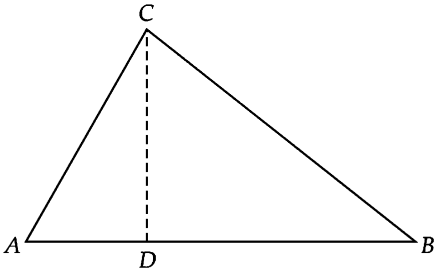
Q. 2. If A, B, C are three non-collinear points with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively, then show that the length perpendicular from C on AB is

$$\frac{|(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})|}{|\vec{b}-\vec{a}|}$$ **[ISC, 2018]**

Sol. In ΔABC,

$$\text{area of } \Delta ABC = \frac{1}{2}|\vec{AB}|.\text{(length of} \perp \text{from } C \text{ to } AB)$$

$$= \frac{1}{2}|\vec{b}-\vec{a}|.\text{ (length of} \perp \text{from } C \text{ to } AB) \qquad ...(i)$$



$$\text{Also, area of } \Delta ABC = \frac{1}{2}|(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})| \qquad ...(ii)$$

From eqs. (i) & (ii), we get

$$\frac{1}{2}|\vec{b}-\vec{a}|.\text{(length of} \perp \text{from } C \text{ to } AB)$$

$$= \frac{1}{2}|(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})|$$

$\Rightarrow$ length of ⊥ from C to AB

$$= \frac{|(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})|}{|\vec{b}-\vec{a}|}$$

Hence Proved.
[ISC Marking Scheme, 2018]

Examiner's Comment

- Very few candidates attempted this question.

Answering Tip

- Learn and understand the formula for the area of triangle by using position vectors.

AI Q. 3. Show that :

$$(\vec{a}\times\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$$

[ISC Specimen Paper, 2018]

Sol. LHS $= (\vec{a}\times\vec{b})^2$

$$= [|\vec{a}||\vec{b}|\sin\theta\,\hat{n}]^2$$

$$= |a|^2|b|^2\sin^2\theta \qquad (\because \hat{n}.\hat{n} = 1)$$

$$= (\vec{a})^2(\vec{b})^2(1-\cos^2\theta) \qquad (\because |\vec{a}|^2 = (\vec{a})^2)$$

$$= (\vec{a})^2(\vec{b})^2 - (\vec{a})^2(\vec{b})^2\cos^2\theta$$

$$= (\vec{a})^2(\vec{b})^2 - [|\vec{a}||\vec{b}|\cos\theta]^2$$

$$\text{LHS} = (\vec{a})^2(\vec{b})^2 - (\vec{a}.\vec{b})^2 \qquad ...(i)$$

$$\text{RHS} = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$$

$$= (\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})(\vec{a}.\vec{b})$$

$$= (\vec{a})^2(\vec{b})^2 - (\vec{a}.\vec{b})^2$$

$$= (\vec{a})^2(\vec{b})^2 - (\vec{a}.\vec{b})^2 \qquad ...(ii)$$

From (i) and (ii)
LHS = RHS **Hence Proved.**

Q. 4. Let $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}, \vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$ Find a vector $\vec{p}$ which perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}\cdot\vec{c} = 18$.

Sol. $\vec{p}$ is ⊥ to both $\vec{a}$ and $\vec{b}$

$$\Rightarrow \qquad \vec{p} = \lambda(\vec{a}\times\vec{b})$$

$$\text{Now,} \qquad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$= 32\hat{i}-\hat{j}-14\hat{k}$$

$$\text{Given that} \qquad \vec{p}\cdot\vec{c} = 18$$

$$\Rightarrow \quad \lambda(32\hat{i}-\hat{j}-14\hat{k})(2\hat{i}-\hat{j}+4\hat{k}) = 18$$

$$\text{or} \qquad \lambda(64+1-56) = 18$$

$$\text{or} \qquad \lambda = 2$$

$$\therefore \qquad \vec{p} = 64\hat{i}-2\hat{j}-28\hat{k}$$

Q. 4. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k}$, find a vector $\vec{c}$, such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}.\vec{c} = 3$.

Sol. Let $\qquad \vec{c} = x\hat{i}+y\hat{j}+z\hat{k}$

Given $\qquad \vec{a} = \hat{i}+\hat{j}+\hat{k}$

$$\vec{b} = \hat{j}-\hat{k}$$

According to the question,

$$\vec{a}\cdot\vec{c} = 3$$

$$\text{or } (\hat{i}+\hat{j}+\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k}) = 3$$

$$\text{or} \qquad x+y+z = 3 \qquad ...(i)$$

$$\text{and} \qquad \vec{a}\times\vec{c} = \vec{b}$$

or $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \vec{b} = \hat{j} - \hat{k}$

or $(z-y)\hat{i} + (x-z)\hat{j} + (y-x)\hat{k} = \hat{j} - \hat{k}$

On equating the coefficients of like terms, we get

$z - y = 0$, or $y = z$ ...(ii)

$x - z = 1$ ...(iii)

and $y - x = -1$ ...(iv)

Solving eqns. (i), (ii), (iii) and (iv), we get

$x = 5/3$

and $y = 2/3 = z$

Hence, $\vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

6 Marks Questions

AI **Q. 1. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$. [ISC, 2016]**

Sol. Given, $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}, \vec{b} = \hat{i}+2\hat{j}-2\hat{k}$

$\therefore \quad \vec{a}+\vec{b} = 4\hat{i}+4\hat{j}$

and $\quad \vec{a}-\vec{b} = 2\hat{i}+4\hat{k}$

Let $\vec{r} = (\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = (4\hat{i}+4\hat{j})\times(2\hat{i}+4\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = 16\hat{i}-16\hat{j}-8\hat{k}$$

$$= 8(2\hat{i}-2\hat{j}-\hat{k})$$

$$\therefore \quad \text{Unit vector} = \frac{\vec{r}}{|\vec{r}|} = \frac{8(2\hat{i}-2\hat{j}-\hat{k})}{8\times 3}$$

$$= \frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}$$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Some candidates used cross product without evaluating and $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$. Some candidates made mistakes while evaluating the unit vector in the final answer.

Answering Tip

- Vector notation, usage, dot and cross product in terms of vectors or their components need to be understand well and in detail.

Q. 2. Find the area of a parallelogram whose diagonals are determined by the vectors

$\vec{a} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}-3\hat{j}+4\hat{k}$

[ISC, 2009]

Sol. Area of parallelogram $= \frac{1}{2}|\vec{a}\times\vec{b}|$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= \frac{1}{2}\left|\hat{i}(4-6)-\hat{j}(12+2)+\hat{k}(-9-1)\right|$$

$$= \frac{1}{2}\left|(-2\hat{i}-14\hat{j}-10\hat{k}\right|$$

$$= \frac{1}{2}(\sqrt{4+196+100})$$

$$= \frac{1}{2}(\sqrt{300})$$

$= 5\sqrt{3}$ sq. units.

Q. 3. Find a unit vector perpendicular to the vector $4\hat{i}+3\hat{j}+\hat{k}$ and $2\hat{i}-\hat{j}+2\hat{k}$. Determine the sine of angle between these two vectors. [ISC, 2007]

Sol. Vector perpendicular to $\vec{A}$ and $\vec{B}$ is given by

$$\frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \hat{i}(6+1)-\hat{j}(8-2)+\hat{k}(-4-6)$$

$$= 7\hat{i}-6\hat{j}-10\hat{k}$$

$$|\vec{A}\times\vec{B}| = \sqrt{(7)^2+(-6)^2+(-10)^2}$$

$$= \sqrt{49+36+100}$$

$$= \sqrt{185}$$

$\therefore$ Unit vector perpendicular to given vectors

$$= \frac{7\hat{i}-6\hat{j}-10\hat{k}}{\sqrt{185}}$$

$$= \frac{7}{\sqrt{185}}\hat{i}-\frac{6}{\sqrt{185}}\hat{j}-\frac{10}{\sqrt{185}}\hat{k}$$

Now, $\sin\theta = \left|\dfrac{\vec{A}\times\vec{B}}{|\vec{A}||\vec{B}|}\right|$

$$= \left|\frac{7\hat{i}-6\hat{j}-10\hat{k}}{\sqrt{26}\sqrt{9}}\right| = \frac{\sqrt{185}}{3\sqrt{26}}$$

AI Q. 4. The vector $-2\hat{i}+4\hat{j}+4\hat{k}$ and $-4\hat{i}-2\hat{k}$ represent the diagonals *BD* and *AC* of a parallelogram *ABCD*. Then, find the area of the parallelogram.

Sol. Let $\vec{BD} = -2\hat{i}+4\hat{j}+4\hat{k}$

and $\vec{AC} = -4\hat{i}-2\hat{k}$

Ar (parallelogram *ABCD*) $= \dfrac{1}{2}\left|\vec{BD}\times\vec{AC}\right|$

$$\therefore \quad \vec{BD}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 4 & 4 \\ -4 & 0 & -2 \end{vmatrix}$$

$$= \hat{i}(-8-0)-\hat{j}(4+16)+\hat{k}(0+16)$$

$$= -8\hat{i}-20\hat{j}+16\hat{k}$$

Now, $Ar(ABCD) = \dfrac{1}{2}|\vec{BD}\times\vec{AC}|$

$$\Rightarrow \quad Ar_{(ABCD)} = \frac{1}{2}\sqrt{(-8)^2+(-20)^2+(16)^2}$$

$$= \frac{1}{2}\sqrt{64+400+256}$$

$$= \frac{1}{2}\sqrt{720}$$

$$= \frac{1}{2}\times 12\sqrt{5}$$

$$Ar(ABCD) = 6\sqrt{5} \text{ sq. units}$$

$\therefore$ Area of parallelogram *ABCD* is $6\sqrt{5}$ sq. units.

Q. 5. If $\vec{a} = 3\hat{i}-\hat{j}$ and $\vec{b} = 2\hat{i}+\hat{j}-3\hat{k}$ then express $\vec{b}$ in the form of $\vec{b} = \vec{b_1}+\vec{b_2}$, where $\vec{b_1} \parallel \vec{a}$ and $\vec{b_2}$ perpendicular to $\vec{a}$.

Sol. Here, $\vec{a} = 3\hat{i}-\hat{j}$

and $\vec{b} = 2\hat{i}+\hat{j}-3\hat{k}$

To express $\vec{b} = \vec{b_1}+\vec{b_2}$

where $\vec{b_1} \parallel \vec{a} \Rightarrow \vec{b_1} = \lambda\vec{a}$

$$\vec{b_1} = \lambda(3\hat{i}-\hat{j})$$

Let $\vec{b_2} = x\hat{i}+y\hat{j}+z\hat{k}$

Now, $\vec{b_2} \perp \vec{a} \Rightarrow \vec{b_2}\cdot\vec{a} = 0$

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k}).(3\hat{i}-\hat{j}) = 0$$

$$\Rightarrow \quad 3x-y = 0 \qquad \text{...(i)}$$

Now, $\vec{b} = \vec{b_1}+\vec{b_2}$

$$2\hat{i}+\hat{j}-3\hat{k} = (3\lambda\hat{i}-\lambda\hat{j})+(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= (3\lambda+x)\hat{i}+(y-\lambda)\hat{j}+z\hat{k}$$

Comparing the corresponding components

$$2 = 3\lambda+x \qquad \text{...(ii)}$$

$$1 = -\lambda+y$$

$$\Rightarrow \quad \lambda = y-1 \qquad \text{... (iii)}$$

$$-3 = z \qquad \text{... (iv)}$$

From (ii), $2 = 3(y-1)+x$

$$\Rightarrow \quad 2 = 3y-3+x$$

$$\Rightarrow \quad x+3y = 5 \qquad \text{... (v)}$$

Solving (i) and (v),

$$x = \frac{1}{2}, y = \frac{3}{2}$$

$\therefore$ From (iii), $\lambda = \dfrac{1}{2}$

$$\therefore \quad \vec{b_1} = \frac{3}{2}\hat{i}-\frac{1}{2}\hat{j}$$

$$\Rightarrow \quad \vec{b_2} = \frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$$

$$\vec{b} = \vec{b_1}+\vec{b_2}$$

$$= 2\hat{i}+\hat{j}-3\hat{k}$$

TOPIC-3

Scalar Triple Product

Revision Notes

- **Scalar Triple Product :**

If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors, then the scalar product of $\vec{a}\times\vec{b}$ with $\vec{c}$ is called scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$.

Scan to know more about this topic

Scalar Triple Product

Thus, $(\vec{a}\times\vec{b})\cdot\vec{c}$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$.

- **Notation for scalar triple product :** The scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$
  i.e., $(\vec{a}\times\vec{b})\cdot\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]$

➢ **Properties of Scalar Triple Product :**

- $(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$ *i.e.,* the position of dot and cross can be interchanged without change in the value of the scalar triple product (provided their cyclic order remains the same).
- $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$ *i.e.,* the value of scalar triple product doesn't change when cyclic order of the vectors is maintained.
- Also, $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$; $[\vec{b}\ \vec{c}\ \vec{a}] = -[\vec{b}\ \vec{a}\ \vec{c}]$ *i.e.,* the value of scalar triple product remains the same in magnitude but changes the sign when cyclic order of the vectors is altered.
- For any three vectors $\vec{a}, \vec{b}, \vec{c}$ and scalar λ we have $[\lambda\vec{a}\ \vec{b}\ \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$.
- The value of scalar triple product is zero if any two of the three vectors are identical. *i.e.,* $[\vec{c}\ \vec{b}\ \vec{c}] = 0 = [\vec{a}\ \vec{b}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{a}]$ etc.
- Value of scalar triple product is zero if any two of the three vectors are parallel or collinear.
- Scalar triple product of $\hat{i}, \hat{j}$ and $\hat{k}$ is 1 (unity) *i.e,.* $[\hat{i}\ \hat{j}\ \hat{k}] = 1$
- If $[\vec{a}, \vec{b}, \vec{c}] = 0$ then the non-parallel and non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

Know the Formulae

➢ **Volume of parallelepiped :**

- If $\vec{a}, \vec{b}$ and $\vec{c}$ represent the three co-terminous edges of a parallelepiped, then its volume can be obtained by $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\times\vec{b})\cdot\vec{c}$
  i.e., $(\vec{a}\times\vec{b})\cdot\vec{c}$ = Base area of Parallelepiped × Height of Parallelepiped on this base.

Note : If for any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ we have $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, then volume of parallelepiped with the co-terminous edges as $\vec{a}, \vec{b}$ and $\vec{c}$ is zero. This is possible only if the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are co-planar.

1 Mark Questions

AI **Q. 1. If vectors $\hat{i}+\hat{j}-3\hat{k}$, $2\hat{i}+\hat{j}-\lambda\hat{k}$ and $5\hat{i}+2\hat{j}+3\hat{k}$ are coplanar, then value of λ is**

(a) 4 **(b)** 0
(c) – 3 **(d)** 2

Ans.Correct option : (b)

Explanation :

$$\begin{vmatrix} 1 & 1 & -3 \\ 2 & 1 & -\lambda \\ 5 & 2 & 3 \end{vmatrix} = 0$$

$\Rightarrow$ $1(3 + 2\lambda) - 1(6 + 5\lambda) - 3(-1) = 0$

$\Rightarrow$ $3 + 2\lambda - 6 - 5\lambda + 3 = 0$

$\Rightarrow$ $3\lambda = 0$

$\Rightarrow$ $\lambda = 0$

Examiner's Comment

- Some candidates fail to apply condition of coplanarity.

Answering Tip

- Scalar triple product and its applications need to be practiced with the help of practical examples.

Q. 2. **[*i k j*] + [*k j i*] + [*j k i*]**

(a) 1 **(b) 3**
(c) – 3 **(d) – 1**

Sol. **Correct option :** (d)

Explanation :

$[i\ k\ j] + [k\ j\ i] + [j\ k\ i] = [i\ k\ j] + [i\ k\ j] - [i\ k\ j]$

$= [i\ k\ j] = -1$

AI **Q. 3. Find the volume of the parallelepiped with co-terminous edges**

$2\hat{i}+\hat{j}-\hat{k},\ \hat{i}+2\hat{j}+3\hat{k}$ **and** $-3\hat{i}-\hat{j}+\hat{k}$

[ISC Specimen Paper, 2021]

Sol. Volume of the parallelepiped

$$= \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -3 & -1 & 1 \end{vmatrix} = -3$$

$\therefore$ Required Volume = 5 cm^3

Q. 4. Find $\vec{a}.(\vec{b}\times\vec{c})$**, if**

$\vec{a}=2\hat{i}+\hat{j}+3\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ **and** $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$**.**

Sol.

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \hat{i}(4-1)-\hat{j}(-2-3)+\hat{k}(-1-6)$$

$$= 3\hat{i}+5\hat{j}-7\hat{k}$$

$$\therefore\quad \vec{a}.(\vec{b}\times\vec{c}) = (2\hat{i}+\hat{j}+3\hat{k}).(3\hat{i}+5\hat{j}-7\hat{k})$$

$$= 6+5-21 = -10$$

2 Marks Questions

Q. 1. Find the value of λ if vectors $\lambda\hat{i}+\hat{j}+2\hat{k},\ \hat{i}+\lambda\hat{j}-\hat{k}$ **and** $2\hat{i}-\hat{j}+\lambda\hat{k}$ **are coplanar.**

Sol. Let $\vec{a}=\lambda\hat{i}+\hat{j}+2\hat{k},\ \vec{b}=\hat{i}+\lambda\hat{j}-\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+\lambda\hat{k}$

For $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar,

$$\begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda^2-1)-1(\lambda+2)+2(-1-2\lambda)=0$$

$$\Rightarrow \lambda^3-\lambda-\lambda-2-2-4\lambda=0$$

$$\Rightarrow \lambda^3-6\lambda-4=0$$

$$\Rightarrow (\lambda+2)(\lambda^2-2\lambda-2)=0$$

$$\Rightarrow \lambda=-2 \text{ or } \lambda=\frac{2\pm\sqrt{12}}{2}$$

$$\Rightarrow \lambda=-2 \text{ or } \lambda=\frac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$$

AI **Q. 2.** $\vec{r}.\vec{a}=0, \vec{r}.\vec{b}=0$ **and** $\vec{r}.\vec{c}=0$ **for some non-zero vector** $\vec{r}$**, then find the value of** $\vec{a}.(\vec{b}\times\vec{c})$**.**

Sol. Since, $\vec{r}$ is a non-zero vector. So, we can say that $\vec{a},\vec{b}$ and $\vec{c}$ are in a same plane.

$$\therefore\quad \vec{a}.(\vec{b}\times\vec{c})=0$$

[Since, angles between $\vec{a},\vec{b}$ and $\vec{c}$ are zero *i.e.*, θ = 0]

Q. 3. If $\vec{a}, \vec{b}$ **and** $\vec{c}$ **are three non-zero vectors, then find the value** $[\vec{a},\vec{b}+\vec{c},\vec{a}+\vec{b}+\vec{c}]$**.**

Sol. $[\vec{a},\vec{b}+\vec{c},\vec{a}+\vec{b}+\vec{c}]$

$$= [\vec{a},\vec{b}+\vec{c},\vec{a}] + [\vec{a},\vec{b}+\vec{c},\vec{b}+\vec{c}]$$

$$= 0+0=0$$

AI **Q. 4. Let** $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}$ **and** $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$**, then then** c_1 **= 1 and** c_2 **= 2, find** c_3 **which makes** $\vec{a}, \vec{b}$ **and** $\vec{c}$ **coplanar.**

Sol.

$$[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_2-c_3$$

$$c_1=1, c_2=2$$

$$[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 2-c_3$$

$\therefore$ $\vec{a}, \vec{b}, \vec{c}$ are coplanar $[\vec{a}\ \ \vec{b}\ \ \vec{c}]$ = 0 or c_3 = 2

4 Marks Questions

Q. 1. Prove that $\vec{a}.\left[(\vec{b}+\vec{c})\times(\vec{a}+3\vec{b}+4\vec{c})\right]=\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$

[ISC, 2020]

Sol. L.H.S. $=\vec{a}\left[(\vec{b}+\vec{c})\times(\vec{a}+3\vec{b}+4\vec{c})\right]$

$$=\vec{a}\left[\vec{b}\times\vec{a}+3\vec{b}\times\vec{b}+4\vec{c}\times\vec{b}+\vec{c}\times\vec{a}+3\vec{b}\times\vec{c}+4\vec{c}\times\vec{c}\right]$$

$$=\vec{a}\left[\vec{b}\times\vec{a}+4\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+3\vec{c}\times\vec{b}\right]$$

$$=\vec{a}\left[\vec{b}\times\vec{a}+4\vec{b}\times\vec{c}+\vec{c}\times\vec{a}-3\vec{b}\times\vec{c}\right]$$

$$=\vec{a}\left[\vec{b}\times\vec{a}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right]$$

$$=\vec{a}.\vec{b}\times\vec{a}+\vec{a}.\vec{b}\times\vec{c}+\vec{a}.\vec{c}\times\vec{a}$$

$$=0+\vec{a}.\vec{b}\times\vec{c}+0$$

$$=\vec{a}.\vec{b}\times\vec{c}=\left[\vec{a}\,\vec{b}\,\vec{c}\right]$$

Hence Proved.

[ISC Marking Scheme, 2020]

Examiner's Comment

- Many candidates performed dot product first, followed by cross product. This was incorrect. They simplified and arrived at the result somehow but it was not driven by logic.

Answering Tip

- Dot product produces a scalar quantity and subsequent cross-product of scalar with vector is meaningless. Vector algebra in totality need to be understood by the students, specifically properties of Scalar Triple Product.

AI Q. 2. If $\vec{a} = i - 2j + 3k$, $\vec{b} = 2i + 3j - 5k$ prove that $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular. [ISC, 2019]

Sol.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ 2 & 3 & -5 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = i + 11j + 7k$$

$$\vec{a}.\left(\vec{a} \times \vec{b}\right)$$

$$(i - 2j + 3k).(i + 11j + 7k)$$

$$1 - 22 + 21 = 0$$

[ISC Marking Scheme, 2019]

Examiner's Comment

- Majority of the candidates calculated *a* × *b* but failed to apply the concept of scalar triple product *a*. (*a* × *b*) =0. A few candidates made calculation errors while finding *a* × *b*.

Answering Tip

- Understand the concept of scalar triple product and its applications with the help of examples.

AI Q. 3. Show that the four points *A, B, C,* and *D* with position vectors

$$4\hat{i} + 5\hat{j} + \hat{k}, -\hat{j} - \hat{k}, 3\hat{i} + 9\hat{j} + 4\hat{k} \text{ and } 4(-\hat{i} + \hat{j} + \hat{k})$$

respectively, are coplanar. [ISC, 2018]

Sol. $\vec{AB} = \vec{B} - \vec{A} = (-\hat{j} - \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\vec{AC} = \vec{C} - \vec{A} = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -\hat{i} + 4\hat{j} + 3\hat{k}$

$\vec{AD} = \vec{D} - \vec{A} = 4(-\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -8\hat{i} - \hat{j} + 3\hat{k}$

Vectors are coplanar, if $[\vec{AB} \quad \vec{AC} \quad \vec{AC}] = 0$

$$\therefore \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(15) + 6(21) - 2(33)$$

$= -60 + 126 - 66$

$= 0$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Majority of the candidates solved the question correctly. A few candidates made calculation errors in the process of simplification of scalar triple product.

Answering Tip

- Students are advised to concentrate on solving a sum step wise to avoid calculation mistakes.

AI Q. 4. Show that :

$$\vec{a}.(\vec{b} + \vec{c}) \times (\vec{a} + 2\vec{b} + 3\vec{c}) = [\vec{a} \quad \vec{b} \quad \vec{c}]$$

[ISC Specimen Paper, 2018] [ISC, 2014]

Sol. LHS $= \vec{a}.(\vec{b} + \vec{c}) \times (\vec{a} + 2\vec{b} + 3\vec{c})$

$= (\vec{a}.\vec{b} + \vec{a}.\vec{c}) \times (\vec{a} + 2\vec{b} + 3\vec{c})$

$= (\vec{a}.\vec{b}) \times \vec{a} + (\vec{a}.\vec{b}) \times 2\vec{b} + (\vec{a}.\vec{b}) \times 3\vec{c}$

$+ (\vec{a}.\vec{c}) \times \vec{a} + (\vec{a}.\vec{c}) \times 2\vec{b} + (\vec{a}.\vec{c}) \times 3\vec{c}$

$= [\vec{a} \quad \vec{b} \quad \vec{a}] + [\vec{a} \quad \vec{b} \quad 2\vec{b}] + [\vec{a} \quad \vec{b} \quad 3\vec{c}]$

$+ [\vec{a} \quad \vec{c} \quad \vec{a}] + [\vec{a} \quad \vec{c} \quad 2\vec{b}] + [\vec{a} \quad \vec{c} \quad 3\vec{c}]$

$= 0 + 2(0) + 3[\vec{a} \quad \vec{b} \quad \vec{c}] + 0$

$+ 2[\vec{a} \quad \vec{c} \quad \vec{b}] + 3(0)$

$(\because [\vec{a} \quad \vec{b} \quad k\vec{c}] = k[\vec{a} \quad \vec{b} \quad \vec{c}])$

LHS $= 3[\vec{a} \ \vec{b} \ \vec{c}] - 2[\vec{a} \ \vec{b} \ \vec{c}]$

$= [\vec{a} \ \vec{b} \ \vec{c}] \qquad (\because [\vec{a} \ \vec{b} \ \vec{c}] = -[\vec{a} \ \vec{b} \ \vec{c}])$

= RHS **Hence Proved.**

[ISC Marking Scheme, 2014]

6 Marks Questions

Q. 1. Find the value of λ for which the four points with position vectors $6\hat{i}-7\hat{j}, 16\hat{i}-19\hat{j}-4\hat{k}, \lambda\hat{j}-6\hat{k}$ and $2\hat{i}-5\hat{j}+10\hat{k}$ are coplanar. [ISC, 2017]

Sol.

$$\vec{OA} = 6\hat{i}-7\hat{j}$$

$$\vec{OB} = 16\hat{i}-19\hat{j}-4\hat{k}$$

$$\vec{OC} = \lambda\hat{j}-6\hat{k}$$

$$\vec{OD} = 2\hat{i}-5\hat{j}+10\hat{k}$$

$$\vec{AB}=\vec{OB}-\vec{OA} = 10\hat{i}-12\hat{j}-4\hat{k}$$

$$\vec{AC}=\vec{OC}-\vec{OA} = -6\hat{i}+(\lambda+7)\hat{j}-6\hat{k}$$

$$\vec{AD}=\vec{OD}-\vec{OA} = -4\hat{i}+2\hat{j}+10\hat{k}$$

$\because$ Points A, B, C and D are coplanar

$$[\vec{AB}\,\vec{AC}\,\vec{AD}] = 0$$

$$\therefore \quad \begin{vmatrix} 10 & -12 & -4 \\ -6 & \lambda+7 & -6 \\ -4 & 2 & 10 \end{vmatrix} = 0$$

$$10\,[10\,(\lambda+7)+12]+12\,[-60-24]-4\,[-12+4(\lambda+7)]=0$$

$$10\,[10\lambda+82]+12\,[-84]-4\,[4\lambda+16]=0$$

$$84\lambda = 252$$

$$\lambda = 3$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Some candidates made errors in finding the vectors $\vec{AB}, \vec{AC}$ and $\vec{AD}$. Some candidates expanded the scalar triple product determinant incorrectly. A few candidates did not know the basic difference between position vector and vector.

Answering Tip

- Understand scalar triple product and its applications with the help of practical examples.

AI **Q. 2. For any three vectors $\vec{a},\vec{b},\vec{c}$ show that $\vec{a}-\vec{b},\vec{b}-\vec{c},\vec{c}-\vec{a}$ are coplanar. [ISC 2016, 12]**

Sol. $[\vec{a}-\vec{b},\vec{b}-\vec{c},\vec{c}-\vec{a}]$

$$=(\vec{a}-\vec{b})\cdot[(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})]$$

$$=(\vec{a}-\vec{b})\cdot[\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$$

$$=(\vec{a}-\vec{b})\cdot[\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}]$$

$$=\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{a}\cdot(\vec{c}\times\vec{a})$$
$$-\vec{b}\cdot(\vec{b}\times\vec{c})-\vec{b}\cdot(\vec{a}\times\vec{b})-\vec{b}\cdot(\vec{c}\times\vec{a})$$

$$=[\vec{a}\;\vec{b}\;\vec{c}]-[\vec{b}\;\vec{c}\;\vec{a}]$$

$$=[\vec{a}\;\vec{b}\;\vec{c}]-[\vec{a}\;\vec{b}\;\vec{c}]=0$$

Hence vectors are co-planar.

OR

If $a+b+c=0$, one vector can be represented as linear combination of other two.

$\therefore$ a, b, c are coplanar

Hence proved that $a-b, b-c, c-a$ are coplanar.

Since, $(a-b)+(b-c)+(c-a)=0$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Some of the candidates failed to apply condition of co-planarity. Many candidates made errors while simplifying the scalar triple product $[\vec{a}-\vec{b},\vec{b}-\vec{c},\vec{c}-\vec{a}]$. A few candidates failed to write the correct order of scalar triple product.

Answering Tip

- Scalar triple product and its applications need to be practised with the help of practical examples.

Q. 3. Find the volume of a parallelepiped whose edges are represented by the vectors :

$\vec{a}=2\hat{i}-3\hat{j}-4\hat{k}, \vec{b}=1\hat{i}+2\hat{j}-1\hat{k},$ and $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$ [ISC, 2015]

Sol. The volume of the parallelepiped is:

$$=\begin{vmatrix} 2 & -3 & -4 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$=2\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}+3\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}-4\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 2\cdot 5 + 3\cdot 5 - 4(-5)$$

$$= 45 \text{ cubic units}$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- The concept of scalar triple product was clear to most of the candidates but some wrote dot product first and then cross product, which was incorrect. Some wrote $[\vec{a}\ \ \vec{b}\ \ \vec{c}]$ in determinant form and made mistakes in calculation.

Answering Tip

- Combination of dot and cross product in scalar triple product needs thorough understanding as well as rigorous practice.

AI Q. 4. Prove that : $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

[ISC 2011, 2007]

Sol. We have

$$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$$

$$\text{L.H.S.} = (\vec{a}+\vec{b})\cdot[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$$

$$= (\vec{a}+\vec{b})\cdot[(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{c})+\vec{c}\times\vec{a}]$$

$$= (\vec{a}+\vec{b})\cdot[\vec{b}\times\vec{c}-\vec{a}\times\vec{b}+\vec{c}\times\vec{a}]$$

$$\left[\begin{array}{l}\therefore\ \vec{c}\times\vec{c}=0\\ \text{and } \vec{b}\times\vec{a}=-(\vec{a}\times\vec{b})\end{array}\right]$$

$$= \vec{a}\cdot(\vec{b}\times\vec{c})-\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{a}\cdot(\vec{c}\times\vec{a})$$
$$+\vec{b}\cdot(\vec{b}\times\vec{c})-\vec{b}\cdot(\vec{a}\times\vec{b})+\vec{b}\cdot(\vec{c}\times\vec{a})$$

$$= [\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{a}\ \vec{b}]+[\vec{a}\ \vec{c}\ \vec{a}]+[\vec{b}\ \vec{b}\ \vec{c}]$$
$$-[\vec{b}\ \vec{a}\ \vec{b}]+[\vec{b}\ \vec{c}\ \vec{a}]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]$$

[Other scalar triple products having two equal vectors being 0.]

$$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{c}]$$

$$= 2[\vec{a}\ \vec{b}\ \vec{c}] = \text{R.H.S.}$$

Q. 5. Show that four points with position vectors $6\hat{i}-7\hat{j}$, $16\hat{i}-19\hat{j}-4\hat{k}$, $3\hat{j}-6\hat{k}$ and $2\hat{i}+5\hat{j}+10\hat{k}$ are not coplanar.

Sol. Let, $A = 6\hat{i}-7\hat{j}, B = 16\hat{i}-19\hat{j}-4\hat{k}, C = 3\hat{j}-6\hat{k}$

and $D = 2\hat{i}+5\hat{j}+10\hat{k}$

$$\therefore\quad \vec{a} = \vec{AB} = (16\hat{i}-19\hat{j}-4\hat{k})-(6\hat{i}-7\hat{j})$$

$$= 10\hat{i}-12\hat{j}-4\hat{k}$$

$$\vec{b} = \vec{AC} = (3\hat{j}-6\hat{k})-(6\hat{i}-7\hat{j})$$

$$= -6\hat{i}+10\hat{j}-6\hat{k}$$

$$\text{and}\quad \vec{c} = \vec{AD} = (2\hat{i}+5\hat{j}+10\hat{k})-(6\hat{i}-7\hat{j})$$

$$= -4\hat{i}+12\hat{j}+10\hat{k}$$

$\vec{a}, \vec{b}, \vec{c}$ are coplanar, if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\text{Now, } [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix}$$

$$= 10(100+72)+12(-60-24)-4(-72+40)$$

$$= 1720-1008+128$$

$$= 840 \neq 0$$

$\therefore\ \vec{a}, \vec{b}, \vec{c}$ are not coplanar.

Thus, points *A*, *B*, *C* and *D* are not coplanar.

Q. 6. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} = 0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$ then prove that :

(i) $\vec{a} = \pm 2(\vec{b}\times\vec{c})$

(ii) $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = \pm 1$

Sol. (i) As given

$\vec{a}\cdot\vec{b} = 0, \vec{a}\cdot\vec{c} = 0 = \vec{a} \perp$ both $\vec{b}$ and $\vec{c}$

(as $\vec{a}, \vec{b}, \vec{c}$ are non-zero vectors)

$$\Rightarrow\quad \vec{a} \parallel (\vec{b}\times\vec{c})$$

$$\text{Let}\quad \vec{a} = \lambda(\vec{b}\times\vec{c})$$

$$\text{then}\quad |\vec{a}| = |\lambda||(\vec{b}\times\vec{c})|$$

$$\Rightarrow\quad \frac{|\vec{a}|}{|(\vec{b}\times\vec{c})|} = |\lambda|$$

$$\Rightarrow\quad |\lambda| = \frac{1}{|\vec{b}||\vec{c}|\sin\theta} = \frac{1}{\sin\frac{\pi}{6}} = 2$$

$$[\because |\vec{b}| = |\vec{c}| = 1]$$

$$\therefore \quad \lambda = \pm 2$$

$$\therefore \quad \vec{a} = \pm 2(\vec{b} \times \vec{c})$$ **Hence Proved.**

(ii) Now, $[\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}]$

$$= [(\vec{a}+\vec{b}) \times (\vec{b}+\vec{c})] \cdot (\vec{c}+\vec{a})$$

$$= (\vec{a} \times \vec{b}) \cdot \vec{c} + (\vec{b} \times \vec{c}) \cdot \vec{a}$$

Now, $\vec{a} \cdot (\vec{b} \times \vec{c}) + (\vec{a} \times \vec{b}) \cdot \vec{c}$

$$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$= 2\vec{a}(\vec{b} \times \vec{c}) \quad = 2\vec{a}\left(\pm\frac{1}{2}\vec{a}\right)$$

$$= \pm 1$$ **Hence Proved.**

SELF ASSESSMENT TEST

UNIT–V

Maximum Marks : 25 **Timing : 1 Hour**

Q. 1. Write a unit vector in the direction of the sum of vectors :

$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ **and** $\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$

Q. 2. Find the position vector of a point which divides the join of points with position vectors $(\vec{a} - 2\vec{b})$ **and** $(2\vec{a} + \vec{b})$ **externally in the ratio 2 : 1.**

Q. 3. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ **and** $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ **are two equal vectors, then the value of** $x + y + z$**.**

Q. 4. Let the vectors $\vec{a}$ **and** $\vec{b}$ **be such that** $|\vec{a}| = 3$ **and** $|\vec{b}| = \frac{\sqrt{2}}{3}$**, then** $\vec{a} \times \vec{b}$ **is a unit vector, if the angle between** $\vec{a}$ **and** $\vec{b}$ **is**

(a) $\frac{\pi}{6}$ **(b)** $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ **(d)** $\frac{\pi}{2}$

Q. 5. Find λ, so that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ **and** $3\hat{i} + \lambda\hat{j} + 5\hat{k}$ **are coplanar.**

Q. 6. The position vectors of points *A*, *B* and *C* are $\lambda\hat{i} + 3\hat{j}$, $12\hat{i} + \mu\hat{j}$ **and** $11\hat{i} - 3\hat{j}$ **respectively. If *C* divides the line segment joining *A* and *B* in the ratio 3 : 1, find the values of λ and** μ**.**

Q. 7. Find the angle between the vectors

$\vec{a} = \hat{i} + \hat{j} - \hat{k}$ **and** $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

Q. 8. The vectors from origin to the points A and B are $\vec{a} = 2\hat{i} - 3\hat{j} + 2\hat{k}$ **and** $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$ **respectively, then find the area of triangle OAB.**

Q. 9. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i}$ **and** $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$**, then If** $c_2 = -1$ **and** $c_3 = 1$**, show that no value of** c_1 **can make** $\vec{a}, \vec{b}$ **and** $\vec{c}$ **coplanar.**

Q. 10. If the sum of two unit vectors $\vec{a}$ **and** $\vec{b}$ **is a unit vector, show that the magnitude of their difference is** $\sqrt{3}$**.**

Q. 11. The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} - 5\hat{k}$ **and** $2\hat{i} + 2\hat{j} + 3\hat{k}$**. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram.**

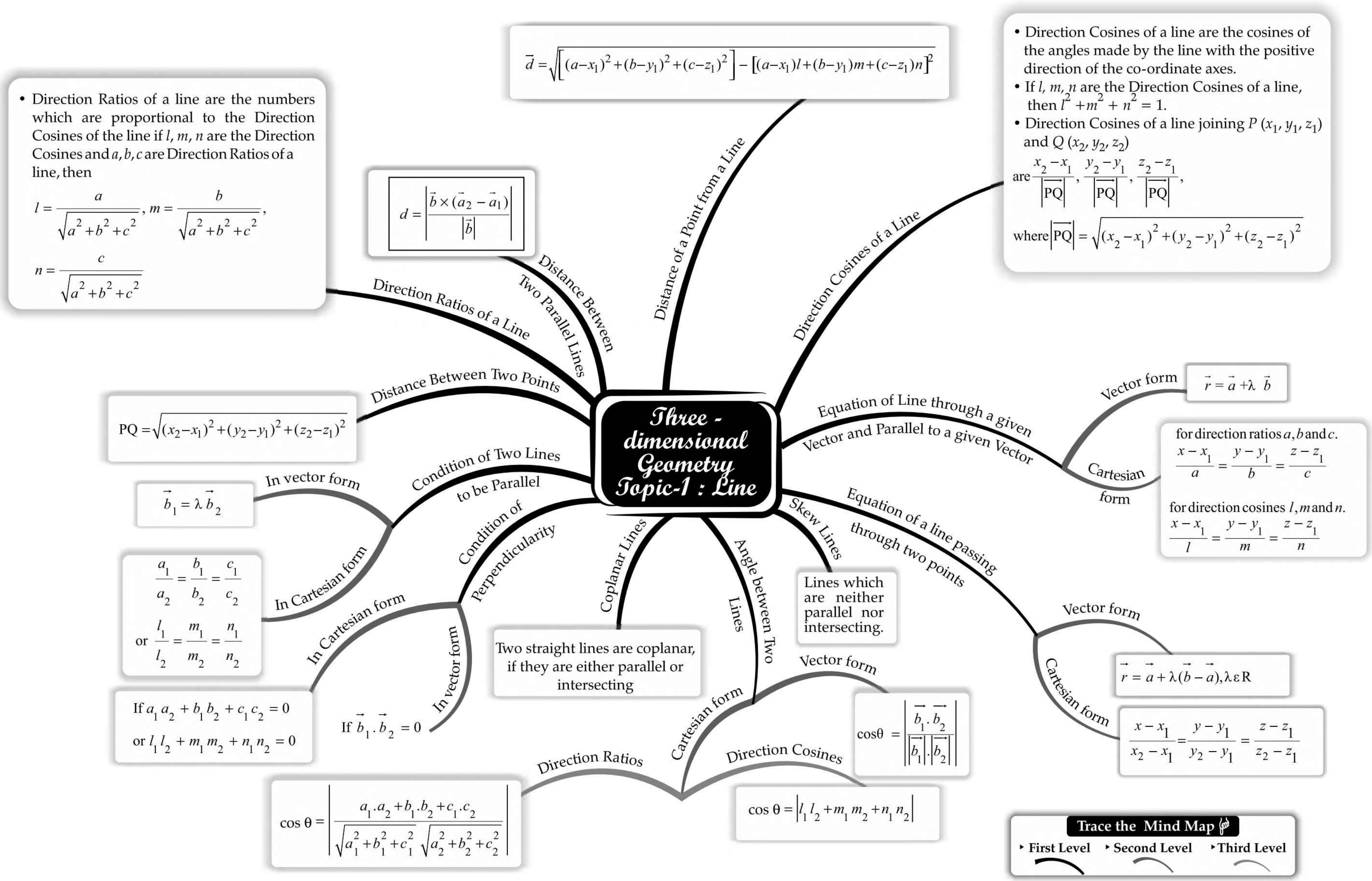
Q. 12. Find the value of λ, if four points with position vectors $3\hat{i} + 6\hat{j} + 9\hat{k}, \hat{i} + 2\hat{j} + 3\hat{k}, 2\hat{i} + 3\hat{j} + \hat{k}$ **and** $4\hat{i} + 6\hat{j} + \lambda\hat{k}$ **are coplanar.**

OSWAAL EXAM TOOLS

OSWAAL PREPARATION TOOLS

Solutions for Self-assessment Test

Visit : *http://qr.w69b.com/g/pi8gdC0qk* **OR Scan the Code**



Three - dimensional Geometry Topic-1 : Line

Direction Cosines of a Line

- Direction Cosines of a line are the cosines of the angles made by the line with the positive direction of the co-ordinate axes.
- If l, m, n are the Direction Cosines of a line, then $l^2 + m^2 + n^2 = 1$.
- Direction Cosines of a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2 - x_1}{|\overrightarrow{PQ}|}, \frac{y_2 - y_1}{|\overrightarrow{PQ}|}, \frac{z_2 - z_1}{|\overrightarrow{PQ}|}$, where $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Distance of a Point from a Line

$$\vec{d} = \sqrt{\left[(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right] - \left[(a-x_1)l+(b-y_1)m+(c-z_1)n\right]^2}$$

Distance Between Two Parallel Lines

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

Direction Ratios of a Line

- Direction Ratios of a line are the numbers which are proportional to the Direction Cosines of the line if l, m, n are the Direction Cosines and a, b, c are Direction Ratios of a line, then

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}},$$
$$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Distance Between Two Points

$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Condition of Two Lines to be Parallel

In vector form

$$\vec{b}_1 = \lambda \vec{b}_2$$

In Cartesian form

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$
$$\text{or } \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

Condition of Perpendicularity

In Cartesian form

$$\text{If } a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$
$$\text{or } l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

In vector form

$$\text{If } \vec{b}_1 . \vec{b}_2 = 0$$

Coplanar Lines

Two straight lines are coplanar, if they are either parallel or intersecting

Angle between Two Lines

Vector form

$$\cos\theta = \left|\frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

Cartesian form

Direction Cosines

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

Direction Ratios

$$\cos\theta = \left|\frac{a_1.a_2 + b_1.b_2 + c_1.c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

Skew Lines

Lines which are neither parallel nor intersecting.

Equation of a line passing through two points

Vector form

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \lambda \varepsilon R$$

Cartesian form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Equation of Line through a given Vector and Parallel to a given Vector

Vector form

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

Cartesian form

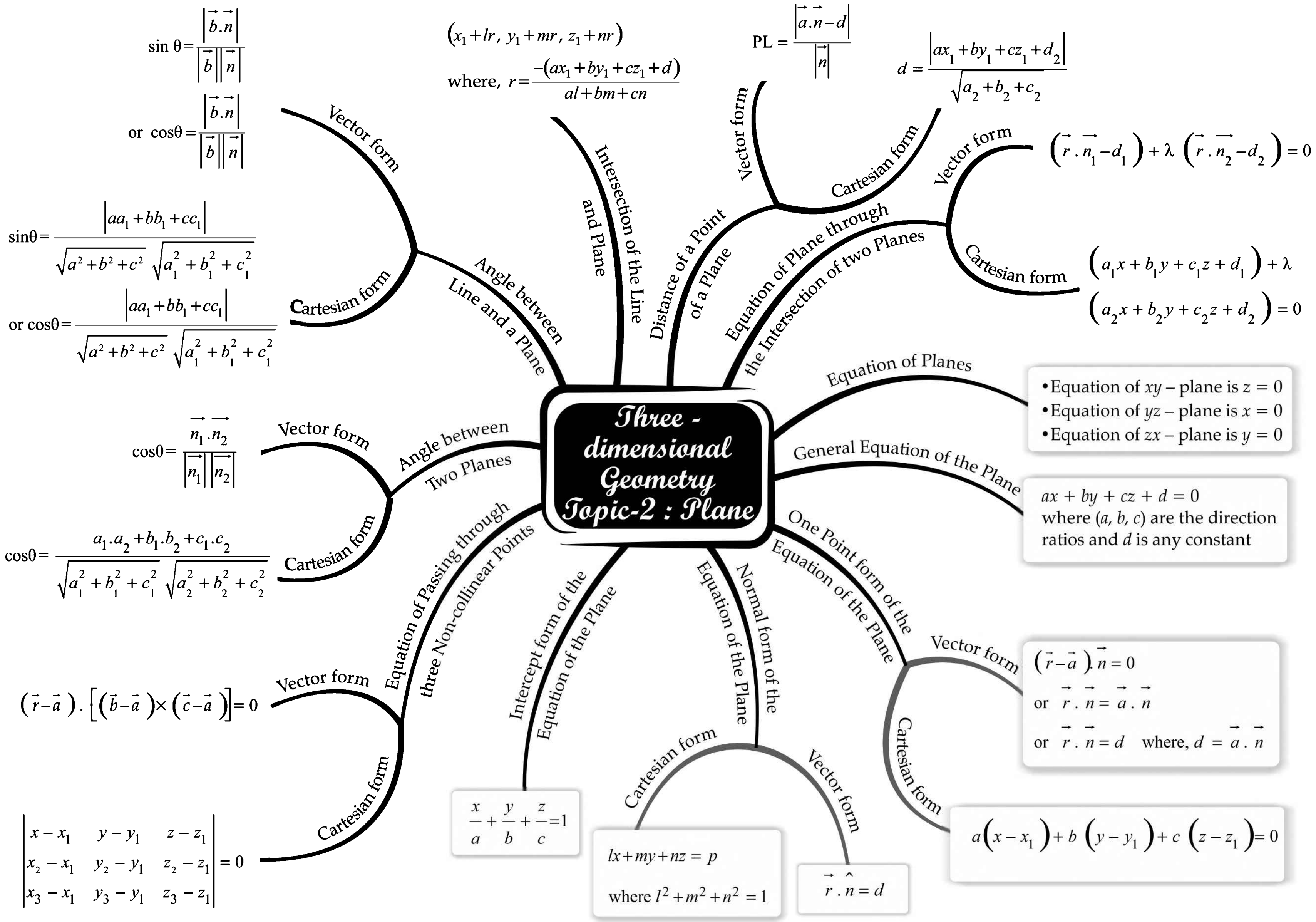
for direction ratios a, b and c.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

for direction cosines l, m and n.

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Trace the Mind Map



Three - dimensional Geometry Topic-2 : Plane

- Equation of Planes
  - Equation of xy – plane is $z = 0$
  - Equation of yz – plane is $x = 0$
  - Equation of zx – plane is $y = 0$
- General Equation of the Plane
  - $ax + by + cz + d = 0$ where (a, b, c) are the direction ratios and d is any constant
- One Point form of the Equation of the Plane
  - Vector form
    - $(\vec{r}-\vec{a}).\vec{n} = 0$
    - or $\vec{r}.\vec{n} = \vec{a}.\vec{n}$
    - or $\vec{r}.\vec{n} = d$ where, $d = \vec{a}.\vec{n}$
  - Cartesian form
    - $a\left(x-x_1\right)+b\left(y-y_1\right)+c\left(z-z_1\right)=0$
- Normal form of the Equation of the Plane
  - Vector form
    - $\vec{r}.\hat{n} = d$
  - Cartesian form
    - $lx+my+nz = p$ where $l^2+m^2+n^2 = 1$
- Intercept form of the Equation of the Plane
  - $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$
- Equation of Passing through three Non-collinear Points
  - Vector form
    - $(\vec{r}-\vec{a}).\left[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right]=0$
  - Cartesian form
    - $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$
- Angle between Two Planes
  - Vector form
    - $\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$
  - Cartesian form
    - $\cos\theta = \frac{a_1.a_2+b_1.b_2+c_1.c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$
- Angle between Line and a Plane
  - Vector form
    - $\sin\theta = \frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$
    - or $\cos\theta = \frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$
  - Cartesian form
    - $\sin\theta = \frac{|aa_1+bb_1+cc_1|}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}$
    - or $\cos\theta = \frac{|aa_1+bb_1+cc_1|}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}$
- Intersection of the Line and Plane
  - (x_1+lr, y_1+mr, z_1+nr) where, $r = \frac{-(ax_1+by_1+cz_1+d)}{al+bm+cn}$
- Distance of a Point of a Plane
  - Vector form
    - $PL = \frac{|\vec{a}.\vec{n}-d|}{|\vec{n}|}$
  - Cartesian form
    - $d = \frac{|ax_1+by_1+cz_1+d_2|}{\sqrt{a_2+b_2+c_2}}$
- Equation of Plane through the Intersection of two Planes
  - Vector form
    - $\left(\vec{r}.\vec{n}_1-d_1\right)+\lambda\left(\vec{r}.\vec{n}_2-d_2\right)=0$
  - Cartesian form
    - $\left(a_1x+b_1y+c_1z+d_1\right)+\lambda\left(a_2x+b_2y+c_2z+d_2\right)=0$

Trace the Mind Map

- First Level
- Second Level
- Third Level

UNIT-VI

Three Dimensional Geometry

CHAPTER 11

THREE-DIMENSIONAL GEOMETRY

SYLLABUS

- *Direction cosines and direction ratios of a line joining two points. Cartesian and vector equation of a line, coplanar and skew lines, shortest distance between two lines. Cartesian and vector equation of a plane. Angle between (i) two lines (ii) two planes, (iii) a line and a plane. Distance of a point from a plane.*
  - *Equation of x-axis, y-axis, z-axis and lines parallel to them.*
  - *Equation of xy-plane, yz-plane and zx-plane.*
  - *Direction cosines, direction ratios.*
  - *Angle between two lines in terms of direction cosines/direction ratios.*
  - *Condition for lines to be perpendicular/parallel.*
- *Lines*
  - *Cartesian and vector equations of a line through one and two points.*
  - *Coplanar and skew lines.*
  - *Condition for intersection of two lines.*
  - *Distance of a point from a line.*
  - *Shortest distance between two lines.*

Note : Symmetric and non-symmetric forms of lines are required to be covered.

- *Planes*
  - *Cartesian and vector equation of a plane.*
  - *Direction ratios of the normal to the plane.*
  - *One point form.*
  - *Normal form.*
  - *Intercept form.*
  - *Distance of a point from a plane.*
  - *Intersection of the line and plane.*
  - *Angle between two planes, a line and a plane.*
  - *Equation of a plane through the intersection of two planes i.e.* $P_1 + kP_2 = 0$.

Trend Analysis

List of Concepts	2016	2017	2018	2019	2020
Direction Ratios, Direction cosines and lines	1 Q (6 marks)	1 Q (6 marks)	1 Q (2 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (2 marks) 1 Q (4 marks)
Plane and Its Equations in various form	1 Q (6 marks)	1 Q (6 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (2 marks) **OR** 1 Q (4 marks)	1 Q (2 marks) **OR** 1 Q (4 marks)

TOPIC - 1
Direction Ratios, Direction Cosines and Lines .... **P. 306**

TOPIC - 2
Plane and Its Equation in Various Forms .... **P. 316**

TOPIC-1

Direction Ratios, Direction Cosines and Lines

Revision Notes

- **Direction Cosines of a Line :**

  If A and B are the two points on a given line L then direction cosines of vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are the direction cosines (d.c.´s) of line L. Thus, if α, β, γ are the direction-angles which the line L makes with the positive direction of x, y, z-axis respectively, then its d.c.´s are $\cos\alpha$, $\cos\beta$, $\cos\gamma$. If direction of line L is reserved, the direction angles are replaced by their supplements, *i.e.*, $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$ and so are the d.c.´s, *i.e.*, the direction cosines become $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$. So, a line in space has two set of d.c.´s *viz* $\pm\cos\alpha$, $\pm\cos\beta$, $\pm\cos\gamma$.

Scan to know more about this topic

Introduction to Three Dimensional Geometry

Scan to know more about this topic

Direction Cosines & Direction Ratios of Vector

- The d.c.´s are generally denoted by l, m, n. Also, $l^2+m^2+n^2=1$ and so we can deduce that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. Also, $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$
- The d.c.´s of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $\pm\frac{x_2-x_1}{AB}, \pm\frac{y_2-y_1}{AB}, \pm\frac{z_2-z_1}{AB}$; where AB is the distance between the points A and B, *i.e.*, $AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

- **Direction Ratios of a Line :**

  Any three numbers a, b, c (say) which are proportional to d.c.´s, *i.e.*, l, m, n of a line are called the **direction ratios** (d.r.´s) of the line. Thus, $a=\lambda l$, $b=\lambda m$, $c=\lambda n$ for any $\lambda \in R-\{0\}$.

Consider,
$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{1}{\lambda} \quad \text{(say)}$$
$$\Rightarrow \quad l=\frac{a}{\lambda},\ m=\frac{b}{\lambda},\ n=\frac{c}{\lambda}$$
$$\Rightarrow \quad \left(\frac{a}{\lambda}\right)^2+\left(\frac{b}{\lambda}\right)^2+\left(\frac{c}{\lambda}\right)^2=1 \quad [\text{Using } l^2+m^2+n^2=1]$$
$$\Rightarrow \quad \lambda=\pm\sqrt{a^2+b^2+c^2}$$

Therefore,
$$l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}},\ n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

- The d.r.´s of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are x_2-x_1, y_2-y_1, z_2-z_1 or x_1-x_2, y_1-y_2, z_1-z_2.
- Direction ratios are sometimes called as **Direction Numbers**.

- **Equation of a Line passing through two given points :**

  Consider the two given points as $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ with position vectors $\vec{a}$ and $\vec{b}$ respectively. Also, assume $\vec{r}$ as the position vector of any arbitrary point $P(x, y, z)$ on the line L passing through A and B.

Thus, $\overrightarrow{OA}=\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, $\overrightarrow{OB}=\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$, $\overrightarrow{OP}=\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

(a) Vector equation of a line : Since the points A, B and P all lie on the same line which means they are all collinear points.

Further it means, $\overrightarrow{AP}=\vec{r}-\vec{a}$ and $\overrightarrow{AB}=\vec{b}-\vec{a}$ are collinear vectors,

i.e.,
$$\overrightarrow{AP}=\lambda\overrightarrow{AB}$$
$$\Rightarrow \quad \vec{r}-\vec{a}=\lambda(\vec{b}-\vec{a})$$
$$\Rightarrow \quad \vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a}),\ \text{where } \lambda \in R.$$

This is the vector equation of the line.

Scan to know more about this topic

Equations of a straight line in space

(b) Cartesian equation of a line : By using the vector equation of the line $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

On equating the coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$, we get

$$x = x_1 + \lambda(x_2 - x_1), y = y_1 + \lambda(y_2 - y_1), z = z_1 + \lambda(z_2 - z_1)$$

On eliminating λ, we have

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

- **Angle between two lines :**

(a) When d.r.´s or d.c.´s of the two lines are given :

Consider two lines L_1 and L_2 with d.r.´s in proportion to a_1, b_1, c_1 and a_2, b_2, c_2 respectively; d.c.´s as l_1, m_1, n_1 and l_2, m_2, n_2. Consider $\vec{b_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$. These, vectors $\vec{b_1}$ and $\vec{b_2}$ are parallel to the given lines L_1 and L_2. So, in order to find the angle between the lines L_1 and L_2, we need to get the angle between the vectors $\vec{b_1}$ and $\vec{b_2}$.

So the acute angle θ between the vectors $\vec{b_1}$ and $\vec{b_2}$ (and hence lines L_1 and L_2) can be obtained as,

$$\vec{b_1} \cdot \vec{b_2} = |\vec{b_1}||\vec{b_2}|\cos\theta$$

$$\therefore \qquad \cos\theta = \frac{\vec{b_1} . \vec{b_2}}{|\vec{b_1}| . |\vec{b_2}|}$$

Thus,
$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

- Also, in terms of d.c.´s : $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$.
- Sine of angle is given as : $\sin\theta = \left|\frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

(b) When vector equations of two lines are given :

Consider vector equations of lines L_1 and L_2 as $\vec{r_1} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r_2} = \vec{a_2} + \mu\vec{b_2}$ respectively.

Then, the acute angle θ between the two lines is given by the relation

$$\cos\theta = \left|\frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$$

(c) When Cartesian equation of two lines are given :

Consider the lines L_1 and L_2 in Cartesian form as,

$$L_1 = \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

$$L_2 = \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

Then the acute angle θ between the Lines L_1 and L_2 can be obtained by,

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

Important Facts

- For two perpendicular lines :
  $a_1a_2 + b_1b_2 + c_1c_2 = 0$, $l_1l_2 + m_1m_2 + n_1n_2 = 0$.
- **For two parallel lines :**
  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}; \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

- **Shortest Distance between two Lines :**

If two lines are in the same plane, *i.e.*, they are coplanar, they will intersect each other if they are non-parallel. Hence, the shortest distance between them is zero. If the lines are parallel then the shortest distance between them will be the perpendicular distance between the lines, *i.e.*, the length of the perpendicular drawn from a point on one line onto the other line. Adding to this discussion, in space, there are lines which are neither intersecting nor parallel. Such pair of lines are non-coplanar and are called the skew lines.

Know the Formulae

- **Distance formula :**

  The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by the expression

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \text{ units}$$

- **Section formula :**

  The co-ordinates of a point Q which divides the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$

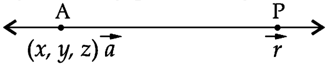
  (a) internally, are $\left(\frac{(mx_2 + nx_1)}{m+n}, \frac{(my_2 + ny_1)}{m+n}, \frac{(mz_2 + nz_1)}{m+n}\right)$

  (b) externally, *i.e.*, internally in the ratio $(m) : (-n)$, are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

Know the Terms

- **Equation of a line in space passing through a given point and parallel to a given vector :**

  Consider the line L is passing through the given point $A(x_1, y_1, z_1)$ with the position vector $\vec{a}$, $\vec{b}$ is the given vector with d.r.'s a, b, c and $\vec{r}$ is the position vector of any arbitrary point $P(x, y, z)$ on the line.



Thus, $\vec{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{b} = a\hat{i} + b\hat{j} + c\hat{k}.$

(a) Vector equation of a line : As the line L is parallel to given vector $\vec{b}$ and points A and P are lying on the line so, $\vec{AP}$ is parallel to the $\vec{b}$.

$\Rightarrow$ $\vec{AP} = \lambda\vec{b}$, where $\lambda \in R$, *i.e.*, set of real numbers

$\Rightarrow$ $\vec{r} - \vec{a} = \lambda\vec{b}$

$\Rightarrow$ $\vec{r} = \vec{a} + \lambda\vec{b}$

This is the vector equation of line.

(b) Parametric equations : If d.r.'s of the line are a, b, c, then by using $\vec{r} = \vec{a} + \lambda\vec{b}$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$$

Now, as we equate the coefficients of $\hat{i}, \hat{j}, \hat{k}$, we get the parametric equations of the line given as,

$$x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c.$$

- Co-ordinates of any point on the line considered here are $(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$.

(c) Cartesian equation of a line : If we eliminate the parameter λ from the parametric equations of a line, we get the Cartesian equation of line as

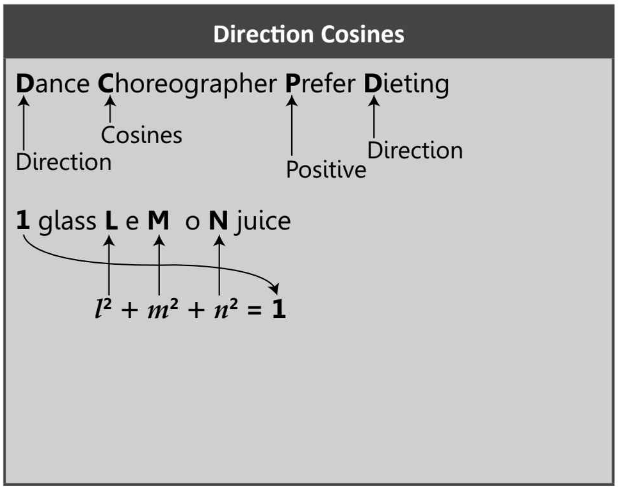
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

- If l, m, n are the d.c.'s of the line, then Cartesian equation of line becomes

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- **Skew Lines :** Two straight lines in space which are neither parallel nor intersecting are known as the skew lines. They lie in different planes and are non-coplanar.
- **Lines of Shortest distance :** There exists unique line perpendicular to each of the skew lines L_1 and L_2, and this line is known as the shortest distance (S.D.).

Mnemonics

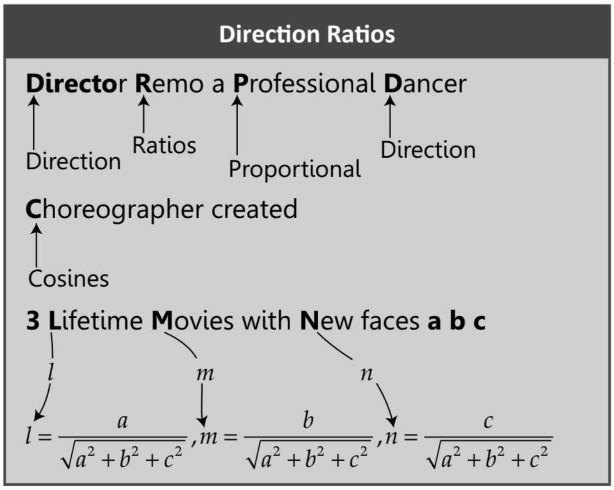


Direction Cosines

Dance **C**horeographer **P**refer **D**ieting

1 glass **L** e **M** o **N** juice

$l^2 + m^2 + n^2 = 1$



Direction Ratios

Director **R**emo a **P**rofessional **D**ancer

Choreographer created

3 Lifetime **M**ovies with **N**ew faces **a b c**

$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$

1 Mark Questions

AI Q. 1. If the directions cosines of a line are k, k, k, then

(a) $k > 0$ **(b) $0 < k < 1$**

(c) $k = 1$ **(d) $k = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$**

Sol. Correct option : (d)

Explanation :

Since, direction cosines of a line are k, k and k.

$\therefore\ l = k, m = k$ and $n = k$

We know that, $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \qquad k^2+k^2+k^2=1$$

$$k^2=\frac{1}{3}$$

$$\therefore \qquad k=\pm\frac{1}{\sqrt{3}}$$

Q. 2. The reflection of the point (α, β, γ) in the xy-plane is

(a) $(\alpha, \beta, 0)$ **(b) $(0, 0, \gamma)$**

(c) $(-\alpha, -\beta, \gamma)$ **(d) $(\alpha, \beta, -\gamma)$**

Sol. Correct option : (d)

Explanation :

In xy-plane, the reflection of the point (α, β, γ) is $(\alpha, \beta, -\gamma)$.

AI Q. 3. If a line makes angles α, β, γ with the positive direction of co-ordinate axes, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is.

(a) 0 **(b) 1**

(c) 2 **(d) 3**

Sol. Correct option : (c)

Explanation :

We know that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or $(1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Q. 4. Distance of the point (α, β, γ) from y-axis is

(a) β **(b) $|\beta|$**

(c) $|\beta|+|\gamma|$ **(d) $\sqrt{\alpha^2+\gamma^2}$**

Sol. Correct option : (d)

Explanation :

The foot of perpendicular from point (α, β, γ) on y-axis is $Q(0, \beta, 0)$.

Required distance,

$$PQ=\sqrt{(\alpha-0)^2+(\beta-\beta)^2+(\gamma-0)^2}=\sqrt{\alpha^2+\gamma^2}$$

Q. 5. The distance of the plane $\vec{r}\cdot\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)=1$ from the origin is

(a) 1 **(b) 7**

(c) 1/7 **(d) None of these**

Sol. Correct option : (a)

Explanation :

The distance of the plane $\vec{r}\cdot\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)=1$ form the origin is 1.

[Since $\vec{r}\cdot\hat{n}=d$ is the form of above equation, where d represents the distance of plane from the origin, *i.e.*, $d = 1$]

AI Q. 6. If a line makes angles 90°, 60° and θ with x, y and z-axis respectively, where θ is acute, then find θ.

Sol. Since,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\text{or} \quad \cos^2 90^\circ + \cos^2 60^\circ + \cos^2\theta = 1$$

$$0 + \frac{1}{4} + \cos^2\theta = 1$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\text{or} \quad \theta = 30^\circ$$

Q. 7. Find the direction cosines of the line :

$$\frac{x-1}{2} = -y = \frac{z+1}{2}$$

Sol. Direction ratios of the given line are 2, –1, 2.

The direction cosines of the line are

$$\pm\frac{2}{\sqrt{2^2+(-1)^2+2^2}},\ \pm\frac{-1}{\sqrt{2^2+(-1)^2+2^2}},\ \pm\frac{2}{\sqrt{2^2+(-1)^2+2^2}}$$

i.e., $\pm\frac{2}{\sqrt{9}},\ \pm\frac{-1}{\sqrt{9}},\ \pm\frac{2}{\sqrt{9}}$

$$\frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \text{ or } \frac{-2}{3}, \frac{1}{3}, \frac{-2}{3}$$

Q. 8. Write the equation of the straight line through the point (α, β, γ) and parallel to *z*-axis.

Sol. $$\frac{x-\alpha}{0} = \frac{y-\beta}{0} = \frac{z-\gamma}{1}$$

Q. 9. If the cartesian equation of a line are $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line.

Sol. Given $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$

$$\frac{x-3}{-5} = \frac{y+4}{7} = \frac{z-3}{2}$$

$$\vec{i} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k})$$

for the line.

Examiner's Comment

- Sometimes candidates transform cartesian form of line into vector equation incorrectly.

Answering Tip

- Learn the concept of transforming cartesian form to vector form and vice-versa in detail.

AI Q. 10. What is the distance of the point (*p, q, r*) from the *x*-axis ?

Sol. Distance of the point (*p, q, r*) from the *x*-axis

= Distance of the point (*p, q, r*) from the point(*p*, 0, 0)

$$= \sqrt{q^2 + r^2}$$

2 Marks Questions

Q 1. Find the angle between the two lines :

$$\frac{x+1}{2} = \frac{y-2}{5} = \frac{z+3}{4} \text{ and } \frac{x-1}{5} = \frac{y+2}{2} = \frac{z-1}{-5}$$

[ISC, 2020]

Sol. Angle between two lines

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

Where $a_1 = 2, b_1 = 5, c_1 = 4$

and $a_2 = 5, b_2 = 2, c_2 = -5$

$$\cos\theta = \left|\frac{2\times5 + 5\times2 + 4(-5)}{\sqrt{2^2+5^2+4^2}\sqrt{5^2+2^2+(-5)^2}}\right|$$

$$\cos\theta = \left|\frac{10+10-20}{\sqrt{4+25+16}\sqrt{25+4+25}}\right|$$

$$\cos\theta = 0$$

$$\cos\theta = \frac{\pi}{2}$$

Lines are perpendicular each other.

[ISC Marking Scheme, 2020]

Examiner's Comment

- Few students do calculation mistakes while solving cos θ. Some leave the question at cos θ = 0, they did not find the value of angle θ.

Answering Tip

- Find the final value of θ, don't leave the question in the middle.

AI Q. 2. Find the angle between the two lines $2x = 3y = -z$ and $6x = -y = -4z$. **[ISC, 2019]**

Sol. Equations are $2x = 3y = -z$, $6x = -y = -4z$

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{-6} \text{ and } \frac{x}{2} = \frac{y}{-12} = \frac{z}{-3}$$

(3, 2, –6) and (2, –12, –3) are direction cosines of the lines.

$$\cos\theta = \frac{6-24+18}{\sqrt{9+4+36}\sqrt{4+144+9}}$$

$$\cos\theta = \frac{0}{7\times\sqrt{157}}$$

$$\cos\theta = 0$$

$$\theta = 90°$$

[ISC Marking Scheme, 2019]

Examiner's Comment

- Most candidates attempted well barring a few who could not express the given equation into the desired form and hence, could not calculate the direction ratios of the lines correctly.

Answering Tip

- Understand the method of identifying direction ratios and conversion of vector form to symmetric form and vice versa.

Q. 3. The Cartesian equation of a line is : $2x - 3 = 3y + 1 = 5 - 6z$. Find the vector equation of a line passing through (7, – 5, 0) and parallel to given line.

[ISC, 2018]

Sol. Given line : $2x - 3 = 3y + 1 = 5 - 6z$

$$\Rightarrow \frac{x-3/2}{1/2} = \frac{y+1/3}{1/3} = \frac{z-5/6}{-1/6}$$

∴ Direction ratios : $\frac{1}{2}, \frac{1}{3}, \frac{-1}{6}$ or 3, 2, – 1

Equation of line passing through (7, – 5, 0) and parallel to the above line

$$\vec{r} = (7\hat{i}-5\hat{j}+0.k)+\lambda(3\hat{i}+2\hat{j}-\hat{k})$$

or $$\vec{r} = (7\hat{i}-5\hat{j})+l(3\hat{i}+2\hat{j}-\hat{k})$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Most of the candidates made errors while writing the equation of a line in the symmetrical form and finding direction ratios of the line. Some candidates transformed cartesian form of line into vector equation incorrectly.

Answering Tip

- The concept of transforming Cartesian form to vector form and vice versa need to understand in detail.

AI Q. 4. Find the cartesian equation of the line passing through the points (– 1, 0, 2) and (3, 4, 6).

[ISC Specimen Paper, 2018]

Sol. Given points are (– 1, 0, 2) and (3, 4, 6). Cartesian equation of a line passing through two points is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\therefore \quad \frac{x-(-1)}{3-(-1)} = \frac{y-0}{4-0} = \frac{z-2}{6-2}$$

$$\Rightarrow \quad \frac{x+1}{4} = \frac{y}{4} = \frac{z-2}{4}$$

$$x + 1 = y = z - 2$$

is the required equation.

Q. 5. If the equation of a line *AB* is

$$\frac{3-x}{-3} = \frac{y+2}{-2} = \frac{z+2}{6},$$

then find the direction cosines of a line parallel to *AB*.

Sol. Given, $\frac{3-x}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$

$$\Rightarrow \quad \frac{x-3}{3} = \frac{y+2}{-2} = \frac{z+2}{6}$$

Direction ratios are 3, –2, 6.

Direction Ratio's of line parallel to *AB* is 3, –2, 6.

∴ Direction Cosines

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}},$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$l = \frac{3}{\sqrt{9+4+36}}, m = \frac{-2}{\sqrt{9+4+36}}, n = \frac{6}{\sqrt{9+4+36}}$$

Hence, direction Cosines are $\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$

Q. 6. Find the cartesian equation of the line which passes through the point (–2, 4, –5) and is parallel to the line $\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}$.

Sol. Given line : $\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}$

[If two lines are parallel, then they both have proportional direction ratio]

Here, direction ratio are 3, – 5, 6.

Therefore, cartesian equation of line through (– 2, 4, – 5) having D.R's 3, – 5, 6 is

$$\frac{x-(-2)}{3} = \frac{y-4}{-5} = \frac{z-(-5)}{6}$$

$$\Rightarrow \quad \frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$$

AI Q. 7. Show that the lines $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$ are coplanar.

Sol. Equations of lines are :

$$\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5} \text{ and } \frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$$

Here, $x_1 = 5, y_1 = 7, z_1 = -3;$

$x_2 = 8, y_2 = 4, z_2 = 5,$

$a_1 = 4, b_1 = 4, c_1 = -5;$

$a_2 = 7, b_2 = 1, c_2 = 3$

$$\text{If} \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0,$$

then lines are coplanar

$$\begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 3(17)+3(47)+8(-24) = 0$$

Hence, lines are co-planar.

4 Marks Questions

AI Q. 1. Find the equation of the lines passing through the point (2, l, 3) and perpendicular to the lines

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} \text{ and } \frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$$

[ISC Specimen Paper, 2021] [ISC, 2019]

Sol. Let the direction ratios of the line be *a*, *b*, *c* and the direction ratios of two given lines are <1, 2, 3> and <–3, 2, 5>

Since the line is perpendicular to the other two lines therefore

$a + 2b + 3c = 0$

$-3a + 2b + 5c = 0$

$$\frac{a}{10-6} = \frac{b}{-9-5} = \frac{c}{2+6} \Rightarrow \frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$$

$$\frac{a}{2} = \frac{b}{-7} = \frac{c}{4}$$

Direction ratios of required line are <2, –7, 4>
And the line passes through the point (2, 1, 3)
Hence the equation will be :

$$\frac{x-2}{2}=\frac{y-1}{-7}=\frac{z-3}{4}$$

[ISC Marking Scheme, 2019]

Q. 2. Determine the equation of the line passing through the point (–1, 3, –2) and perpendicular to the lines :

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3} \text{ and } \frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5}$$ **[ISC, 2020]**

Sol. Equation of the line passes through (– 1, 3, – 2) is :

$$\frac{x+1}{a}=\frac{y-3}{b}=\frac{z+2}{c}$$

it is perpendicular to

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

$\therefore \quad a + 2b + 3c = 0$ ...(i)

it is also perpendicular to

$$\frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5}$$

$\therefore \quad -3a + 2b + 5c = 0$ ...(ii)

$a + 2b + 3c = 0$ ...(iii)

$$\frac{a}{6-10}=\frac{b}{5+9}=\frac{c}{-6-2}$$

$$\frac{a}{-4}=\frac{b}{14}=\frac{c}{-8}$$

$$\frac{a}{2}=\frac{b}{-7}=\frac{c}{4}$$

Hence, required equation of the line is :

$$\frac{x+1}{2}=\frac{y-3}{-7}=\frac{z+2}{4}$$

Examiner's Comment

- Most candidates did not apply the concept of perpendicular lines and condition for the same correctly. Some candidates made calculation errors while applying cross multiplication rule to find direction ratios.

Answering Tip

- Understand the different methods of finding the equation of straight line by applying different types of conditions.

Q. 3 Find the shortest distance between the lines $\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$ **and** $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

[ISC Specimen Paper, 2018]

Sol. Given lines are

$$\frac{x-8}{3}=\frac{y-(-9)}{-16}=\frac{z-10}{7}$$

and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

The shortest distance between the lines

$$=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$$

$$=\left|\frac{\begin{vmatrix} 15-8 & 29+9 & 5-10 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}}{\sqrt{[(-16)(-5)-(8)(7)]^2+[(7)(3)+(5)(3)]^2 +[(3)(8)-(3)(-16)]^2}}\right|$$

$$=\left|\frac{\begin{vmatrix} 7 & 38 & -5 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}}{\sqrt{(80-56)^2+(21+15)^2+(24+48)^2}}\right|$$

$$=\left|\frac{7(80-56)-38(-15-21)-5(24+48)}{\sqrt{(24)^2+(36)^2+(72)^2}}\right|$$

Shortest distance $=\left|\frac{|7(24)-38(-36)-5(72)|}{\sqrt{12^2(2^2+3^2+6^2)}}\right|$

$$=\left|\frac{168+1368-360}{12\sqrt{4+9+36}}\right|$$

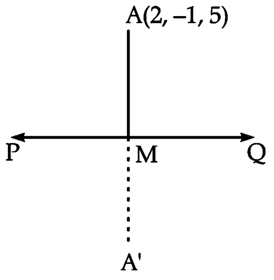
$$=\frac{1176}{12\sqrt{49}}$$

$$=\frac{98}{7}=14 \text{ units.}$$

AI Q. 4. Find the image of the point (2, –1, 5) in the line $\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}$**. Also, find the length of the perpendicular from the point (2, –1, 5) to the line.**

[ISC, 2016]

Sol. Let $A(2, -1, 5)$ be the point and PQ be the line $\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}$.



Let M be any point on the line PQ is

$M(10r + 11, -4r - 2, -11r - 8)$

d.r.'s of AM $10r + 9, -4r - 1, -11r - 13$

AM is perpendicular to PQ

$$\therefore \qquad a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$10(10r + 9) - 4(-4r - 1) - 11(-11r - 13) = 0$$

$$100r + 90 + 16r + 4 + 121r + 143 = 0$$

$$237r = -237$$

$$\therefore \qquad r = -1$$

∴ M is (1, 2, 3)

M is the mid point of AA'

$$\frac{2+x}{2} = 1, \frac{-1+y}{2} = 2, \frac{5+z}{2} = 3$$

Coordinates of A' are (0, 5, 1)

$$\text{Length of the perpendicular} = \sqrt{(-1)^2 + 3^2 + (-2)^2}$$

$$= \sqrt{14} \text{ units}$$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Candidate made mistakes in calculating a point M on the line PQ and then applying the condition to prove that $AM \perp PQ$, *i.e.*, $a_1a_2 + b_1b_2 + c_1c_2 = 0$. Some made errors while calculating the length of perpendicular.

Answering Tip

- Need to emphasize on complete understanding of perpendicularity condition at different stages. Do practice in solving problems based on the concept of Image of a given point with reference to a line.

Q. 5. Find the shortest distance between the lines $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{r} = 2\hat{i} + 4\hat{j} + 5\hat{k} + \mu(4\hat{i} + 6\hat{j} + 8\hat{k})$ [ISC, 2015]

Sol. Here, $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{a_2} = 2\hat{i} + 4\hat{j} + 5\hat{k}$

$$\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{a_2} - \vec{a_1} = (2\hat{i} + 4\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= \hat{i} + 2\hat{j} + 2\hat{k}$$

$$|\vec{b}| = \sqrt{4 + 9 + 16} = \sqrt{29}$$

$$(\vec{a_2} - \vec{a_1}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = 2\hat{i} - 0\hat{j} - \hat{k}$$

$$|(\vec{a_2} - \vec{a_1}) \times \vec{b}| = \sqrt{4 + 1} = \sqrt{5}$$

$$\frac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|} = \sqrt{\frac{5}{29}} \text{ units}$$

$$\therefore \text{Shortest distance} = \sqrt{\frac{5}{29}} \text{ units}$$

[ISC Marking Scheme, 2015]

Examiner's Comment

- A number of candidates were unable to calculate the correct values of $\bar{a}_1, \bar{a}_2$ and $\bar{b}$. Some made mistakes in calculating $(\bar{a}_2 - \bar{a}_1)$. The concepts of skew lines and parallel lines were not clear to many candidates.
- Some candidates calculated $\bar{b}_1 \times \bar{b} = 0$. They were unable to understand that the given lines are parallel. A few candidates applied wrong formula to calculate the shortest distance between the given lines.

Answering Tip

- Clearly understand the concept of parallel and non-parallel lines.

AI **Q. 6. Find the equation of a line passing through the points $P(-1, 3, 2)$ and $Q(-4, 2, -2)$. Also, if the point $R(5, 5, \lambda)$ is collinear with the points P and Q, then find the value of λ. [ISC, 2014]**

Sol. The equation of a line passing through two points $P(-1, 3, 2)$ and $Q(-4, 2, -2)$ is given by :

$$\frac{x+1}{-4+1} = \frac{y-3}{2-3} = \frac{z-2}{-2-2}$$

$$\text{or} \qquad \frac{x+1}{-3} = \frac{y-3}{-1} = \frac{z-2}{-4}$$

$$\text{or} \qquad \frac{x+1}{3} = \frac{y-3}{1} = \frac{z-2}{4} \qquad \ldots(1)$$

Now, if the three points $P(-1, 3, 2)$, $Q(-4, 2, -2)$ and $R(5, 5, \lambda)$ are collinear, then the coordinates of point R must satisfy the equation (1).

$$\frac{5+1}{3} = \frac{5-3}{1} = \frac{\lambda-2}{4}$$

$$\Rightarrow \qquad \frac{6}{3} = \frac{2}{1} = \frac{\lambda-2}{4}$$

$$\Rightarrow \qquad \lambda - 2 = 8$$

$$\textit{i.e.,} \qquad \lambda = 10$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- After expressing the equation of the line in Cartesian symmetric form, a few candidates failed to find λ by substituting coordinates of (5, 5, λ) in the equation of the line. Candidates who used vector form of the line also made a similar mistake.

Answering Tip

- Equation of a line in all forms needs to be understood well. Direction ratios of line and parallel, perpendicular lines need clarity of understanding.

Q. 7. Find the equation of a line passing through the point (–1, 3, –2) and perpendicular to the lines : $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$. [ISC, 2013]

Sol. Any line through the point (–1, 3, –2) is given by

$$\frac{x-(-1)}{a} = \frac{y-3}{b} = \frac{z-(-2)}{c} \qquad \ldots(i)$$

It is perpendicular to the lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \qquad \ldots(ii)$$

$$\text{and} \qquad \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5} \qquad \ldots(iii)$$

$$\text{then} \qquad 1\cdot a + 2\cdot b + 3\cdot c = 0$$

$$\Rightarrow \qquad a + 2b + 3c = 0 \qquad \ldots(iv)$$

$$\text{and} \qquad -3\cdot a + 2\cdot b + 5\cdot c = 0$$

$$\Rightarrow \qquad -3a + 2b + 5c = 0 \qquad \ldots(v)$$

Subtracting (v) from (iv), we get

$$4a - 2c = 0$$

$$\Rightarrow \quad c = 2a \quad \text{...(vi)}$$

From (iv),

$$a + 2b + 3\cdot 2a = 0$$

$$\Rightarrow \quad 2b = -7a$$

$$\Rightarrow \quad b = \frac{-7a}{2} \quad \text{...(vii)}$$

From (vi) and (vii),

$$a : b : c = a : -\frac{7a}{2} : 2a$$

$$\Rightarrow \quad a : b : c = 2 : -7 : 4$$

Putting these values in (i), the equation of the required line is

$$\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

Q. 8. Find the shortest distance between the lines :

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{5-z}{5}$$

[ISC, 2012]

Sol. Given lines are : $\dfrac{x-8}{3} = \dfrac{y+9}{-16} = \dfrac{z-10}{7}$ ...(i)

and $\dfrac{x-15}{3} = \dfrac{y-29}{8} = \dfrac{5-z}{5}$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad \text{...(ii)}$$

Any point on (i) is $P(8 + 3t, -9 - 16t, 10 + 7t)$

and any point on (ii) is $Q(15 + 3s, 29 + 8s, 5 - 5s)$

$\therefore$ Direction ratios of the line PQ are

$(15 + 3s - 8 - 3t, 29 + 8s + 9 + 16t, 5 - 5s - 10 - 7t)$

$(7+ 3s - 3t, 38 + 8s + 16t, -5 - 5s - 7t)$

Now, PQ will be the shortest distance between (i) and (ii).

If PQ is perpendicular to both.

i.e., if $3(7 + 3s - 3t) + (-16)(38 + 8s + 16t) + 7(-5 - 5s - 7t) = 0$

$\therefore \quad -154s - 314t - 622 = 0$

i.e., $\quad 77s + 157t + 311 = 0$ ...(iii)

and $\quad 3(7 + 3s - 3t) + 8(38 + 8s + 16t) (-5)(-5-5s - 7t) = 0$

and $\quad 98s + 154t + 350 = 0$

i.e., $\quad 49s + 77t + 175 = 0$ ...(iv)

On solving (iii) and (iv), we get

$$t = -1 \text{ and } s = -2$$

$t = -1$ gives $P(5, 7, 3)$ and $s = -2$ gives $Q(9, 13, 15)$

$\therefore$ The shortest distance between the given lines

$$PQ = \sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2}$$

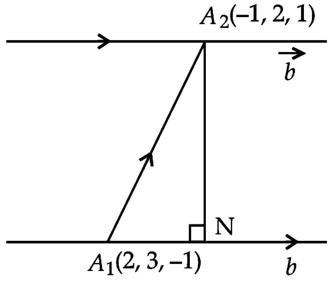
$$= \sqrt{196} = 14 \text{ units}$$

AI **Q. 9. Find the vector equation of the line passing through the point (–1, 2, 1) and parallel to the line $\vec{r} = 2\hat{i} + 3\hat{j} - \hat{k} + \lambda(\hat{i} - 2\hat{j} + \hat{k})$. Also, find the distance between these lines.**

Sol. The given line is $\vec{r} = 2\hat{i} + 3\hat{j} - \hat{k} + \lambda(\hat{i} - 2\hat{j} + \hat{k})$ ...(i)

It passes through the point A_1 with the position vector $\vec{a_1} = 2\hat{i} + 3\hat{j} - \hat{k}$ and is parallel to the vector

$$\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$$



The equation of the line passing through the point $A_2(-1, 2, 1)$ with position vector

$$\vec{a_2} = -\hat{i} + 2\hat{j} + \hat{k}$$

and parallel to the the line (i) is

$$\vec{r} = -\hat{i} + 2\hat{j} + \hat{k} + \mu(\hat{i} - 2\hat{j} + \hat{k}) \quad \text{...(ii)}$$

Also, $\quad \bar{a}_2 - \bar{a}_1 = -3\hat{i} - \hat{j} + 2\hat{k}$

$$|\bar{b}| = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$$

and $\quad \vec{b} \times (\vec{a_2} - \vec{a_1}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ -3 & -1 & 2 \end{vmatrix}$

$$= \hat{i}(-4+1) - \hat{j}(2+3) + \hat{k}(-1-6)$$

$$= -3\hat{i} - 5\hat{j} - 7\hat{k}$$

$\therefore \quad |\vec{b} \times (\vec{a_2} - \vec{a_1})| = \sqrt{(-3)^2 + (-5)^2 + (-7)^2}$

$$= \sqrt{9 + 25 + 49}$$

$$= \sqrt{83}$$

$\therefore$ The distance between the parallel lines (i) and (ii)

$$= \frac{|\vec{b} \times (\vec{a_2} - \vec{a_1})|}{|\vec{b}|}$$

$$= \frac{\sqrt{83}}{\sqrt{6}} = \sqrt{\frac{83}{6}} \text{ units}$$

Q. 10. Find the value of p, so that the lines

$$l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$$

and $l_2 : \dfrac{7-7x}{3p} = \dfrac{y-5}{1} = \dfrac{6-z}{5}$

are perpendicular to each other.

Also, find the equation of a line passing through a point (3, 2, –4) and parallel to line l_1.

Sol. Given lines can be written as

$$l_1 = \frac{x-1}{-3} = \frac{y-2}{\frac{p}{7}} = \frac{z-3}{2}$$

and $$l_2 = \frac{x-1}{\left(\frac{-3p}{7}\right)} = \frac{y-5}{1} = \frac{z-6}{-5}$$

Since, the lines are perpendicular.

$$\therefore \quad (-3)\left(-\frac{3p}{7}\right)+\left(\frac{p}{7}\right)(1)+(2)(-5)=0$$

$$\Rightarrow \qquad p = 7$$

Equation of line passing through (3, 2, –4) and parallel to l_1 is

$$\frac{x-3}{-3}=\frac{y-2}{1}=\frac{z+4}{2}$$

6 Marks Questions

AI Q. 1. Show that the lines $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ and $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ intersect. Find the coordinates of their point of intersection.

[ISC, 2017]

Sol. Sol. Given lines are :

$$\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7} \qquad \text{...(i)}$$

and $$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8} \qquad \text{...(ii)}$$

General point on (i) $P = (\lambda + 4, -4\lambda - 3, 7\lambda - 1)$ and general point on (ii) $Q = (2\mu + 1, -3\mu - 1, 8\mu - 10)$

If the lines intersect for some value of λ and μ the points P and Q coincide.

$\therefore \qquad \lambda + 4 = 2\mu + 1$

$\Rightarrow \qquad \lambda - 2\mu = -3 \qquad$ ...(iii)

$\qquad -4\lambda - 3 = -3\mu - 1$

$\Rightarrow \qquad -4\lambda + 3\mu = 2 \qquad$ ...(iv)

$\qquad 7\lambda - 1 = 8\mu - 10$

$\Rightarrow \qquad -7\lambda + 8\mu = 9 \qquad$ ...(v)

Solving (iii) and (iv), we get

$\lambda = 1$ and $\mu = 2$

Substituting λ and μ in eq. (v),

$-7 \times 1 + 8 \times 2 = -7 + 16 = 9$

Satisfied the eq. (v), *i.e.*, the lines intersect.

∴ Point of intersection is $P(5, -7, 6)$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates formed correct equations from the general points and some of them found the values of constants λ and μ correctly. However, they did not do the verification to prove that the lines are intersecting. Many candidates used incorrect formula to find the shortest distance, to prove the lines are intersecting some candidates applied vector method and proved lines are intersecting but failed to calculate the point of intersection.

Answering Tip

- Revise all the forms and related formulae of 3D geometry.

Q. 2. Using vectors show that the points $A(-2, 3, 5)$, $B(7, 0, -1)$, $C(-3, -2, -5)$ and $D(3, 4, 7)$ are such that AB and CD intersect at the point $P(1, 2, 3)$.

Sol. Given: $A(-2, 3, 5)$, $B(7, 0, -1)$
$C(-3, -2, -5)$, $D(3, 4, 7)$

Equation of line AB is :

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\Rightarrow \quad \frac{x+2}{7+2}=\frac{y-3}{0-3}=\frac{z-5}{-1-5}$$

$$\Rightarrow \quad \frac{x+2}{9}=\frac{y-3}{-3}=\frac{z-5}{-6}=\alpha \text{ (Let)}$$

Equation of line CD is :

$$\Rightarrow \quad \frac{x+3}{3+3}=\frac{y+2}{4+2}=\frac{z+5}{7+5}$$

$$\Rightarrow \quad \frac{x+3}{6}=\frac{y+2}{6}=\frac{z+5}{12}=\beta \text{ (Let)}$$

∴ General co-ordinates of any point on line AB are :

$x = 9\alpha - 2, y = 3 - 3\alpha$ and $z = 5 - 6\alpha$

And general co-ordinates of any point on line CD are :

$x = 6\beta - 3, y = 6\beta - 2$ and $z = 12\beta - 5$

AB and CD to intersect, they must have same common point. Hence, for some values of α and β:

$9\alpha - 2 = 6\beta - 3 \Rightarrow 9\alpha - 6\beta = -1 \qquad$ ...(i)

$3 - 3\alpha = 6\beta - 2 \Rightarrow -3\alpha - 6\beta = -5 \qquad$ ...(ii)

$5 - 6\alpha = 12\beta - 5 \Rightarrow -6\alpha - 12\beta = -10 \qquad$ ...(iii)

Let us solve first two equations to find the values of α and β

$\Rightarrow \qquad 12\alpha = 4$

$\Rightarrow \qquad \alpha = \frac{1}{3}$

$\therefore \qquad 9 \times \frac{1}{3} - 6\beta = -1$

$\Rightarrow \qquad 3 + 1 = 6\beta$

$\Rightarrow \qquad 4 = 6\beta$

$\Rightarrow \qquad \beta = \frac{2}{3}$

These values of α and β also satisfy eq. (iii). Hence, the given lines intersect.

∴ Co-ordinates of point of intersection are :

$$x = 9 \times \frac{1}{3} - 2 = 1,$$

$$y = 3 - 3 \times \frac{1}{3} = 2,$$

and $$z = 5 - 6 \times \frac{1}{3} = 3$$

$(x, y, z) = (1, 2, 3)$

TOPIC-2

Plane and Its Equation in Various Forms

Revision Notes

- **Plane and its equation :**

  A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface.

  (a) Equation of a plane in Normal form :

Scan to know more about this topic

Plane Equation Normal Form

Scan to know more about this topic

Plane through intersecting planes

Consider a plane at distance *d* from the origin such that $\vec{ON}$ is the normal from the origin to the plane and the $\hat{n}$ is a unit vector along $\vec{ON}$. Then $\vec{ON} = d\hat{n}$ if $ON = d$. Consider $\vec{r}$ be the position vector of any arbitrary point $P(x, y, z)$ on the plane.

- **Vector form of the equation of plane :** Since *P* lies on the plane $\vec{NP}$ is perpendicular to the vector $\vec{ON}$. which implies $\vec{NP}\cdot\vec{ON} = 0$.

$$\Rightarrow \qquad (\vec{r} - d\hat{n})\cdot\hat{n} = 0$$

$$\Rightarrow \qquad \vec{r}\cdot\hat{n} - d\hat{n}\cdot\hat{n} = 0$$

$$\Rightarrow \qquad \vec{r}\cdot\hat{n} - d(\hat{n}\cdot\hat{n}) = 0$$

$$\Rightarrow \qquad \vec{r}\cdot\hat{n} - d|\hat{n}|^2 = 0$$

$$\Rightarrow \qquad \vec{r}\cdot\hat{n} - d = 0$$

$$\Rightarrow \qquad \vec{r}\cdot\hat{n} = d \qquad [\because |\hat{n}| = 1]$$

Scan to know more about this topic

Plane in intercept form

Scan to know more about this topic

Line in 3 dimensions

This is the vector equation of the plane.

- **Cartesian form of the equation of plane :** If *l*, *m*, *n* are d.c.´s of the normal $\hat{n}$ to the given plane. Then by using $\vec{r}\cdot\hat{n} = d$

  We get, $(x\hat{i} + y\hat{j} + z\hat{k})\cdot(l\hat{i} + m\hat{j} + n\hat{k}) = d$

  $\Rightarrow \qquad lx + my + nz = d$

  This is the Cartesian equation of the plane.

- Also, if *a*, *b*, *c* are the d.r.´s of the normal $\hat{n}$ to the plane, then the Cartesian equation of plane becomes $ax + by + cz = d$.

(b) Equation of plane perpendicular to a given vector and passing through a given point :

Assume that the plane passes through a point $A(x_1, y_1, z_1)$ with the position vector $\vec{a}$ and is perpendicular to the vector $\vec{m}$ with d.r.´s as *A*, *B*, *C* ($\therefore\ \vec{m} = A\hat{i} + B\hat{j} + C\hat{k}$).

Also, consider $P(x, y, z)$ as any arbitrary point on the plane with position vector as $\vec{r}$.

- **Vector form of the equation of plane :** As $\vec{AP}$ lies in the plane and $\vec{m}$ is perpendicular to the plane. So, $\vec{AP}$ is perpendicular to $\vec{m}$.

$$\Rightarrow \qquad \vec{AP}\cdot\vec{m} = 0$$

$$\Rightarrow \qquad (\vec{r} - \vec{a})\cdot\vec{m} = 0$$

This is the vector equation of the plane.

- The above obtained equation of plane can also be expressed as $\vec{r}\cdot\vec{m} = \vec{a}\cdot\vec{m}$.
- **Cartesian form of the equation of plane :** As $\vec{AP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$, and let $\vec{m}$ be $A\hat{i} + B\hat{j} + C\hat{k}$ so by using $(\vec{r} - \vec{a})\cdot\vec{m} = 0$, we get

$$[(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}]\cdot(A\hat{i}+B\hat{j}+C\hat{k})=0$$

$$\Rightarrow \quad A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

This is the Cartesian equation of the plane.

(c) Equation of plane passing through three non-collinear points :

- **Cartesian form :**

In order to find the equation of a plane passing through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), the following steps are used :

Step 1 : Write the equation of a plane passing through (x_1, y_1, z_1) as

$$a\,(x-x_1)+b\,(y-y_1)+c\,(z-z_1)=0 \qquad \text{...(i)}$$

Step 2 : If the plane (i) passes through (x_2, y_2, z_2) and (x_3, y_3, z_3), then

$$a\,(x_2-x_1)+b\,(y_2-y_1)+c\,(z_2-z_1)=0 \qquad \text{...(ii)}$$

and
$$a\,(x_3-x_1)+b\,(y_3-y_1)+c\,(z_3-z_1)=0 \qquad \text{...(iii)}$$

Step 3 : Solve equations (ii) and (iii) by cross multiplication method to obtain the values of a, b and c. Substitute the values of a, b and c in eq (i) to get the equation of the required plane.

Note : On eliminating a, b, c from eqs (i), (ii) and (iii), we get

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

as the equation of the plane passing through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

- **Coplanarity of four points :**

In order to prove the co-planarity of four points, first find the equation of a plane passing through any three out of the given four points :

Now, show that the coordinates of the fourth point satisfies the obtained equation.

- **Vector form :** Assume that the plane contains three non-collinear points $R(x_1, y_1, z_1)$, $S(x_2, y_2, z_2)$ and $T(x_3, y_3, z_3)$ with the position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Let $P(x, y, z)$ be any arbitrary point in the plane whose position vector is $\vec{r}$.

  Hence vector equation of the required plane is given by

$$[\vec{r}\ \vec{a}\ \vec{b}]+[\vec{r}\ \vec{b}\ \vec{c}]+[\vec{r}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]$$

or
$$(\vec{r}-\vec{a}).[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$$

- **Co-planarity of two Lines :**

Assume that the given lines $L_1 : \vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $L_2 : \vec{r}=\vec{a_2}+\mu\vec{b_2}$ such that L_1 passes through $A(x_1, y_1, z_1)$ with position vector $\vec{a_1}$ and is parallel to $\vec{b_1}$ with d.r.'s a_1, b_1, c_1. Also, L_2 passes through $B(x_2, y_2, z_2)$ with position vector $\vec{a_2}$ and is parallel to $\vec{b_2}$ with the d.r.'s a_2, b_2, c_2.

(a) Vector form of co-planarity of lines :

We know that $\vec{AB}=\vec{a_2}-\vec{a_1}$. Now the lines L_1 and L_2 are coplanar if $\vec{AB}$ is perpendicular to $\vec{b_1}\times\vec{b_2}$. That implies, $\vec{AB}\cdot(\vec{b_1}\times\vec{b_2})=0$

$$\Rightarrow \quad (\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})=0$$

(b) Cartesian form of co-planarity of lines :

We know that $\vec{AB}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$, $\vec{b_1}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{b_2}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$. So, by using $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})=0$, we get

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

- Note that only coplanar lines can intersect each other in the plane they exist.

(c) Distance between two parallel planes :

Assume that the two planes are $\vec{r}\cdot\vec{m}=d_1, i.e., Ax+By+Cz+D_1=0$ and $\vec{r}\cdot\vec{m}=d_2, i.e., Ax+By+Cz+D_2=0$. Then the distance p (say) between them is given as

(i) Vector form : $p=\dfrac{|d_1-d_2|}{|\vec{m}|}$

(ii) Cartesian form : $p=\dfrac{|D_1-D_2|}{\sqrt{A^2+B^2+C^2}}$

Know the Formulae

- **Angle between a plane and a line**
  - If θ is the angle between line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$, then

$$\sin\theta=\left|\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}\right|$$

  - Vector form : If θ is the angle between a line $\vec{r}=(\vec{a}+\lambda\vec{b})$ and $\vec{r}.\vec{n}=d$, then

$$\sin\theta=\left|\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

  - Condition for perpendicularity : $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}, \vec{b}\times\vec{n}=0$
  - Condition for parallel : $al+bm+cn=0$, $\vec{b}.\vec{n}=0$

- **Angle between two planes**
  - Consider two planes $ax+b+cz+d=0$ and $a'x+b'y+c'z+d'=0$. Angle between these plane is the angle between their normals. Since direction ratios of their normals are (a, b, c) and (a', b', c') respectively. Hence, θ the angle between them is given by

$$\cos\theta=\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}$$

Planes are perpendicular, if $aa'+bb'+cc'=0$ and planes are parallel if $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$

  - The angle θ between the planes $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$ is given by

$$\sin\theta=\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}|.|\vec{n_2}|}$$

Planes are perpendicular, is $\vec{n_1}.\vec{n_2}=0$ and planes are parallel if $\vec{n_1}=\lambda\vec{n_2}$.

- **Equation the plane in intercept form :**

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Note :
- Equation of XY-plane : $z=0$
- Equation of YZ-plane : $x=0$
- Equation of ZX-plane : $y=0$

1 Mark Questions

AI **Q. 1. The intercepts made by the plane $3x - 2y + 4z = 12$ on the coordinates axes are :**

(a) 6, –4, 3 **(b) 4, –6, 3**

(c) 2, –3, 4 **(d) $\frac{1}{4}, -\frac{1}{6}, \frac{1}{3}$**

[ISC Specimen Paper, 2021]

Sol. Correct option : (b)

Explanation :

$$\Rightarrow \frac{x}{9} + \frac{y}{-6} + \frac{z}{3} = 1$$

$\therefore$ x-intercept = 4, y-intercept = –6, z-intercept = 3

Q. 2. What is the distance (in units) between the two planes $3x + 5y + 7z = 3$ and $9x + 15y + 21z = 9$?

(a) 0 **(b) 3**

(c) $\frac{6}{\sqrt{83}}$ **(d) 6**

Sol. Correct option : (a)

Explanation :

Equation of P_1 : $3x + 5y + 7z - 3 = 0$ ...(i)

Equation of P_2 : $9x + 15y + 21z - 9 = 0$

or $3x + 5y + 7z - 3 = 0$ ...(ii)

Equations (i) and (ii) represent the same plane, *i.e.*, they coincide.

Distance between them = 0

AI **Q. 3. The planes : $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are**

(a) Perpendicular **(b) Parallel**

(c) Intersect y-axis

(d) Passes through $\left(0, 0, \frac{5}{4}\right)$

Sol. Correct option : (b)

Explanation :

$$\frac{a_1}{a_2} = \frac{2}{5}$$

$$\frac{b_1}{b_2} = \frac{-1}{-2.5} = \frac{2}{5}$$

$$\frac{c_1}{c_2} = \frac{4}{10} = \frac{2}{5}$$

$$\because \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2},$$

the planes are parallel.

Q. 4. Find the equation of the plane passing through (–2, 1, 3) and perpendicular to the line having direction ratios <3, 1, 5>.

[ISC Specimen Paper, 2021]

Sol. Given that, the equation of the plane is perpendicular to the line having direction ratios <3, 1, 5>

$\therefore$ Normal to the line is parallel to the line

$\therefore$ Required equation of plane passing through (–2, 1, 3) is $3(x + 2) + 1(y - 1) + 5(z - 3) = 0$

$\therefore$ $3x + y + 5z = 10$

Q. 5. Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Sol. Here $\vec{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$, then

$$|\vec{n}| = \sqrt{4 + 9 + 36} = \sqrt{49}$$

$$= 7 \text{ units}$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

So, the required equation is

$$r \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) = 5$$

Q. 6. Find the vector equation of the plane with intercepts 3, – 4 and 2 on x, y and z-axis respectively.

Sol. The equation is $\frac{x}{3} + \frac{y}{-4} + \frac{z}{2} = 1$ or $4x - 3y + 6z = 12$

The vector equation is $\vec{r} \cdot (4\hat{i} - 3\hat{j} + 6\hat{k}) = 12$

AI **Q. 7. Write the vector equation of the plane passing through the point (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$**

Sol. Since (a, b, c) is parallel to the given plane,

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2 \quad \text{(constant varies)}$$

$$\{\vec{r} - (a\hat{i} + b\hat{j} + c\hat{k})\} \cdot (\hat{i} + \hat{j} + \hat{k}) = 0$$

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = (a + b + c)$$

Q. 8. A plane passes through the points (2,0,0), (0,3,0) and (0,0,4). Then find the equation of plane.

Sol. We know that, equation of a plane that cuts the coordinate axes at (a, 0, 0), (0, b, 0) and (0, 0, c) is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Hence, the equation of plane passes through the points (2, 0, 0), (0, 3, 0) and (0, 0, 4) is $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$.

Q. 9. Find the cartesian equation of the plane $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$

Sol. We have, $r \cdot (\hat{i} + \hat{j} - \hat{k}) =$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$$

$$\Rightarrow \quad x + y - z = 2$$

This is the required Cartesian equation.

AI **Q. 10. Find the sum of intercepts cut off by the plane $2x + y - z = 5$ on the coordinate axes.**

Sol. Equation of the plane is $2x + y - z = 5$ Dividing by 5

$$\Rightarrow \quad \frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1$$

$$\Rightarrow \quad \frac{x}{\frac{5}{2}}+\frac{y}{5}+\frac{z}{-5}=1$$

$$a=\frac{5}{2},\ b=5,\ c=-5$$

Sum of intercepts $= a + b + c$

$$= \frac{5}{2}$$

2 Marks Questions

Q. 1. Find the equation of the plane passing through the point (2, –3, 1) and perpendicular to the line joining the points (4, 5, 0) and (1, –2, 4). [ISC, 2020]

Sol. Plane passing through the point (2, –3, 1) and perpendicular to the line joining the point (4, 5, 0) and (1, –2, 4) direction ratio of the line is

$\{(1-4), (-2-5), (4-0)\}$

$= (-3, -7, 4)$

direction ratio of the plane

= direction ratio of the line

$= (-3, -7, 4)$

Equation of the plane

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$
$$-3(x-2)-7(y+3)+4(z-1)=0$$
$$-3x+6-7y-21+4z-4=0$$
$$-3x-7y+4z-19=0$$
$$3x+7y-4z+19=0$$

Required equation of the plane is

$$3x+7y-4z+19=0$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Most of the candidate didn't apply the concept of perpendicular lines and equation of plane correctly.

Answering Tip

- Understand the different situation of finding the equation of the plane by applying different condition.

Q. 2 Find the length of the perpendicular from origin to the plane $r(3\hat{i}+4\hat{j}-12\hat{k})+39=0$. [ISC 2019]

Sol. The equation may be written as

$3x-4y-12z+39=0$

Distance from origin

$$=\frac{39}{\sqrt{3^2+(-4)^2+(-12)^2}}=\frac{39}{\sqrt{9+16+144}}=\frac{39}{13}=3 \text{ units}$$

[ISC Marking Scheme, 2019]

Examiner's Comment

- Many candidates did not apply the perpendicular distance formula from a given point to a plane. Some candidates attempted to express it in the normal from but made calculation errors.

Answering Tip

- Understand the concept of converting cartesian form of equations of plane / straight line to vector form and vice versa and formulae for perpendicular distance with the help of illustrations.

Q. 3. Find the equation of the plane through the intersection of planes $\vec{r}\,.\ r(3\hat{i}+4\hat{j}-12\hat{k})+39=0$ and $\vec{r}\,.\ (2\hat{i}-\hat{j}+\hat{k})=3$ and passing through the origin. [ISC 2018]

Sol. The equation of a plane through the intersection of :

$$\vec{r}\,.(\hat{i}+3\hat{j}-\hat{k})=9 \text{ and } \vec{r}\,.(2\hat{i}-\hat{j}+\hat{k})=3 \text{ is}$$

$$[\vec{r}\,.(\hat{i}+3\hat{j}-\hat{k})-9]+\lambda[\vec{r}\,.(2\hat{i}-\hat{j}+\hat{k})-3)]=0$$

$$\Rightarrow \quad \vec{r}\,.(\hat{i}+3\hat{j}-\hat{k})+\lambda[\vec{r}\,.(2\hat{i}-\hat{j}+\hat{k})]=3\lambda+9$$

$$\Rightarrow \quad \vec{r}\,[(\hat{i}+3\hat{j}-\hat{k})+\lambda(2\hat{i}-\hat{j}+\hat{k})]=3\lambda+9$$

∵ Passing through origin

$$\therefore \quad 3\lambda+9=0 \Rightarrow \lambda=-3$$

Required equation, $\vec{r}\,.(-5\hat{i}+6\hat{j}-4\hat{k})=0$

[ISC Marking Scheme, 2018]

Alternate solution :

$x+3y-z-9=0$ and $2x-y+z-3=0$

$\therefore\ (x+3y-z-9)+\lambda(2x-y+z-3)=0$

$\Rightarrow \quad -9-3\lambda=0$

$\Rightarrow \quad \lambda=-3$

∴ Required equation, $-5x+6y-4z=0$ or $5x-6y+4z=0$

Q. 4. Find the angle between the line $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ and the plane $2x+3y-5z=4$.

[ISC Specimen Paper, 2018]

Sol. Equation of line is $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$

and plane is $2x+3y-5z=4$

Here, $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

and $\vec{n}=2\hat{i}+3\hat{j}-5\hat{k}$

$$\therefore \quad \sin\theta=\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$$

$$=\frac{|(2\hat{i}+3\hat{j}+6\hat{k}).(2\hat{i}+3\hat{j}-5\hat{k})|}{\sqrt{2^2+3^2+6^2}\ \sqrt{2^2+3^2+(-5)^2}}$$

$$=\frac{|(2)(2)+(3)(3)+(6)(-5)|}{\sqrt{49}\ \sqrt{38}}$$

$$= \frac{|4+9-30|}{7\sqrt{38}} = \frac{17}{7\sqrt{38}}$$

$$\therefore \quad \sin\theta = \frac{17}{7\sqrt{38}}$$

$$\theta = \sin^{-1}\left(\frac{17}{7\sqrt{38}}\right)$$

Q. 5. Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i}+3\hat{j}+6\hat{k}$.

Sol. Equation of the plane is given by

$$\vec{r}\cdot\hat{n} = p$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+36}}$$

$$= \frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$$

∴ Equation of the plane is $\vec{r}\cdot\hat{n} = p$

$$\text{or} \quad \vec{r}\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right) = 5$$

$$\text{or} \quad \vec{r}\cdot(2\hat{i}+3\hat{j}+6\hat{k}) = 35$$

AI **Q. 6. Find the vector equation of the plane with intercepts 3, – 4 and 2 on *x*, *y* and *z*-axis respectively.**

Sol. Intercepts are (3, 0, 0), (0, –4, 0), (0, 0, 2)

$$\vec{a} = 3\hat{i},\ \vec{b} = -4\hat{j},\ \vec{c} = 2\hat{k}$$

Vector equation of the plane passing through $\vec{a}, \vec{b}$ and $\vec{c}$ is

$$(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})] = 0$$

Here, $(\vec{b}-\vec{a})\times(\vec{c}-\vec{a}) = [-4\hat{j}+(-3\hat{i})]\times(2\hat{k}-3\hat{i})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -4 & 0 \\ -3 & 0 & 2 \end{vmatrix}$$

$$= -8\hat{i}+6\hat{j}-12\hat{k}$$

∴ Vector equation becomes

$$(\vec{r}-3\hat{i})\cdot(-8\hat{i}+6\hat{j}-12\hat{k}) = 0$$

Q. 7. Write the distance between the parallel planes $2x - y + 3z = 4$ and $2x - y + 3z = 18$.

Sol. Required distance(d) $= \dfrac{|D_1 - D_2|}{\sqrt{a^2+b^2+c^2}}$

$$\sqrt{a^2+b^2+c^2} = \sqrt{(2)^2+(-1)^2+(3)^2}$$

$$= \sqrt{4+1+9} = \sqrt{14} \text{ units}$$

$$\therefore \quad d = \frac{18-4}{\sqrt{14}} = \frac{14}{\sqrt{14}} = \sqrt{14} \text{ units}$$

Q. 8. Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$.

Sol. The length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$ is

$$d = \frac{2(0)-3(0)+6(0)+21}{\sqrt{2^2+(-3)^2+6^2}} = \frac{21}{7} = 3 \text{ units}$$

4 Marks Questions

AI **Q. 1. Find the equation of the plane passing through the intersection of the planes $2x + 2y - 3z - 7 = 0$ and $2x + 5y + 3z - 9 = 0$ such that the intercepts made by the resulting plane on the *x*-axis and the *z*-axis are equal. [ISC Specimen, 2021] [ISC, 2019]**

Sol. The required plane will be :

$2x + 2y - 3z - 7 + k(2x + 5y + 3z - 9) = 0$

$(2 + 2k)x + (2 + 5k)y + (3k - 3)z = 7 + 9k$

since x intercept = z intercept

$$3k - 3 = 2 + 2k$$

$$k = 5$$

and $(2 + 10)x + (2 + 25)y + 12z = 52$

$$12x + 27y + 12z = 52$$

[ISC Marking Scheme, 2019]

Examiner's Comment

- Majority of the candidates could not find the correct values of *x*-intercept and *z*-intercept.

Answering Tip

- Understand the concepts of family of planes and all possible conditions that may be involved in problems on this topic.

Q. 2. Find the image of the point (3, –2, 1) in the plane $3x - y + 4z = 2$. [ISC, 2020]

Sol. Equation of the plane

$3x - y + 4z = 2$

∴ direction ratio of perpendicular line to the plane is (3, – 1, 4)

Equation of the line is passing through (3, – 2, 1) and direction ratios are (3, – 1, 4) is

$$\frac{x-3}{3} = \frac{y+2}{-1} = \frac{z-1}{4}$$

General point on the line is $(3\lambda + 3, -\lambda - 2, 4\lambda + 1)$

Foot of perpendicular lies in the plane

$\therefore 3(3\lambda + 3) - (-\lambda - 2) + 4(4\lambda + 1) = 2$

$$9\lambda + 9 + \lambda + 2 + 16\lambda + 4 = 2$$

$$26\lambda = 2 - 15$$

$$\lambda = \frac{-13}{26} = \frac{-1}{2}$$

Hence point of intersection of the line and plane is

$$\left(\frac{-3}{2}+3, \frac{1}{2}-2, -2+1\right) = \left(\frac{3}{2}, \frac{-3}{2}, -1\right)$$

Let the image of the point (3, – 2, 1) be (*x*, *y*, *z*)

$$\therefore \qquad \frac{x+3}{2}=\frac{3}{2}\Rightarrow x=0$$

$$\frac{y-2}{2}=\frac{-3}{2}\Rightarrow y=-1$$

$$\frac{z-1}{2}=-1\Rightarrow z=-1$$

Hence, coordinate of image (0, – 1, – 1).

Examiner's Comment

- Most of the candidates applied the condition for perpendicularity incorrectly and used incorrect direction ratios. It led to calculate incorrect foot of the perpendicular and incorrect image of the given point.

Answering Tip

- Do enough practice of problems based on applications of distance of a point etc. with reference to a line or a plane.

Q. 3. Find the cartesian equation of the plane passing through the intersection of the planes $\vec{r}\,.\,(2\hat{i}+6\hat{j})+12=0$ and $3x-y+4z=0$ and at a unit distance from origin.

[ISC Specimen Paper, 2021]

Sol. $$P_1 = \vec{r}.(2\vec{i}+6\vec{j})+12=0$$

$$= 2x+6y+12=0$$

$$P_2 = 3x-y+4z=0$$

Equation of plane passing through the intersection of P_1 and P_2 is

$$P_1+\lambda P_2=0$$

$$(2x+6y+12)+\lambda(3x-y+4z)=4$$

$$(2+3\lambda)x+(6-\lambda)y+4\lambda z+12=0 \qquad \text{...(i)}$$

Given that its distance from origin is 1 units

$$\Rightarrow \qquad 1=\left|\frac{12}{\sqrt{(2+3\lambda)^2+(6-\lambda)^2+(4\lambda)^2}}\right|$$

$$4+9\lambda^2+12\lambda+36+\lambda^2-12\lambda+16\lambda^2=144$$

$$26\lambda^2=104$$

$$\lambda^2=4$$

$$\lambda=\pm 2$$

∴ Equation of plane is

$$8x+4y+8z+12=0$$

$$\Rightarrow \quad 2x+y+2z+3=0$$

$$\text{or} \quad -4x+8y-8z+12=0$$

$$\Rightarrow \quad -x+2y-2z+3=0$$

AI Q. 4. Find the equation of the plane through the point (–1, –1, 2) and perpendicular to the planes $3x+2y-3z=1$ and $5x-4y+z=5$.

Sol. Given that, required plane passes through the point (–1, –1, 2).

$$\therefore \quad a(x+1)+b(y+1)+c(z-2)=0 \qquad \text{...(i)}$$

is the required equation of the plane.

As (i) is perpendicular to the planes

$3x+2y-3z=1$ and $5x-4y+z=5$,

We have $\quad 3a+2b-3c=0 \qquad$ ...(ii)

and $\quad 5a-4b+c=0 \qquad$ ...(iii)

(∵ Condition of perpendicularity,

$a_1a_2+b_1b_2+c_1c_2=0$)

Solving eq. (ii) and (iii), we get

$$\frac{a}{2-12}=\frac{b}{-3-15}=\frac{c}{-12-10}$$

$$\Rightarrow \quad \frac{a}{-10}=\frac{b}{-18}=\frac{c}{-22}$$

$$\Rightarrow \quad a=\frac{5c}{11},\ b=\frac{9c}{11}$$

Substitute these values in eq. (i), we get

$$\frac{5c}{11}(x+1)+\frac{9c}{11}(y+1)+c(z-2)=0$$

$$\Rightarrow \quad \frac{c}{11}[5x+5+9y+9+11z-22]=0$$

$$\Rightarrow \quad 5x+9y+11z-8=0$$

is the required equation of the plane.

Q. 5. Find the equation of the plane passing through (1, 2, 3) and perpendicular to the straight line $\frac{x}{-2}=\frac{y}{-4}=\frac{z}{3}$.

Sol. Equation of the plane passing through (1, 2, 3) is given by

$$a(x-1)+b(y-2)+c(z-3)=0 \qquad \text{...(i)}$$

(i) is perpendicular to $\frac{x}{-2}=\frac{y}{-4}=\frac{z}{3}$ (given)

Normal to the plane is parallel to the line, *i.e.*, direction ratios of normal to the plane is proportional to the line.

i.e., $$\frac{a}{-2}=\frac{b}{-4}=\frac{c}{3}=k$$

$$\Rightarrow \quad a=-2k,\ b=-4k,\ c=3k$$

Substitute these values in eq (i),

$$-2k(x-1)-4k(y-2)+3k(z-3)=0$$

$$\Rightarrow \quad -2x+2-4y+8+3z-9=0$$

$$\Rightarrow \quad -2x-4y+3z+1=0$$

$$\Rightarrow \quad 2x+4y-3z-1=0$$

is the required equation of the plane.

6 Marks Questions

AI Q. 1. Find the equation of plane passing through the point (1, – 2, 1) and perpendicular to the line joining the points *A* (3, 2, 1) and *B* (1, 4, 2).

[ISC, 2017]

Sol. Any plane passing through the point (1, – 2, 1) is

$$a(x-1)+b(y+2)+c(z-1)=0$$

D.R.'s of normal to the plane : *a*, *b*, *c*

This plane is perpendicular to the line joining points *A*(3, 2, 1) and *B*(1, 4, 2).

∴ D.R.'s of line perpendicular to plane :

3 – 1, 2 – 4, 1 – 2 *i.e.*, 2, – 2, – 1

∴ D.R.'s of normal to the plane and D.R.'s of line perpendicular to the plane are proportional

$$\frac{a}{2}=\frac{b}{-2}=\frac{c}{-1}=k$$

or $a = 2k, b = -2k$ and $c = -k$

∴ Required equation of the plane is :

$$2k(x-1)+(-2k)(y+2)+(-k)(z-1)=0$$

$$\Rightarrow \quad 2x-2-2y-4-z+1=0$$

$$\Rightarrow \quad 2x-2y-z-5=0$$

[ISC Marking Scheme, 2017]

Q. 2. Find the image of a point having position vector : $3\hat{i}-2\hat{j}+\hat{k}$ in the plane $\vec{r}.(3\hat{i}-\hat{j}+4\hat{k})=2$.

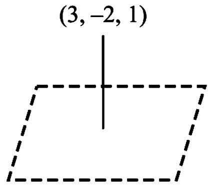
[ISC, 2018]

Sol. P.V. of a point : $3i-2j+k$: (3, – 2, 1)

Eq. of plane : $\vec{r}.(3i-j+4k)=2$

$$\Rightarrow \quad 3x-y+4z=2$$

D.R. of normal to the plane : 3, – 1, 4



∴ Eq. of straight line passing through (3, – 2, 1) is

$$\frac{x-3}{3}=\frac{y+2}{-1}=\frac{z-1}{4}=\text{say }(\lambda)$$

For some value of λ the foot of the perpendicular is

$$(3\lambda+3, -\lambda-2, 4\lambda+1)$$

Now, $3(3\lambda+3)-1(-\lambda-2)+4(4\lambda+1)=2$

$$\Rightarrow \quad 26\lambda+15=2$$

$$\Rightarrow \quad 26\lambda=-13$$

$$\Rightarrow \quad \lambda=\frac{-13}{26}=\frac{-1}{2}$$

∴ Foot of the perpendicular $(3\lambda+3, -\lambda-2, 4\lambda+1)$

i.e., $\left(\frac{3}{2},\frac{-3}{2},-1\right)$

Foot of perpendicular $\frac{x_1+3}{2},\frac{y_1-2}{2},\frac{z_1+1}{2}$

Let (x_1, y_1, z_1) be the image of the point with reference to the give plane.

$x_1+3=3, y_1-2=-3, z+1=-2$

$\Rightarrow x_1=0, y_1=-1, z_1=-3$

∴ Required image of a point is (0, – 1, 3).

[ISC Marking Scheme, 2018]

Q. 3. Find the Cartesian equation of the plane, passing through the line of intersection of the planes : $\vec{r}\cdot(2\hat{i}+3\hat{j}-4\hat{k})+5=0$ and $\vec{r}\cdot(\hat{i}-5\hat{j}+7\hat{k})+2=0$ and intersecting y-axis at (0, 3, 0). **[ISC, 2016]**

Sol. Any plane passing through the line of intersection of the given planes is

$$[\vec{r}.(2\hat{i}+3\hat{j}-4\hat{k})+5]+\lambda[\vec{r}.(i-5\hat{j}+7\hat{k})+2]=0$$

$$\Rightarrow \quad \vec{r}[(2+\lambda)\hat{i}+(3-5\lambda)\hat{j}+(-4+7\lambda)\hat{k}]=-5-2\lambda \quad \text{...(i)}$$

Given equation (i) intersects the y-axis at (0, 3, 0) so at y-axis

coordinate of $x = 0$

coordinate of $y = 3$

coordinate of $z = 0$

As at y-axis coordinate of x and z will be.

$$\therefore \quad \vec{r}=0\hat{i}+3\hat{j}+0\hat{k}$$

$$(0\hat{i}+3\hat{j}+0\hat{k}).[(2+\lambda)\hat{i}+(3-5\lambda)\hat{j}+(-4+7\lambda)\hat{k}]=-5-2\lambda$$

$$\Rightarrow \quad 0(2+\lambda)+3(3-5\lambda)+0(-4+7\lambda)+5+2\lambda=0$$

$$\lambda=\frac{14}{13}$$

Putting value of $\lambda=\frac{14}{13}$ eq (i), we get

$$\vec{r}.\left[\left(2+\frac{14}{13}\right)\hat{i}+\left(3-5\times\frac{14}{13}\right)\hat{j}+\left(-4+7\times\frac{14}{13}\right)\hat{k}\right]=-5-2\times\frac{14}{13}$$

$$\Rightarrow \quad \vec{r}.\left[\left(\frac{26+14}{13}\right)\hat{i}+\left(\frac{39-70}{13}\right)\hat{j}+\left(\frac{-52+98}{13}\right)\hat{k}\right]=\frac{-65-28}{13}$$

$$\Rightarrow \quad \vec{r}.\left[\frac{40}{13}\hat{i}+\left(\frac{-31}{13}\right)\hat{j}+\frac{46}{13}\hat{k}\right]=\frac{-93}{13}$$

For Cartesian form, put $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

∴ Required equation,

$$(x\hat{i}+y\hat{j}+z\hat{k}).\left(\frac{40}{13}\hat{i}-\frac{31}{13}\hat{j}+\frac{46}{13}\hat{k}\right)=\frac{-93}{13}$$

$$\Rightarrow \quad \frac{40}{13}x-\frac{31}{13}y+\frac{46}{13}z=\frac{-93}{13}$$

$$\Rightarrow \quad 40x-31y+46z+93=0$$

[ISC Marking Scheme, 2016]

Q. 4. Find the equation of the plane passing through the intersection of the planes : $x+y+z+1=0$ and $2x-3y+5z-2=0$ and the point (–1, 2, 1).

[ISC, 2015]

Sol. Equation of plane passing through the intersection of the given plane is :

$$(x+y+z+1)+k(2x-3y+5z-2)=0$$

If this plane passes through (–1, 2, 1) then

$$(-1+2+1+1)+k(-2-6+5-2)=0$$

$$3=5k$$

$$k=3/5$$

$$5(x+y+z+1)+3(2x-3y+5z-2)=0$$

or $11x-4y+20z-1=0$

[ISC Marking Scheme, 2015]

Examiner's Comment

- A number of candidates wrote incorrect equation of the plane passing through the intersection of planes. Some made mistakes in calculating the value of k. A few candidates applied the condition of perpendicularity in this question which was incorrect.

Answering Tip

- Learn the equation of plane thoroughly. Cartesian and vector form of plane should be revised by practicing different types of questions.

Q. 5. Find the equation of the plane passing through the points (2, –3, 1) and (–1, 1, –7) and perpendicular to the plane $x - 2y + 5z = 0$. [ISC, 2014]

Sol. Equation of plane passing through (2, –3, 1)

$$a(x-2)+b(y+3)+c(z-1)=0$$

$$a(-1-2)+b(1+3)+c(-7-1)=0$$

$$-3a+4b-8c=0 \qquad \ldots(1)$$

The given plane is perpendicular to $x - 2y + 5z = 0$

$$a-2b+5c=0 \qquad \ldots(2)$$

Solving (1) and (2), we get $a = 4k, b = 7k, c = 2k$

$$4(x-2)+7(y+3)+2(z-1)=0$$

$$4x+7y+2z+11=0$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- Some candidates wrote the equation of the Plane in Cartesian form but made mistakes while calculating a, b, c values by rule of cross-multiplication.

Answering Tip

- Equations of plane is Cartesian and vector form of a plane needs to be understand with patience and perseverance.

Detailed Answer :

Equation of plane passing through point (2, – 3, 1) is

$$a(x-2)+b(y+3)+c(z-1)=0 \qquad \text{...(i)}$$

Since, this plane also pass through point (– 1, 1, – 7), therefore

$$a(-1-2)+b(1+3)+c(-7-1)=0$$

i.e., $$-3a+4b-8c=0 \qquad \text{...(ii)}$$

Given, the plane is perpendicular to $x - 2y + 5z = 0$

$$\therefore \quad a-2b+5c=0 \qquad \text{...(iii)}$$

Solving eqs. (ii) and (iii), we get

$$\frac{a}{20-16}=\frac{b}{-8+15}=\frac{c}{6-4}=k$$

i.e., $a = 4k, b = 7k, c = 2k$

Substituting values of a, b and c is eq (i), we get

$$4k(x-2)+7k(y+3)+2k(z-1)=0$$

$$\Rightarrow \quad 4x-8+7y+21+2z-2=0$$

$$\Rightarrow \quad 4x+7y+2z+11=0,$$

which is the required equation to plane.

AI **Q. 6. Find the equation of planes parallel to the plane $2x - 4y + 4z = 7$ and which are at a distance of five units from the point (3, –1, 2). [ISC, 2013]**

Sol. The given plane is $2x - 4y + 4z = 7$ …(i)

Equation of any plane parallel to (i) is given by

$$2x-4y+4z+k=0 \qquad \ldots(ii)$$

Given that (ii) is at a distance of 5 units from the point (3, –1, 2) if

$$\frac{|2\times3-4(-1)+4\times2+k|}{\sqrt{(2)^2+(-4)^2+(4)^2}}=5$$

$$\Rightarrow \quad \frac{|18+k|}{6}=5$$

$$\Rightarrow \quad |18+k|=30$$

$$\Rightarrow \quad 18+k=30$$

$$\Rightarrow \quad k=12$$

$$\text{or} \quad 18+k=-30$$

$$\text{or} \quad k=-48$$

Substituting these values of k in (ii), the equation of the required planes

$$2x-4y+4z+12=0$$

$$\text{or} \quad x-2y+2z+6=0$$

$$\text{and} \quad 2x-4y+4z-48=0$$

$$\text{or} \quad x-2y+2z-24=0$$

Q. 7. Find the equation of the plane passing through the line of intersection of the planes $x + 2y + 3z - 5 = 0$ and $3x - 2y - z + 1 = 0$ and cutting off equal intercepts on the x and z axes. [ISC, 2012]

Sol. The equation of any plane passing through the lines of intersection of the given planes $x + 2y + 3z - 5 = 0$ and $3x - 2y - z + 1 = 0$ is given by

$$(x+2y+3z-5)+k(3x-2y-z+1)=0 \qquad \ldots(i)$$

For the intercept on x-axis, on putting $y = 0$ and $z = 0$, we get

$$x+3kx-5+k=0$$

$$\Rightarrow \quad x=\frac{5-k}{3k+1}$$

$\therefore$ Intercept on x-axis and z-axis made by the plane (i) are $\frac{5-k}{3k+1}$ and $\frac{5-k}{3-k}$ respectively.

Since, the intercepts on x-axis and z-axis are equal.

$$\frac{5-k}{3k+1}=\frac{5-k}{3-k}$$

$$\Rightarrow \quad -8k+k^2+15=-3k^2+14k+5$$

$$\Rightarrow \quad -4k^2+22k-10=0$$

$$\Rightarrow \quad 4k^2-22k+10=0$$

$$\Rightarrow \quad (4k-2)(k-5)=0$$

$$\Rightarrow \quad k=\frac{1}{2},5$$

Now, substitute $k = 5$ in (i), since the plane passes through the origin, it cannot make intercepts on axes. Therefore, $k = 1/2$ is the only admissible value. Substituting $k = 1/2$ in equation (i) the equation of the required plane is

$$x+2y+3z-5+\frac{1}{2}(3x-2y-z+1)=0$$

$$\Rightarrow \quad 2x+4y+6z-10+3x-2y-z+1=0$$

$$\Rightarrow \quad 5x+2y+5z-9=0$$

Q. 8. Find the equation of the plane passing through the points $A(2, 1, -3)$, $B(-3, -2, 1)$ and $C(2, 4, -1)$. [ISC, 2011]

Sol. Equation of any plane passing through $A(2, 1, -3)$ is

$$a(x-2) + b(y-1) + c(z+3) = 0 \quad \ldots(1)$$

Since it passes through $B(-3, -2, 1)$ and $C(2, 4, -1)$,

we get, $a(-3-2) + b(-2-1) + c(1+3) = 0$

i.e., $-5a - 3b + 4c = 0$

$\Rightarrow \quad 5a + 3b - 4c = 0 \quad \ldots(2)$

and $\quad a(2-2) + b(4-1) + c(-1+3) = 0$

i.e., $3b + 2c = 0 \quad \ldots(3)$

Solving (2) and (3), $a : b : c = 18 : -10 : 15$

Replacing in (1) required equation is

$$18(x-2) - 10(y-1) + 15(z+3) = 0$$

$$18x - 36 - 10y + 10 + 15z + 45 = 0$$

$$18x - 10y + 15z + 19 = 0.$$

AI **Q. 9. Find the equation of the plane passing through the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $3z - y = 0$ and perpendicular to the plane $3x + 4y - 2z + 6 = 0$. [ISC, 2010]**

Sol. Equation of any plane passing through the intersection of given planes is given by

$$(x + 2y + 3z - 4) + \lambda(3z - y) = 0$$

$$\Rightarrow \quad x + (2-\lambda)y + (3+3\lambda)z - 4 = 0 \quad \ldots(1)$$

But this is perpendicular to the plane $3x + 4y - 2z + 6 = 0$

$$\Rightarrow \quad 3 + 4(2-\lambda) - 2(3+3\lambda) = 0$$

[$\because$ Condition of Perpendicularity is $a_1a_2 + b_1b_2 + c_1c_2 = 0$]

$$\Rightarrow \quad 3 + 8 - 4\lambda - 6 - 6\lambda = 0$$

$$\Rightarrow \quad -10\lambda = -5$$

$$\Rightarrow \quad \lambda = \frac{1}{2}$$

Hence, required equation of plane is

$$x + \left(2 - \frac{1}{2}\right)y + \left(3 + \frac{3}{2}\right)z - 4 = 0$$

$$\Rightarrow \quad 2x + 3y + 9z - 8 = 0$$

$$\Rightarrow \quad 2x + 3y + 9z = 8$$

Q. 10. Find the equation of the plane which contains the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$. [ISC, 2009]

Sol. Equation of the plane passing through $(1, -1, 3)$ is

$$A(x-1) + B(y+1) + C(z-3) = 0$$

This is $\perp$ to the given plane,

$\therefore \quad 2A - B + 4C = 0$

and $\quad A + 2B + C = 0$

Eliminating A, B and C from equation, we get

$$\begin{vmatrix} x-1 & y+1 & z-3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$\therefore \ (x-1)(-9) - (y+1)(-2) + (z-3)5 = 0$$

$$\Rightarrow \quad 9x - 2y - 5z + 4 = 0$$

Q. 11. Find the coordinates of the point where the line joining the points $(1, -2, 3)$ and $(2, -1, 5)$ cuts the plane $x - 2y + 3z = 19$. Hence, find the distance of this point from the point $(5, 4, 1)$. [ISC, 2008]

Sol. Equation of line passing through $(1, -2, 3)$ and $(2, -1, 5)$ is

$$\frac{x-1}{2-1} = \frac{y+2}{-1+2} = \frac{z-3}{5-3}$$

$$\frac{x-1}{1} = \frac{y+2}{1} = \frac{z-3}{2} = k \text{ (say)}$$

Let any point R on this line be

$$(k+1, k-2, 2k+3)$$

This point R lies on the plane

$$x - 2y + 3z = 19$$

$$\therefore \quad k + 1 - 2k + 4 + 6k + 9 = 19$$

$$\Rightarrow \quad 5k = 19 - 14 = 5$$

$$\Rightarrow \quad k = 1$$

$\therefore$ Point is $R(2, -1, 5)$.

$$\text{Distance} = \sqrt{(5-2)^2 + (4+1)^2 + (1-5)^2}$$

$$= \sqrt{9 + 25 + 16}$$

$$= \sqrt{50} = 5\sqrt{2} \text{ units}$$

Q. 12. Find the equation of the plane through the point $(1, 2, 3)$ and perpendicular to the planes $x + y + 2z = 3$ and $3x + 2y + z = 4$.

Sol. The equation of plane passing through $(1, 2, 3)$ is given by

$$A(x-1) + B(y-2) + C(z-3) = 0 \quad \ldots(\text{i})$$

Plane is perpendicular to the planes given,

$\Rightarrow \quad A + B + 2C = 0 \quad \ldots(\text{ii})$

and $\quad 3A + 2B + C = 0 \quad \ldots(\text{iii})$

Eliminating A, B, C from (i), (ii) and (iii),

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(1-4) - (y-2)(1-6) + (z-3)(2-3) = 0$$

$$\Rightarrow \quad -3x + 3 + 5y - 10 - z + 3 = 0$$

$$\Rightarrow \quad 3x - 5y + z + 4 = 0$$

Q. 13. Find the equation of the two planes passing through the points $(0, 4, -3)$ and $(6, -4, 3)$, if the sum of their intercepts on the three axes is zero.

Sol. Let the equation of the plane be

$$ax + by + cz + d = 0 \quad \ldots(\text{i})$$

Since, plane passes through the points $(0, 4, -3)$ and $(6, -4, 3)$

(i) becomes

$$a(0) + b(4) + c(-3) + d = 0$$

$\Rightarrow \quad 4b - 3c + d = 0 \quad \ldots(\text{ii})$

and $\quad a(6) + b(-4) + c(3) + d = 0$

$\Rightarrow \quad 6a - 4b + 3c + d = 0 \quad \ldots(\text{iii})$

Add equations (ii) and (iii), we get

$$6a + 2d = 0$$

$$\Rightarrow \quad a = \frac{-2d}{6}$$

$$\Rightarrow \quad a = \frac{-d}{3}$$

Substitute $a = \frac{-d}{3}$ in (iii), we get

$$6\left(\frac{-d}{3}\right) - 4b + 3c + d = 0$$

$$\Rightarrow \quad -2d - 4b + 3c + d = 0$$

$$\Rightarrow \quad 4b - 3c = -d$$

$$\Rightarrow \quad b = \frac{3c-d}{4}$$

Equation of the plane $ax + by + cz + d = 0$ can be written as

$$\frac{x}{-d/a} + \frac{y}{-d/b} + \frac{z}{-d/c} = 1$$

where $\frac{-d}{a}$, $\frac{-d}{b}$ and $\frac{-d}{c}$ are the intercepts on x, y and z-axes respectively.

Given that, sum of intercepts = 0

$$\Rightarrow \quad \frac{-d}{c} + \frac{-d}{b} + \frac{-d}{c} = 0$$

$$\Rightarrow \quad -d\left[\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right] = 0$$

$$\Rightarrow \quad -d\left[\frac{-3}{d} + \frac{4}{3c-d} + \frac{1}{c}\right] = 0$$

$$\Rightarrow \quad -d\left[\frac{-3c(3c-d) + 4cd + d(3c-d)}{cd(3c-d)}\right] = 0$$

$$\Rightarrow -9c^2 + 3cd + 4cd + 3cd - d^2 = 0$$

$$\Rightarrow \quad 9c^2 - 10cd + d^2 = 0$$

$$\Rightarrow \quad 9c^2 - 9cd - cd + d^2 = 0$$

$$\Rightarrow \quad 9c(c-d) - d(c-d) = 0$$

$$\Rightarrow \quad (c-d)(9c-d) = 0$$

$$\Rightarrow \quad c = d \text{ or } c = \frac{d}{9}$$

When $c = \frac{d}{9}$,

$$b = \frac{3\left(\frac{d}{9}\right) - d}{4} = \frac{1}{4}\left(\frac{d}{3} - d\right)$$

$$= \frac{d-3d}{12} = \frac{-d}{6}$$

When $c = d$,

$$b = \frac{3d-d}{4} = \frac{d}{2}$$

$\therefore$ Required equation of planes are

$$-\frac{d}{3}x + \left(-\frac{d}{6}\right)y + \frac{d}{9}z + d = 0$$

$$\Rightarrow \quad \frac{-d(6x + 3y - 2z - 18)}{18} = 0$$

$$\Rightarrow \quad 6x + 3y - 2z - 18 = 0$$

and

$$-\frac{d}{3}x + \frac{d}{2}y + dz + d = 0$$

$$\Rightarrow \quad \frac{-d(2x - 3y - 6z - 6)}{6} = 0$$

$$\Rightarrow \quad 2x - 3y - 6z - 6 = 0$$

AI Q. 14. A plane passes through the point (4, 2, 4) and is perpendicular to the planes $2x + 5y + 4z + 1 = 0$ and $4x + 7y + 6z + 12 = 0$. Then, find the equation of the plane.

Sol. Equation of the plane passing through (4, 2, 4) is

$$a(x-4) + b(y-2) + c(z-4) = 0 \qquad \ldots\text{(i)}$$

(i) is perpendicular to the planes

$$2x + 5y + 4z + 1 = 0$$

and

$$4x + 7y + 6z + 2 = 0$$

$$\therefore \quad 2a + 5b + 4c = 0 \qquad \ldots\text{(ii)}$$

$$4a + 7b + 6c = 0 \qquad \ldots\text{(iii)}$$

(Condition of perpendicular $a_1a_2 + b_1b_2 + c_1c_2 = 0$)

Multiply equation (ii) by 2, we get

$$4a + 10b + 8c = 0 \qquad \ldots\text{(iv)}$$

Subtract (iii) from (iv), we get

$$3b + 2c = 0$$

$$\Rightarrow \quad b = \frac{-2c}{3}$$

Substitute $b = \frac{-2c}{3}$ in equation (ii), we get

$$2a + 5\left(\frac{-2c}{3}\right) + 4c = 0$$

$$\Rightarrow \quad 6a - 10c + 12c = 0$$

$$\Rightarrow \quad 6a + 2c = 0$$

$$\Rightarrow \quad a = \frac{-c}{3}$$

Substitute $a = \frac{-c}{3}$ and $b = \frac{-2c}{3}$ in eq (i),

$$\frac{-c}{3}(x-4) - \frac{2}{3}c(y-2) + c(z-4) = 0$$

$$\Rightarrow \quad \frac{-c}{3}[(x-4) + 2(y-2) - 3(z-4)] = 0$$

$$\Rightarrow \quad x + 2y - 3z - 4 - 4 + 12 = 0$$

$$\Rightarrow \quad x + 2y - 3z + 4 = 0$$

is the required equation of the plane.

SELF ASSESSMENT TEST

UNIT–VI

Maximum Marks : 20 **Timing : 1 Hour**

Q. 1. **Find the acute angle which the line with direction cosines $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}$, n makes with positive direction of z-axis.**

Q. 2. **If the lines $\frac{x-1}{-2}=\frac{y-4}{3p}=\frac{z-3}{4}$ and $\frac{x-2}{4p}=\frac{y-5}{2}=\frac{z-1}{-7}$ are perpendicular to each other, then find the value of p.**

Q. 3. **Find the angle between the lines**

$\vec{r}=(2\hat{i}-5\hat{j}+\hat{k})+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ **and**

$\vec{r}=(7\hat{i}-6\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$.

Q. 4. **Find the cosine of the angle between the planes $x+2y-2z+6=0$ and $2x+2y+z+8=0$.**

Q. 5. **Find the equation of the perpendicular drawn from the point $P(2, 4, -1)$ to the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$. Also, write down the co-ordinates of the foot of the perpendicular from P to the line.**

Q. 6. **Find the distance between the point $(-1, -5, -10)$ and the point of intersection of line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and plane $x-y+z=5$.**

Q. 7. **A plane passes through the point (4, 2, 4) and is perpendicular to the planes $2x+5y+4z+1=0$ and $4x+7y+6z+12=0$. Then, find the equation of the plane.**

OSWAAL EXAM TOOLS
OSWAAL PREPARATION TOOLS
Solutions for Self-assessment Test
Visit : *http://qr.w69b.com/g/pi8gdC0qk* **OR Scan the Code**

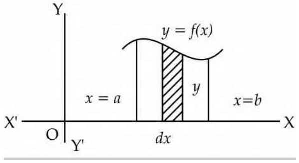
Application of the Integrals

Area Under Simple Curves and Coordinate Axes

Case I

The area of the region bounded by the curve $y = f(x)$ above X – axis and between the lines $x=a$ and $x=b$ $(b > a)$ is given by

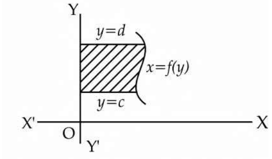
$$A=\int_a^b y\,dx \quad or \quad \int_a^b f(x)\,dx.$$



Case II

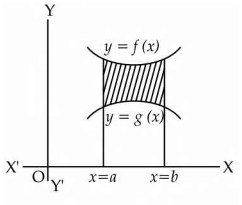
The area of the region bounded by the curve $x = f(y)$ right to the Y – axis and between the lines $y=c$ and $y=d$ $(d > c)$ is given by

$$A=\int_c^d x\,dy \quad or \quad \int_c^d f(y)\,dy.$$



Area enclosed between two curves

Case I



The areas of the region enclosed between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by

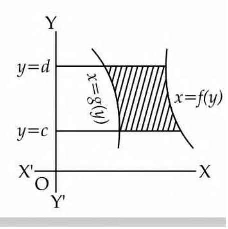
$$A=\int_a^b [f(x)-g(x)]dx, \text{where } f(x) \geq g(x) \text{ in } [a,b]$$

Case II

The area of the region enclosed between curves $x = f(y)$, $x = g(y)$ and the lines $y = c$, $y = d$ is given by

$$A=\int_c^d [f(y) - g(y)]\,dy$$

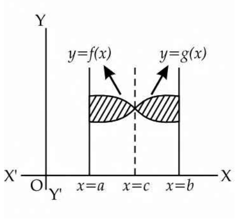
where $f(y) \geq g(y)$ in $[d, c]$



Case III

If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$ where $a < c < b$, then the area of the region bounded by the curves is

$$A=\int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$$



Trace the Mind Map

- First Level
- Second Level
- Third Level

UNIT-VII

Application of Integrals

CHAPTER

12 APPLICATION OF INTEGRALS

SYLLABUS

- ***Application in finding the area bounded by simple curves and coordinate axes. Area enclosed between two curves.***
  - ***Application of definite integrals- area bounded by curves, lines and coordinate axes is required to be covered.***
  - ***Simple curves: lines, circles/ parabolas/ ellipses, polynomial functions, modulus function, trigonometric function, exponential functions, logarithmic functions***

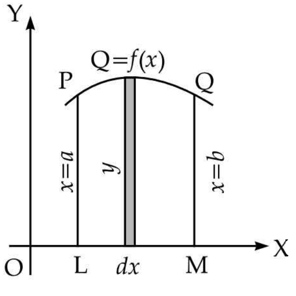
Trend Analysis

List of Concept	2016	2017	2018	2019	2020
Application of Integrals	1 Q (6 marks)	1 Q (6 marks)	2 Q (4 marks)	1 Q (6 marks)	1 Q (6 marks)

Revision Notes

- **Area under Simple curves** :

(i) Let us find the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$. Consider the area under the curve as composed by large number of thin vertical strips.



Scan to know more about this topic

Area Under the Curve

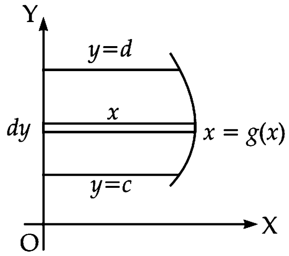
Let there be an arbitrary strip of height y and width dx.

Area of elementary strip $dA = ydx$, where $y = f(x)$. Total area A of the region between x-axis, ordinates $x = a$, $x = b$ and the curve $(y = f(x))$ = sum of areas of elementary thin strips across the region $PQML$.

$$A = \int_a^b ydx = \int_a^b f(x)dx$$

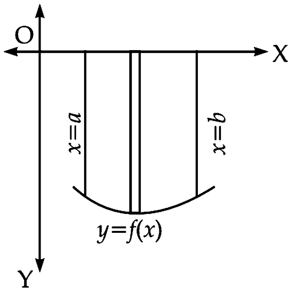
(ii) The area A of the region bounded by the curve $x = g(x)$, y-axis and the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d xdy$$



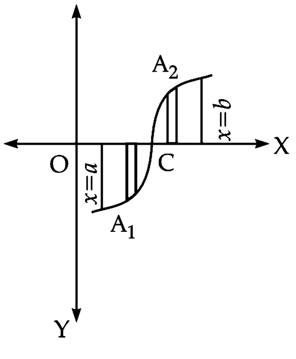
(iii) If the curve under consideration lies below x-axis, then $f(x) < 0$ from $x = a$ to $x = b$, the area bounded by the curve $y = f(x)$ and the ordinates $x = a$, $x = b$ and x-axis is negative. But, if the numerical value of the area is to be taken into consideration, then

$$\text{Area} = \left|\int_a^b f(x)dx\right|$$



(iv) It may also happen that some portion of the curve is above x-axis and some portion is below x-axis as shown in the figure. Let A_1 be the area below x-axis and A_2 be the area above the x-axis. Therefore, area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by

$$A = |A_1| + |A_2|$$



➢ **Area between two curves :**

(i) Let the two curves be $y = f(x)$ and $y = g(x)$, as shown in the figure. Suppose these curves intersect at $f(x)$ with width dx.

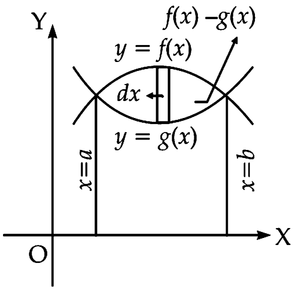
Consider the elementary strip of height y, where $y = f(x)$

$\therefore \quad dA = ydx$

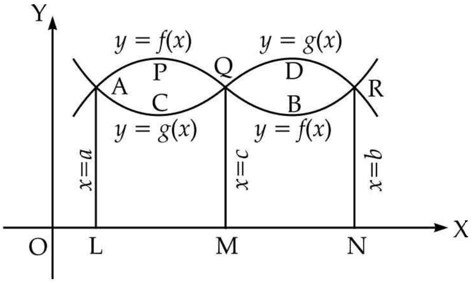
$\Rightarrow \quad A = \int_a^b [f(x) - g(x)]dx$

$= \int_a^b f(x)dx - \int_a^b g(x)dx$

= Area bounded by the curve $\{y = f(x)\}$ – Area bounded by the curve $\{y = g(x)\}$, where $f(x) > g(x)$.



(ii) If the two curves $y = f(x)$ and $y = g(x)$ intersect at $x = a$, $x = c$ and $x = b$, such that $a < c < b$.



If $f(x) > g(x)$ in $[a, c]$ and $g(x) \leq f(x)$ in $[c, b]$, then the area of the regions bounded by the curve :

$$= \text{Area of region } PACQP + \text{Area of region } QDRBQ$$

$$= \int_a^c |f(x) - g(x)|\, dx + \int_c^b |g(x) - f(x)|\, dx$$

4 Marks Questions

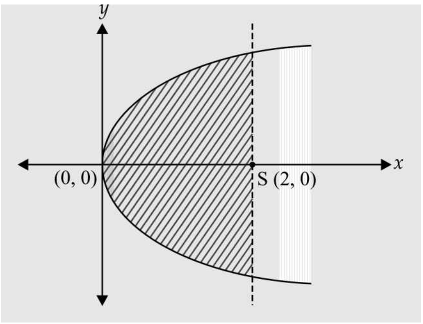
AI **Q. 1. Draw a rough sketch of the curve and find the area of the region bounded by curve $y^2 = 8x$ and the line $x = 2$. [ISC, 2018]**

Sol. Given curve $y^2 = 8x$ and line $x = 2$

$$y^2 = 4.2x$$

$$\Rightarrow \quad y = \sqrt{8}.\sqrt{x} \text{ is in I quadrant}$$

We have, $\quad \text{Area} = 2\int_0^2 y\, dx$



$$= 2\sqrt{8}\int_0^2 \sqrt{x}\, dx$$

$$= 2\sqrt{8}\ \frac{[x^{3/2}]_0^2}{3/2}$$

$$= \frac{4.\sqrt{8}}{3}[2^{3/2} - 0]$$

$$= \frac{4.2\sqrt{2}\,.\,2\sqrt{2}}{3}$$

$$= \frac{32}{3}$$

Hence, $\quad \text{area} = \frac{32}{3}$ square units.

[ISC Marking Scheme, 2018]

Examiner's Comment

- Many candidates identified the symmetry of the curve about x-axis incorrectly and either did not apply correct limits or considered area twice the result. In most of the cases, rough sketch of the curve was not drawn correctly.

Answering Tip

- Sufficient practice needed in solving problems based on area under a given curve and straight line, etc.

Q. 2. Find the area of the figure, bounded by the graph functions $y = x^2$ and $y = 2x - x^2$. [ISC, 2001]

Sol. Given, $\quad y = x^2$

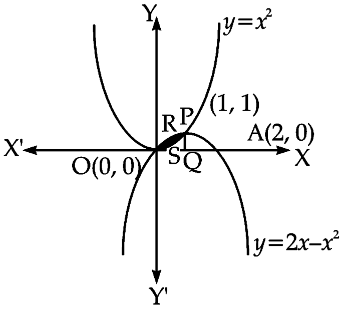
and $\quad y = 2x - x^2$

$\therefore \quad x^2 = 2x - x^2$

$\Rightarrow \quad 2x^2 - 2x = 0$

$\Rightarrow \quad 2x(x - 1) = 0$

$\Rightarrow \quad x = 0$ or $x = 1$



When $x = 0, y = 0$

When $x = 1, y = 1$

$\therefore$ Points of intersection of the two given curves are (0, 0) and (1, 1)

$$y = 2x - x^2$$

$\Rightarrow y - 1 = -(x - 1)^2$ represents a parabola with vertex (1, 1)

Required Area = shaded area ($ORPSO$)

= Area of ($ORPQO$)

– Area of ($OSPQO$)

$$= \int_0^1 (2x - x^2)dx - \int_0^1 x^2 dx$$

$$= \int_0^1 (2x - x^2 - x^2)dx$$

$$= \int_0^1 (2x - 2x^2)dx$$

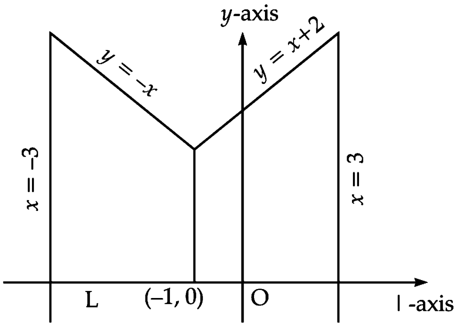
$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = 2\left(\frac{1}{6}\right)$$

$$= \frac{1}{3} \text{ sq units}$$

AI Q. 3. Find the area of the region bounded by $y = |x + 1| + 1, x = -3, x = 3$ and $y = 0$.

Sol. To obtain the graph of the given function, $y = |x + 1| + 1$, shift the graph of $y = |x + 1|$ vertically upwards by 1 unit.



$$\therefore \quad y = \begin{cases} (x+1)+1, & \text{if } x+1 \geq 0 \\ -(x+1)+1, & \text{if } x+1 < 0 \end{cases}$$

$$= \begin{cases} (x+2), & \text{if } x \geq -1 \\ -x, & \text{if } x < -1 \end{cases}$$

$$\therefore \quad \text{Required area } = \int_{-3}^{-1}(-x)dx + \int_{-1}^{3}(x+2)dx$$

$$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$

$$= -\left[\frac{1}{2} - \frac{9}{2}\right] + \left[\frac{9}{2} + 6 - \frac{1}{2} + 2\right]$$

$$= -(-4) + [8 + 4]$$

$$= 16 \text{ sq units}$$

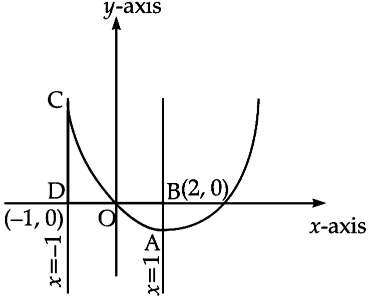
Q. 4. Find the area of the region bounded by the curve $y = x^2 - 2x$, the x-axis and the lines $x = 1$ and $x = -1$.

Sol. Given equation of the curve is

$$y = x^2 - 2x$$

$$\Rightarrow \quad y + 1 = x^2 - 2x + 1$$

$$\Rightarrow \quad y + 1 = (x - 1)^2$$



which represents a parabola with vertex at (1, –1) which opens upward.

Shaded region in the given figure represents the required area

$\therefore$ Required area = Area of (ODC) + Area of (OAB)

$$= \int_{-1}^{0} y\,dx + \left|\int_0^1 (y)dx\right|$$

[Since Area of (OAB) is below the x-axis we take modulus]

$$= \int_{-1}^{0}(x^2 - 2x)dx + \left|\int_0^1 (x^2 - 2x)dx\right|$$

$$\therefore \text{ Required area } = \left[\frac{x^3}{3} - \frac{2x^2}{2}\right]_{-1}^{0} + \left|\left[\frac{x^3}{3} - \frac{2x^2}{2}\right]_0^1\right|$$

$$= \left[0 - \frac{(-1)}{3} + 1\right] + \left|\left[\frac{1}{3} - 1 - 0\right]\right|$$

$$= \frac{4}{3} + \frac{2}{3} = 2 \text{ sq. units}$$

Q. 5. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.

Sol. Equation of the parabola

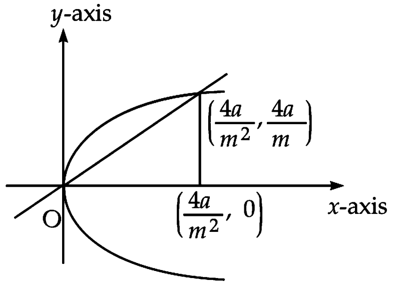
$$y^2 = 4ax$$

represents a parabola with vertex at the origin.

The chord $y = mx$ passes through the origin.

The point of intersection of the curve and the line is $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

Shaded region represents the required area


y-axis
$\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$
O
$\left(\frac{4a}{m^2}, 0\right)$
x-axis

$$\text{Required area } = \int_0^{4a/m^2} 2\sqrt{ax}\,dx - \int_0^{4a/m^2} mx\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{4a/m^2} - m\left[\frac{x^2}{2}\right]_0^{4a/m^2}$$

$$= 2\sqrt{a} \times \frac{2}{3}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2$$

$$= \frac{4\sqrt{a}}{3}\frac{8a\sqrt{a}}{m^3} - \frac{m}{2}\frac{16a^2}{m^4}$$

$$= \frac{32}{3}\frac{a^2}{m^3} - \frac{8a^2}{m^3}$$

$$= \frac{a^2}{m^3}\left(\frac{32}{3} - 8\right)$$

$$= \frac{a^2}{m^3}\left(\frac{32 - 24}{3}\right)$$

$$= \frac{8a^2}{3m^3} \text{ sq. units}$$

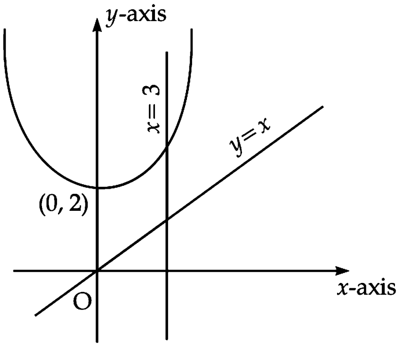
AI **Q. 6. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.**

Sol. Given curve is $y = x^2 + 2$

$\Rightarrow \quad x^2 = y - 2$

It represents a parabola with vertex at (0, 2), which opens upward.

Shaded region represents the required area



$$\text{Required area } = \int_0^3 (x^2+2)dy - \int_0^3 x\,dx$$

$$= \left[\frac{x^3}{3}+2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3$$

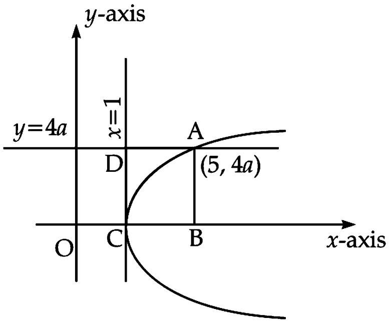
$$= \frac{27}{3}+6-0-\left(\frac{9}{2}-0\right)$$

$$= 9+6-\frac{9}{2} = \frac{21}{2} \text{ sq. units}$$

Q. 7. Find the area of the region bounded by the curves $y^2 = 4a^2(x - 1)$ and the lines $x = 1$ and $y = 4a$.

Sol. Given curve $y^2 = 4a^2(x - 1)$ represents a parabola with vertex at (1, 0), which opens on the right side. Given parabola $y^2 = 4a^2(x - 1)$ and the line $y = 4a$ intersect at $(5, 4a)$.

Shaded region represents the required area



$\therefore$ Required area = Area of (CBAD) – Area of (CBA)

$$= 4\times 4a - \int_1^5 2a\sqrt{x-1}\,dx$$

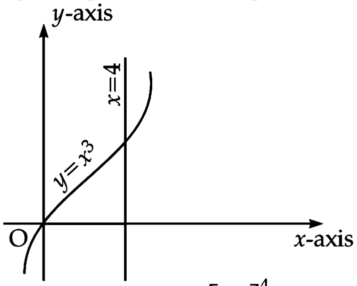
$$= 16a - 2a\left[\frac{(x-1)^{3/2}}{3/2}\right]_1^5$$

$$= 16a - 2a\times\frac{2}{3}(4^{3/2}-0)$$

$$= \frac{16a}{3} \text{ sq. units}$$

Q. 8. Find the area of the region bounded by the curves $y = x^3$, x-axis and the line $x = 4$.

Sol. Graph of the curve $y = x^3$ is shown in the figure. Shaded region represents the required area.



$$\therefore \quad \text{Required area} = \int_0^4 x^3 dx = \left[\frac{x^4}{4}\right]_0^4$$

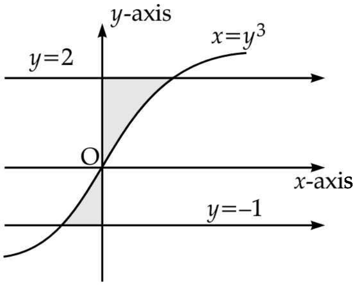
$$= \left(\frac{4^4}{4}-0\right)$$

$$= 4^3 = 64 \text{ sq. units}$$

AI **Q. 9. Find the area of the region bounded by $y = -1$, $y = 2$, $x = y^3$ and $x = 0$.**

Sol. Graph of the curve $x = y^3$ is shown in the figure.

Shaded region represents the required area.



$$\text{Required area } = \left|\int_{-1}^0 x\,dy\right| + \int_0^2 x\,dy$$

$$= \left|\int_{-1}^0 y^3 dy\right| + \int_0^2 y^3 dy$$

$$= \left|\left[\frac{y^4}{4}\right]_{-1}^0\right| + \left[\frac{y^4}{4}\right]_0^2$$

$$= \left|\left(0-\frac{1}{4}\right)\right| + \left(\frac{2^4}{4}-0\right)$$

$$= \frac{1}{4}+4 = \frac{17}{4} \text{ sq. units}$$

Q. 10. Find the area of the region bounded by $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$

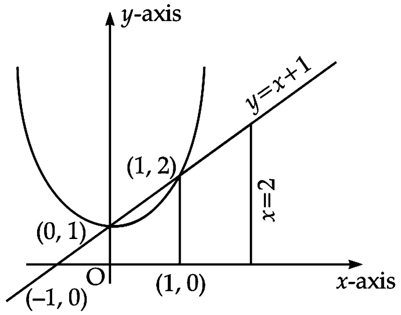
Sol. Given curve is $y = x^2 + 1$

and lines are $y = x + 1$ and $x = 2$

$y = x^2 + 1$ represents a parabola with vertex at (0, 1) which opens upward.

Points of intersection of $y = x^2 + 1$ and $y = x + 1$ are (0, 1) and (1, 2).

Shaded region represents the required area



$\therefore$ Required area $= \int_0^1 (x^2+1)dx + \int_1^2 (x+1)dx$

$$= \left[\frac{x^3}{3}+x\right]_0^1 + \left[\frac{x^2}{2}+x\right]_1^2$$

$$= \frac{1}{3}+1-0+\left(\frac{4}{2}+2-\frac{1}{2}-1\right)$$

$$= \frac{1}{3}+1+4-\frac{1}{2}-1$$

$$= \frac{23}{6} \text{ sq. units}$$

6 Marks Questions

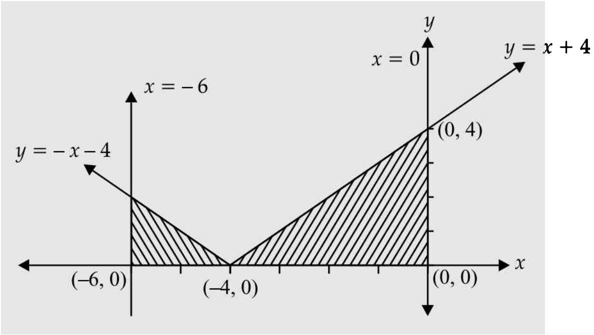
AI **Q. 1. Sketch the graph of $y = |x+4|$. Using integration, find the area of the region bounded by the curve $y = |x+4|$ and $x = -6$ and $x = 0$.**

[ISC Specimen Paper, 2021] [ISC, 2018]

Sol. Given, $y = |x+4|$

$$\because \ |x+4| = \begin{matrix} x+4>0 \Rightarrow x>-4 \\ x+4<0 \Rightarrow x<-4 \end{matrix}$$

$$\therefore \quad y = \begin{cases} x+4, & \forall\, x>-4 \\ -(x+4), & \forall\, x<-4 \end{cases}$$



Required area is

$$= \int_{-6}^{-4} -(x+4)\,dx + \int_{-4}^{0} (x+4)\,dx$$

$$= \left[-\frac{x^2}{2}+4x\right]_{-6}^{-4} + \left[\frac{x^2}{2}+4x\right]_{-4}^{0}$$

$$= [(-8+16)-(-18+24)]+[(0+0)-(8-16)]$$

$$= [-(-8)-6]+[8]$$

$$= 10 \text{ sq. units.}$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Quite a few candidate attempted this question. Many candidates did not understand the concept of absolute value function and could not identify the limits of it by using the given conditions.

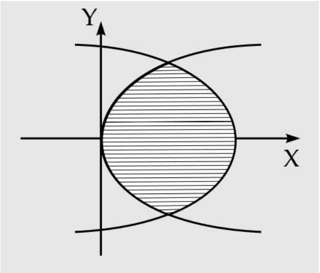
Answering Tip

- Learn and understand the concept of absolute value function. Also, do practice to draw the rough sketch of the same by applying the given conditions.

Q. 2. Sketch the graphs of the curves $y^2 = x$ and $y^2 = 4 - 3x$ and find the area enclosed between them.

[ISC, 2020, 2014, 2007]

Sol.



$$y^2 = 4-3x \qquad \ldots\text{(i)}$$

$$y^2 = x \qquad \ldots\text{(ii)}$$

Solving (i) and (ii) we get $x = 4-3x$

$$4x = 4,\ x = 1$$

$$\text{Required area} = 2\left[\int_0^1 \sqrt{x}dx + \int_1^{4/3} \sqrt{4-3x}dx\right]$$

$$= 2\left(\frac{2}{3}+\frac{2}{9}\right)$$

$$= \frac{16}{9} \text{ sq. units}$$

[ISC Marking Scheme, 2014]

Examiner's Comment

- Sketching of both parabolas in the same diagram were not done by many candidates. Their points of integration were not found by some. The area required was not correctly identified in several cases. Some candidates expressed the definite integrals and their limits incorrectly.

Answering Tip

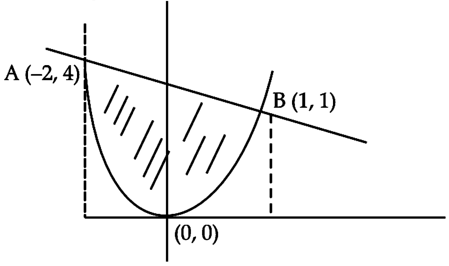
- The parabolas, $y^2 = x$ and $y^2 = 4 - 3x$ required accurate sketching. Their points of intersection need to be calculated by solving the equations simultaneously.

AI **Q. 3. Draw a rough sketch and find the area bounded by the curve $x^2 = y$ and $x + y = 2$.**

[ISC, 2019]

Sol. Curve $y = x^2$

Line $x + y = 2$



points of intersection

$x + x^2 = 2$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2, x = 1$ A (–2, 4)

$y = 4, y = 1$ B (1, 1)

Area of the shaded region

$$\underset{\text{[Line]}}{\int_{-2}^{1} y dx} - \underset{\text{[Curve]}}{\int_{-2}^{1} y dx} = \int_{-2}^{1}(2-x)dx - \int_{-2}^{1} x^2 dx$$

$$\left[\left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\right]_{-2}^{1}$$

$$\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right)$$

$$\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right)$$

$$2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - \frac{8}{3}$$

$$4\frac{1}{2} \text{ sq.units}$$

[ISC Marking Scheme, 2019]

Examiner's Comment

- A number of candidates attempted this question well. Some candidates made mistakes in sketching the two curves. Hence, they could not find the common portion and limits correctly.

Answering Tips

- Do adequate practice of sketching different types of curves and interpretation of the graph.
- Understand the method of identifying the limits of the definite integral and area bounded by the region for the given curves and X-axis, etc.

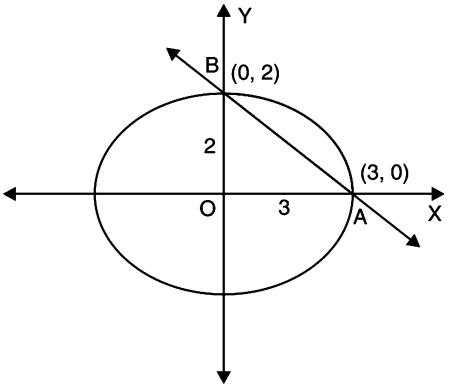
Q. 4. Using integration, find the area of the following region :

$$\left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \le 1 \le \frac{x}{3} + \frac{y}{2}\right\}$$ **[ISC Specimen, 2018]**

Sol. Given ellipse is $\frac{x^2}{9} + \frac{y^2}{4} \le 1$ ... (i)

and given line is $\frac{x}{3} + \frac{y}{2} \ge 1$ ... (ii)

Required area is shown shaded in the given figure.



$$\therefore \text{ Area} = \int_0^3 2\sqrt{1 - \frac{x^2}{9}}\, dx - \int_0^3 2\left(1 - \frac{x}{3}\right) dx$$

$$= \frac{2}{3}\int_0^3 \sqrt{3^2 - x^2}\, dx - \frac{2}{3}\int_0^3 (3 - x)\, dx$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{3^2 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3 - \frac{2}{3}\left[3x - \frac{x^2}{2}\right]_0^3$$

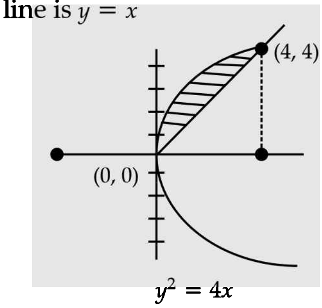
$$\textbf{Area} = \frac{2}{3}\left[0 + \frac{9}{2}\sin^{-1}(1) - 0\right] - \frac{2}{3}\left[9 - \frac{9}{2} - 0\right]$$

$$= 3\left(\frac{\pi}{2}\right) - 3 = 3\left(\frac{\pi}{2} - 1\right) \text{ sq. units.}$$

Q. 5. Draw a rough sketch of the curve $y^2 = 4x$ and find the area of the region enclosed by the curve and the line $y = x$. **[ISC, 2017]**

Sol. Given curve is $y^2 = 4x$ (parabola)

and line is $y = x$



$y^2 = 4x$

x	0	1	1	4	4
y	0	2	–2	4	–4

$y = x$

x	0	1	2	4
y	0	1	2	4

Area of the shaded region

$$= 2\int_0^4 \sqrt{x}\, dx - \int_0^4 x\, dx$$

$$= 2 \times \frac{2}{3}[x^{3/2}]_0^4 - \left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{4}{3}[4^{3/2} - 0] - \frac{1}{2}[4^2 - 0]$$

$$= \frac{32}{3} - 8 = \frac{8}{3} \text{sq units}$$

[ISC Marking Scheme, 2017]

Examiner's Comment

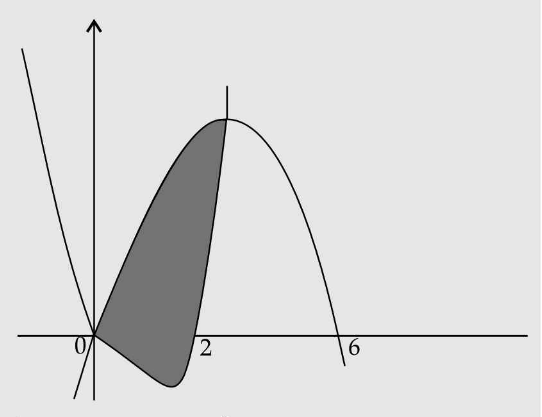
- Many candidates got correct solution for this question. A few candidates solved the question with incorrect limits. Candidates made mistakes in sketching the curve, finding limits and required area bounded by the curves.

Answering Tip

- Sketch the rough curve to understand the requirement like the area required to be found, the points of intersection and the limits of definite integral. Adequate practice of solving (Sketching) different types of curves should be needed.

Q. 6. Find the area of the region bound by the curves $y = 6x - x^2$ and $y = x^2 - 2x$. **[ISC, 2016]**

Sol.



The curve $y = 6x - x^2$

$y = -(x-3)^2 + 9$

represents a parabola with vertex at (3, 9) and it opens downward.

The curve $y = x^2 - 2x$

$y = (x-1)^2 - 1$

represents a parabola with vertex at (1 – 1) and it opens upward.

Both the curves pass through origin and intersect in the first quadrant at (4, 8)

$$\text{Required area} = \int_0^4 (6x - x^2)dx - \int_0^4 (x^2 - 2x)dx$$

$$= \int_0^4 (8x - 2x^2)dx = \left[8.\frac{x^2}{2} - 2.\frac{x^3}{3}\right]_0^4$$

$$= 64 - \frac{128}{3}$$

$$= \frac{64}{3} \text{ sq units}$$

[ISC Marking Scheme, 2016]

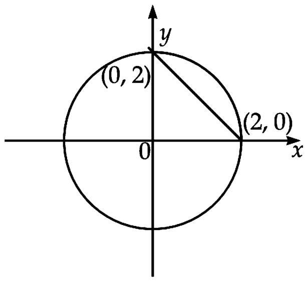
Examiner's Comment

- Many candidates got the correct solution for this question. A few candidates solved the question with incorrect limits. Some candidates made mistakes in sketching the curve and finding limits.

Answering Tip

- Sketching of curves may not be necessary always but a rough sketch will always help to understand the requirements, the area required to be found, the points of intersection and the limits of the definite integral. Since the functions given was simple algebraic the integration of such functions will not cause problems and the solution can be easily found.

AI Q. 7. Find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$. **[ISC, 2015]**



Sol. The required area

$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)dx$$

$$= \left|\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right|_0^2 - \left|2x - \frac{x^2}{2}\right|_0^2$$

$$= 2\times\frac{\pi}{2} - 4 + 2$$

$= \pi - 2$ sq. units **[ISC Marking Scheme, 2015]**

Detailed Answer :

Given the equation of the circle $x^2 + y^2 = 4$ and the equation of the line $x + y = 2$.

The curves intersect at (2, 0) and (0, 2).

The required area

$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)dx$$

$$= \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[\frac{x\sqrt{4-x^2}}{2} + 2\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2}\right]_0^2$$

$$= \frac{2\cdot\sqrt{4-4}}{2} + 2\sin^{-1}1 - 2(2) + \frac{4}{2} - 0$$

$$= 2\cdot\frac{\pi}{2} - 4 + 2$$

$= \pi - 2$ sq. units

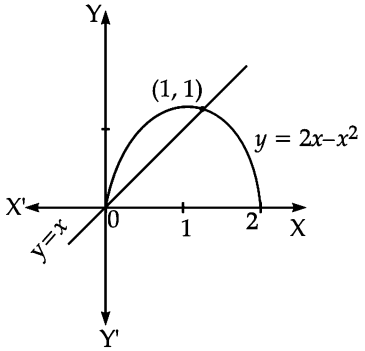
Q. 8. Find the area bounded by the curve $y = 2x - x^2$ and the line $y = x$. **[ISC 2013]**

Sol. Given curve $y = 2x - x^2$

$-y = x^2 - 2x$

$-y + 1 = x^2 - 2x + 1$

$-(y-1) = (x-1)^2$



represents a downward parabola with vertex at (1, 1)

Point of intersection of the parabola and the line $y = x$:

Put $y = x$

$$-(x-1) = (x-1)^2$$

$$-x + 1 = x^2 - 2x + 1$$

$$\Rightarrow \quad x^2 - x = 0$$

$$x(x-1) = 0$$

$$\Rightarrow \quad x = 0, 1$$

∴ Points of intersections are (0, 0) and (1, 1)

∴ Required area is the shaded region in the figure.

$$\therefore \quad \text{Required area} = \int_0^1 (2x - x^2)dx - \int_0^1 x\,dx$$

$$= \int_0^1 (2x - x^2 - x)dx$$

$$= \int_0^1 (x - x^2)dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - 0$$

$$= \frac{1}{6} \text{ sq. units}$$

Q. 9. Find the area of the region bounded by the curve $x = 4y - y^2$ and the y-axis. [ISC, 2012]

Sol. Given, $x = 4y - y^2$

i.e., $y^2 - 4y = -x$

$$(y-2)^2 = -(x-4)$$

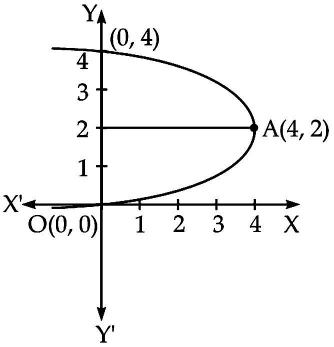
∴ Given curve represents a left parabola with vertex at $A(4, 2)$.

The parabola meets y-axis, *i.e.*, $x = 0$

$$4y - y^2 = 0 \Rightarrow y(4-y) = 0$$

$$\therefore \quad y = 0 \text{ or } 4$$

The area enclosed between the curve and the y-axis is shaded in the figure.



$$\text{Required area} = \int_0^4 x\,dy = \int_0^4 (4y - y^2)dy$$

$$= \left[4 \cdot \frac{y^2}{2} - \frac{y^3}{3}\right]_0^4$$

$$= \left[32 - \frac{64}{3}\right] - [0 - 0]$$

$$= \frac{32}{3} \text{ sq. unit}$$

Q. 10. Draw a rough sketch of the curves $y = (x - 1)^2$ and $y = |x - 1|$. Hence, find the area of the region bounded by these curves. [ISC, 2011]

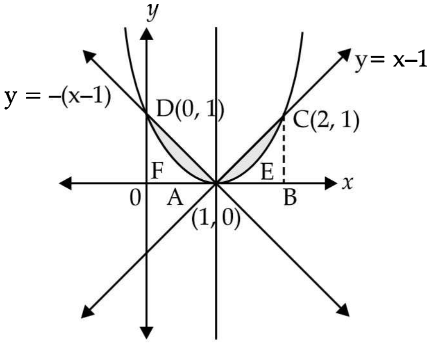
Sol. $y = (x - 1)^2$ is a parabola whose axis is a line $x = 1$ and vertex is at (1, 0).

$y = (x - 1)^2$

x	0	1	2
y	1	0	1

$$\text{Also,} \quad y = |x-1| = \begin{cases} (x-1) & \text{if } x - 1 \geq 0 \\ -(x-1) & \text{if } x - 1 < 0 \end{cases}$$

$$= \begin{cases} x-1 & \text{if } x \geq 1 \\ -(x-1) & \text{if } x < 1 \end{cases}$$



$y = x - 1$

x	0	1	2
y	–1	0	1

Also, $y = -(x - 1)$

x	1	0
y	0	1

Here, we have to find the area enclosed by $y = (x - 1)^2$ and the lines $y = (x - 1)$ and $y = -(x - 1)$

∴ Required area = Area ABC – Area $ABCEA$ + Area AOD – Area $AODFA$

$$= \int_1^2 (\text{Line } y = x - 1)dx$$

$$- \int_1^2 [\text{Curve } y = (x-1)^2]dx$$

$$+ \int_0^1 (\text{Line } y = -(x-1)dx$$

$$- \int_0^1 [\text{Curve } y = (x-1)^2]dx$$

$$= \int_1^2 (x-1)dx - \int_1^2 (x-1)^2 dx$$

$$+ \int_0^1 -(x-1)dx - \int_0^1 (x-1)^2 dx$$

$$=\left[\frac{x^2}{2}-x\right]_1^2-\left[\frac{x^3}{3}-x^2+x\right]_1^2$$

$$-\left[\frac{x^2}{2}-x\right]_0^1-\left[\frac{x^3}{3}-x^2+x\right]_0^1$$

$$=0-\left(-\frac{1}{2}\right)-\left[\frac{8}{3}-4+2-\frac{1}{3}+1-1\right]$$

$$-\left[\frac{1}{2}-1\right]-\left[\frac{1}{3}-1+1\right]$$

$$=\frac{1}{2}-\frac{1}{3}+\frac{1}{2}-\frac{1}{3}$$

$$=1-\frac{2}{3}=\frac{1}{3}\text{ sq. units}$$

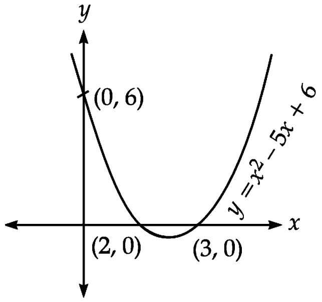
Q. 11. Draw a rough sketch of the curve $y = x^2 - 5x + 6$ and find the area bounded by the curve and the x-axis. [ISC, 2010]

Sol. $y = x^2 - 5x + 6 = (x-3)(x-2)$

Given, curve does not pass through the origin.

The curve cuts the x-axis at (2, 0) and (3, 0) and the curve cuts the y-axis at (0, 6).

The area bounded by the curve and the x-axis is given by



Since, the curve is below the x-axis, we take modulus while finding the area.

$$\left|\int_2^3 y\,dx\right|=\left|\int_2^3(x^2-5x+6)dx\right|$$

$$=\left|\left[\frac{x^3}{3}-\frac{5x^2}{2}+6x\right]_2^3\right|$$

$$=\left|\left[\frac{27}{3}-5\times\frac{9}{2}+6\times3\right]\right.$$

$$\left.-\left[\frac{8}{3}-5\times\frac{4}{2}+6\times2\right]\right|$$

$$=\left|9-\frac{45}{2}+18-\frac{8}{3}+10-12\right|$$

$$=\left|25-\frac{151}{6}\right|=\frac{1}{6}\text{ sq. units}$$

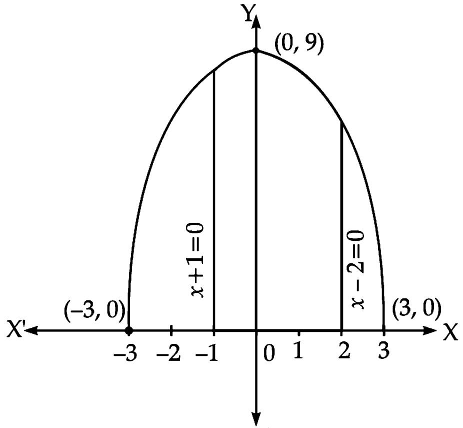
AI **Q. 12. Draw a rough sketch of the curve $x^2 + y = 9$ and find the area enclosed by the curve, the x-axis and the lines $x + 1 = 0$ and $x - 2 = 0$. [ISC, 2009]**

Sol. The given equation of the curve

$$x^2 + y = 9$$

$$\Rightarrow \quad x^2 = -(y-9)$$

represents a parabola with vertex (0, 9)



$$\text{Required area }=\int_{-1}^{2}(9-x^2)dx$$

$$=\left[9x-\frac{x^3}{3}\right]_{-1}^{2}$$

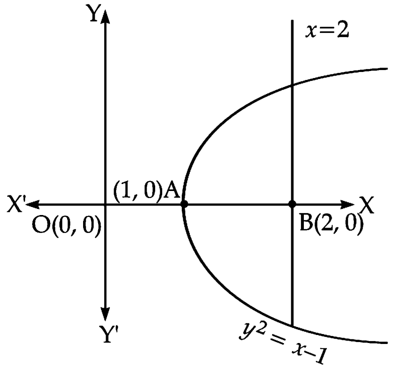
$$=\left[\left(18-\frac{8}{3}\right)-\left(-9+\frac{1}{3}\right)\right]$$

$$=[27-3]=24\text{ sq. units}$$

Q. 13. Draw a rough sketch of the curve $y^2 + 1 = x$, $x \le 2$. Find the area enclosed by the curve and the line $x = 2$. [ISC, 2008]

Sol. Given curve : $y^2 + 1 = x, x \le 2$

$$\Rightarrow \quad y^2 = x - 1$$



$$\text{Required area }=2\int_1^2\sqrt{x-1}\,dx$$

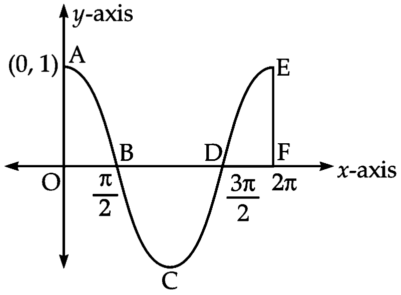
$$=\frac{2\left[(x-1)^{3/2}\right]_1^2}{\frac{3}{2}}$$

$$=2\times\frac{2}{3}[(1)^{3/2}]$$

$$=\frac{4}{3}\text{ sq. units}$$

Q. 14. Find the area of the region bounded by the curve $y = \cos x, x = 0$ and $x = 2\pi$.

Sol. Given, $y = \cos x, 0 \le x \le 2\pi$ ...(i)



Area required = Area OAB + Area BCD + Area DEF

Now, area of $OAB = \int_0^{\pi/2} y\,dx$

$= \int_0^{\pi/2} \cos x dx$ [from (i)]

$= [\sin x]_0^{\pi/2}$

$= \sin\frac{\pi}{2} - \sin 0$

$= 1 - 0 = 1$

Area of $BCD = \int_{\pi/2}^{3\pi/2} y\,dx$

$= \int_{\pi/2}^{3\pi/2} \cos x\,dx$

$= [\sin x]_{\pi/2}^{3\pi/2}$

$= \sin\frac{3\pi}{2} - \sin\frac{\pi}{2}$

$= -1 - 1 = -2$

$\therefore$ Area of $BCD = 2$ ($\because$ area can't be negative)

$\therefore$ Area of $DEF = \int_{\frac{3\pi}{2}}^{2\pi} y\,dx$

$= \int_{\frac{3\pi}{2}}^{2\pi} \cos dx$

$= [\sin x]_{\frac{3\pi}{2}}^{2\pi}$

$= \sin 2\pi - \sin\frac{3\pi}{2}$

$= 0 - (-1)$

$= 1$

Therefore,

Area required = Area OAB + Area BCD + Area DEF

= 1 + 2 + 1

= 4 sq. units.

AI **Q. 15. Using integration, find the area bounded by the tangent to the curve $4y = x^2$ at the point (2, 1) and the lines whose equations are $x = 2y$ and $x = 3y - 3$.**

Sol. Given, $4y = x^2$

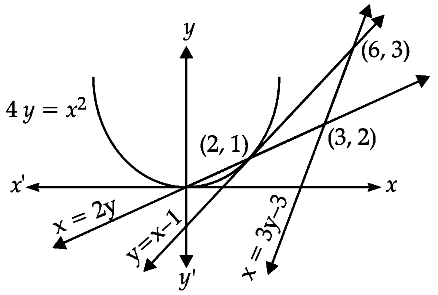
$\Rightarrow \quad 4\frac{dy}{dx} = 2x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{2}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=2} = 1$

The equation of tangent is $y = x - 1$

The required area = Shaded area of graph



$$= \left[\int_2^3\left\{(x-1)-\frac{x}{2}\right\}dx+\int_3^6\left[\frac{(x+3)}{3}-\frac{x}{2}\right]dx\right]$$

$$= \left[\int_2^3(x-1)dx+\frac{1}{3}\int_3^6(x+3)dx-\frac{1}{2}\int_2^6 xdx\right]$$

$$= \left[\left[\frac{x^2}{2}-x\right]_2^3+\frac{1}{3}\left[\frac{x^2}{2}+3x\right]_3^6-\frac{1}{4}[x^2]_2^6\right]$$

$$= \left[\frac{9}{2}-3-2+2\right]+\frac{1}{3}\left[18+18-\frac{9}{2}-9\right]-\frac{1}{4}[36-4]$$

$$= \left[\frac{3}{2}\right]+\frac{1}{3}\left[\frac{45}{2}\right]-\frac{1}{4}[32]$$

$$= \frac{3}{2}+\frac{15}{2}-8$$

$= 1$ sq. unit

SELF ASSESSMENT TEST

UNIT–VII

Maximum Marks : 20 **Timing : 1 Hour**

Q. 1. Find the area of the region bounded by the curve $y = \sin x$, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

Q. 2. Using integration, find the area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

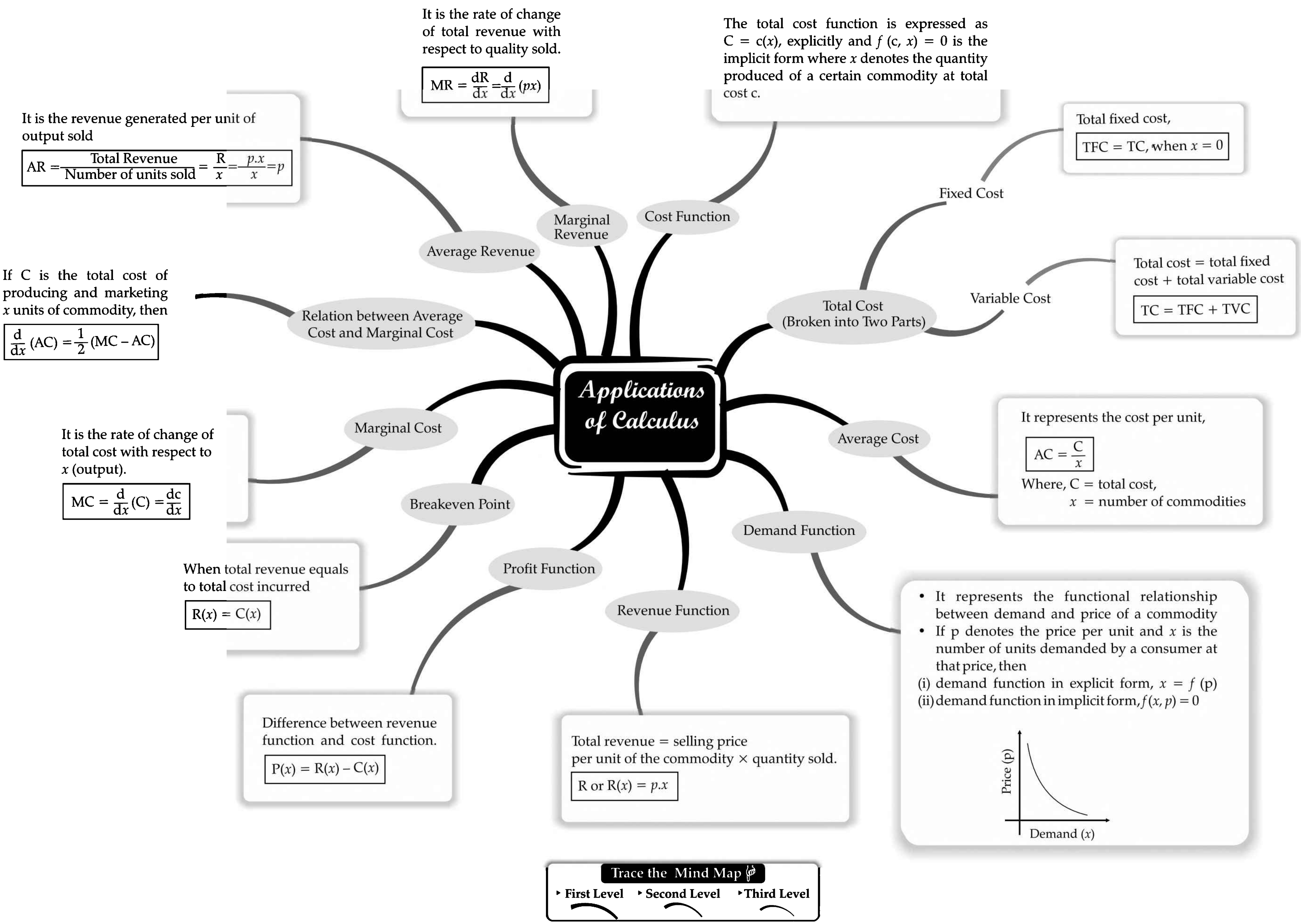
Q. 3. Sketch and shade the area of the region lying in first quadrant and bounded by $y = 9x^2$, $x = 0$, $y = 1$ and $y = 4$. Find the area of the shaded region.

Q. 4. Show that the area included between the x-axis and the curve $a^2y = x^2(x + a)$ is $\frac{a^2}{12}$.

OSWAAL PREPARATION TOOLS

Solutions for Self-assessment Test

Visit : *http://qr.w69b.com/g/pi8gdC0qk* OR Scan the Code



Applications of Calculus

Cost Function

The total cost function is expressed as C = c(x), explicitly and f (c, x) = 0 is the implicit form where x denotes the quantity produced of a certain commodity at total cost c.

Total Cost (Broken into Two Parts)

Fixed Cost

Total fixed cost,

$\boxed{\text{TFC} = \text{TC, when } x = 0}$

Variable Cost

Total cost = total fixed cost + total variable cost

$\boxed{\text{TC} = \text{TFC} + \text{TVC}}$

Average Cost

It represents the cost per unit,

$\boxed{\text{AC} = \frac{\text{C}}{x}}$

Where, C = total cost,
x = number of commodities

Demand Function

- It represents the functional relationship between demand and price of a commodity
- If p denotes the price per unit and x is the number of units demanded by a consumer at that price, then

(i) demand function in explicit form, $x = f$ (p)
(ii) demand function in implicit form, $f(x, p) = 0$

Revenue Function

Total revenue = selling price per unit of the commodity × quantity sold.

$\boxed{\text{R or R}(x) = p.x}$

Profit Function

Difference between revenue function and cost function.

$\boxed{\text{P}(x) = \text{R}(x) - \text{C}(x)}$

Breakeven Point

When total revenue equals to total cost incurred

$\boxed{\text{R}(x) = \text{C}(x)}$

Marginal Cost

It is the rate of change of total cost with respect to x (output).

$\boxed{\text{MC} = \frac{d}{dx}(\text{C}) = \frac{dc}{dx}}$

Relation between Average Cost and Marginal Cost

If C is the total cost of producing and marketing x units of commodity, then

$\boxed{\frac{d}{dx}(\text{AC}) = \frac{1}{2}(\text{MC} - \text{AC})}$

Average Revenue

It is the revenue generated per unit of output sold

$\boxed{\text{AR} = \frac{\text{Total Revenue}}{\text{Number of units sold}} = \frac{\text{R}}{x} = \frac{p.x}{x} = p}$

Marginal Revenue

It is the rate of change of total revenue with respect to quality sold.

$\boxed{\text{MR} = \frac{d\text{R}}{dx} = \frac{d}{dx}(px)}$

UNIT-VIII

Application of Calculus

CHAPTER

13

APPLICATION OF CALCULUS

SYLLABUS

- *Application of calculus in Commerce and Economics in the following :*
  - *Cost function,*
  - *Average cost,*
  - *Marginal cost and its interpretation,*
  - *Demand function,*
  - *Revenue function,*
  - *Marginal revenue function and its interpretation*
  - *Profit function and breakeven point.*
  - *Rough sketching of the following curves : AR, MR, R, C AC, MC and their mathematical interpretation using the concept of maxima and minima and increasing-decreasing functions.*

    Self-explanatory

Note : Application involving differentiation, integration, increasing and decreasing function and maxima and minima to be covered.

Trend Analysis

List of Concept	2016	2017	2018	2019	2020
Application of Calculus	1 Q (6 marks)	1 Q (4 marks)	2 Q (2 marks) 2 Q (4 marks)	2 Q (2 marks) 1 Q (4 marks)	2 Q (2 marks) 1 Q (4 marks)

Revision Notes

Scan to know more about this topic

Application of Calculus

Scan to know more about this topic

Application of Calculus in Commerce

- **Cost function :**

  If x denotes the quantity produced of a certain commodity at total cost c, then the total cost function is given as $C = c(x)$ (explicit form) and $f(c, x) = 0$ in the implicit form. The total cost is divided into two parts, the fixed cost and the variable cost.

  (1) Fixed Cost :

  Fixed costs are those which incurred regardless of the level of production - like interest, rent, wages of permanent staff etc.

  Thus, Total Fixed Cost (TFC) $= TC$, when $x = 0$

  Fixed cost does not change whether there is any increase or decrease in level of production.

  (2) Variable Cost :

  Variable costs are those which vary with output.

  e.g., Raw materials and wages of casual labour etc. Thus,

  $$\text{Total Cost} = \text{Total Fixed Cost} + \text{Total Variable Cost}$$

  or $$TC = TFC + TVC$$

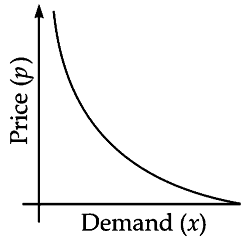
  (3) Average Cost :

  The average cost represents the cost per unit, *i.e.*, $AC = \dfrac{TC}{x} = \dfrac{\text{Total cost}}{\text{number of commodities}}$

(4) Demand Function :

The demand function is the functional relationship between demand and price of a commodity. If p denotes the price per unit and x is the number of units demanded by a consumer at that price, then demand function in explicit form will be written as $x = f(p)$ and in implicit form as $f(x, p) = 0$.

$\therefore$ Price (p) is shown against demand x in the following figure :



(5) Revenue Function :

Revenue is the amount received by a company on selling a certain number of units of a commodity. Let p be the price per unit and x be the number of units sold. Then, total revenue

$$R \text{ or } R(x) = p \times x$$

Total revenue = Selling price per unit of the commodity × Quantity sold

(6) Profit Function :

Profit function is the difference of revenue function and cost function, *i.e.*,

$$P(x) = R(x) - C(x)$$

(7) Breakeven Point :

The breakeven point is that point, where total revenue equals to the total cost incurred, *i.e.*, $P(x) = 0$ or $R(x) = C(x)$. At breakeven point, a company begins to earn profit.

(8) Marginal Cost :

The marginal cost is the rate of change of the total cost with respect to x (Output).

$$MC = \frac{d}{dx}(C) = \frac{dC}{dx}, x > 0$$

(9) Relation between Average Cost (AC) and Marginal Cost (MC) :

If C is the total cost of producing and marketing x units of commodity, then

$$\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC)$$

Here, three cases arise :

Case 1 : For $MC > AC \Rightarrow AC$ increases with x and AC curve is rising.

Case 2 : For $MC = AC \Rightarrow AC$ is constant at all levels of output.

Case 3 : For $MC < AC \Rightarrow AC$ decreases with x and AC curve is falling.

(10) Average Revenue and Marginal Revenue :

(i) Average Revenue is the revenue generated per unit of output sold. It is denoted by AR, *i.e.*,

$$AR = \frac{\text{Total revenue}}{\text{Number of units sold}} = \frac{R}{x} = \frac{p \cdot x}{x} = p$$

(ii) Marginal Revenue is the rate of change of total revenue with respect to quantity sold.

i.e., $$MR = \frac{dR}{dx} = \frac{d}{dx}(px)$$

(11) Marginal Average Cost :

Marginal average cost (MAC) is $\frac{d}{dx}$ (AC), *i.e.*, derivative of the average cost function. This is also called as slope of average cost curve.

➢ **Minimisation of AC and MC Functions :**

Using the concept of maxima and minima, we can determine the level of output where per unit cost is minimum corresponding to a given total or average cost function. Following steps are used to find the optimal level of output :

Step 1 : From AC, determine $\frac{d}{dx}(AC)$.

Step 2 : Let $\frac{d}{dx}(AC) = 0$ and solve for x.

Step 3 : Find $\frac{d^2}{dx^2}(AC)$.

Step 4 : The value of x for which $\frac{d^2}{dx^2}(AC) > 0$ is the minimum per unit cost.

Step 5 : Minimum average cost can be determined by substituting this value of x in $AC(x)$. Similarly, we can find the minimum marginal cost or minimum cost

- **Maximisation of Total Revenue :**

It is possible to determine the level of output at which the total revenue is maximum, if demand function is given. Total revenue, $R(x)$, is the maximum, when marginal revenue is zero.

To maximise total revenue, following steps are to be followed :

Step 1 : Determine $\frac{dR}{dx}$.

Step 2 : Let $\frac{dR}{dx} = 0$ and solve for x.

Step 3 : Determine $\frac{d^2R}{dx^2}$.

Step 4 : The value for which $\frac{d^2R}{dx^2} < 0$ gives the maximum total revenue.

Step 5 : The maximum revenue can be obtained by putting this value of x in $R(x)$. Similarly, maximum profit can be determined.

Mnemonics

Types of Function in Application of calculus

Trained	**Dogs**	**Pluck**	**Roadblocks**
Total Cost Function	Demand Function	Profit Function	Revenue Function

Types of Cost in Application of calculus

Antisocial	**F**ishes	**M**anufacture	**V**oices
Average Cost	Fixed Cost	Marginal Cost	Variable Cost

1 Mark Questions

AI Q. 1 The demand function of a monopolist is given by $x = 100 - 4p$. The quantity at which $MR = 0$ will be :

(a) 25 **(b)** 10

(c) 50 **(d)** 30

[ISC Specimen Paper, 2021]

Sol. Correct Option : (c)

Explanation:

$$x = 100 - 4p$$

$$\Rightarrow \quad p = \frac{100 - x}{4}$$

$$\therefore \quad R(x) = p.x = \frac{100x - x^2}{4}$$

$$\therefore \quad M.R. = \frac{100 - 2x}{4}$$

Now, for $M.R. = 0$

$$\frac{100 - 2x}{4} = 0$$

$$\Rightarrow \quad x = 50$$

Q. 2 Find the marginal cost function (MC), if the cost function is : $C(x) = \frac{x^3}{3} + 5x^2 - 16x + 2$ [ISC Specimen Paper, 2021]

Sol.
$$C(x) = \frac{x^3}{3} + 5x^2 - 16x + 2$$
$$M.C. = \frac{3x^2}{3} + 10x - 16$$
$$= x^2 + 10x - 16$$

AI Q. 3 The marginal revenue function of a commodity is given by MR = 11 – 3x + 4x^2. Find the revenue function. [ISC Specimen Paper, 2021]

Sol. $M.R. = 11 - 3x + 4x^2$
$$R(x) = \int M.R.\, dx + c$$
$$= \int (11 - 3x + 4x^2) + c$$
$$= 11x - 3\frac{x^2}{2} + 4\frac{x^3}{3} + c$$

when, $x = 0$; $R(x) = 0$

$\therefore$ $c = 0$

$\therefore$ $R(x) = 11x - \frac{3}{2}x^2 + \frac{4}{3}x^3$

Q. 4 If the marginal revenue function for a commodity is MR = 9 – 4x^2 . Find the demand function.

Sol. Given marginal revenue
$$MR = 9 - 4x^2$$
$$\Rightarrow \quad \frac{dR}{dx} = 9 - 4x^2$$
$$\Rightarrow \quad dR = (9 - 4x^2)dx$$
$$\Rightarrow \quad \int dR = (9 - 4x^2)dx$$
$$\Rightarrow \quad R = 9x - \frac{4x^3}{3} + k \quad ...(1)$$

When no product is sold, revenue is zero

$\therefore$ When $x = 0, R = 0$

$\Rightarrow \quad 0 = 0 - 0 + k \quad k = 0$

$\therefore$ (1) become
$$R = 9x - \frac{4x^3}{3}$$

We know that $R = Px$ where P is the demand function
$$Px = x\left(9 - \frac{4x^2}{3}\right)$$
$$P = 9 - \frac{4x^2}{3}$$

Hence, the demand function is $9 - \frac{4x^2}{3}$

AI Q. 5 The cost of over haul of an engine is ₹10,000 The operating cost per hour is at the rate of 2x – 240 where the engine has run x km. Find out the total cost if the engine run for 300 hours after overhaul.

Sol. Given cost of overhaul of an engine is ₹10,000

Operating cost per hour = $2x - 240$.

Total cost for the engine to run for 300 hours after overhaul
$$= 10000 + \int_0^{300} (2x - 240)dx$$
$$= 10{,}000 + \left[\frac{2x^2}{2} - 240x\right]_0^{300}$$
$$= 10{,}000 + \left[x^2 - 240x\right]_0^{300}$$
$$= 10{,}000 + \left[(300)^2 - (240)(300)\right] - 0$$
$$= 10{,}000 + 90{,}000 - 72{,}000$$
$$= 1{,}00{,}000 - 72{,}000$$
$$= ₹\ 28{,}000$$

Q. 6 A company receives a shipment of 500 scooters every 30 days. From experience it is known that the inventory on hand is related to the number of days x. Since the shipment. I (x) = 500 – 0.03x^2 , the daily holding cost per scooter is ₹ 0.3. Determine the total cost for maintaining inventory for 30 days.

Sol. Given inventory on hand $I(x) = 500 - 0.03x^2$

Holding cost per scooter is C_1 = ₹ 0.3

Time period T = 30 days

Total inventory carrying cost
$$= C_1 \int_0^{r} I(x)dx$$
$$= 0.3\int_0^{30} (500 - 0.03x^2)dx$$
$$= 0.3\int_0^{30} (500 - 0.03x^2)dx$$
$$= 0.3\left[500x - \frac{0.03x^3}{3}\right]_0^{30}$$
$$= 0.3\left[500x - 0.01x^3\right]_0^{30}$$
$$= 0.3\left\{\left[500(30) - 0.01(30)^3\right] - 0\right\}$$
$$= 0.3[15000 - 0.01(27000)]$$
$$= 0.3[15000 - 270)]$$
$$= 0.3[14730]$$
$$= ₹\ 4419$$

Hence, the total cost or maintaining inventory for 30 days =₹ 4419

2 Marks Questions

AI Q. 1. A company produces a commodity with ₹24,000 as fixed cost. The variable cost estimated to be 25% of the total revenue received on selling the product, is at the rate of ₹8 per unit. Find the break-even point.

[ISC Specimen Paper, 2021] [ISC, 2019]

Sol. Given fixed cost = ₹ 24000

Let number of units be : x

Selling price per unit $p(x)$ = ₹ 8

$\therefore$ Revenue function $R(x) = 8x$

Cost function $= C(x) = 24000 + 25\%$ of $8x$

$= 24000 + 2x$

For breakeven points: $R(x) = C(x)$

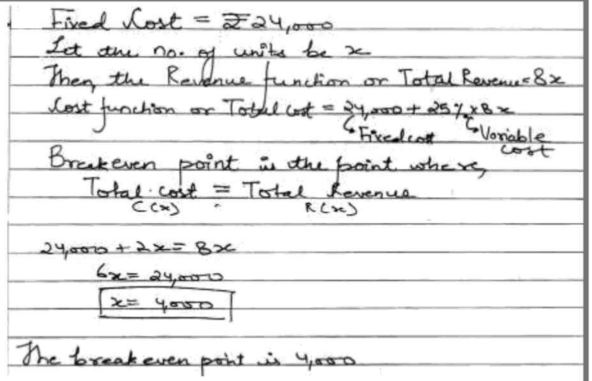
$8x = 24000 + 2x$

$6x = 24000$

$x = 4000$ units

[ISC Marking Scheme, 2019]

OR



Fixed Cost = ₹24,000

Let the no. of units be x

Then the Revenue function or Total Revenue = $8x$

Cost function or Total Cost = $\underset{\text{Fixed cost}}{24{,}000} + \underset{\text{Variable cost}}{25\% \times 8x}$

Breakeven point is the point where,

$\underset{C(x)}{\text{Total cost}} = \underset{R(x)}{\text{Total Revenue}}$

$24{,}000 + 2x = 8x$

$6x = 24{,}000$

$\boxed{x = 4{,}000}$

The breakeven point is 4,000

[Topper's Answer, 2019]

Q. 2. The total cost function for a production is given by $C(x) = \frac{3}{4}x^2 - 7x + 27$.

Find the number of units produced for which M.C. = A.C. (M.C.= Marginal Cost and A.C. = Average Cost.)

[ISC, Specimen Paper, 2021] [ISC, 2019]

Sol.

$$C(x) = \frac{3}{4}x^2 - 7x + 27.$$

$$\text{A.C.} = \frac{C(x)}{x} = \frac{3}{4}x - 7 + \frac{27}{x}$$

$$\text{M.C.} = \frac{d}{dx}(C(x)) = \frac{3}{4}.2x - 7$$

$$= \frac{3}{2}x - 7$$

$$\because \quad AC = MC$$

$$\frac{3}{4} - 7 + \frac{27}{x} = \frac{3}{2}x - 7$$

$$3x^2 + 108 = 6x^2$$

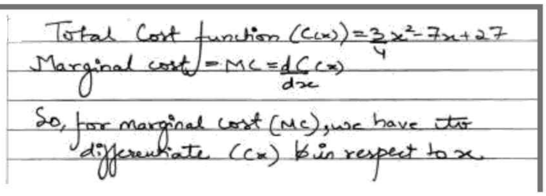
$$3x^2 = 108$$

$$x^2 = 36$$

$$x = 6$$

[ISC Marking Scheme, 2019]

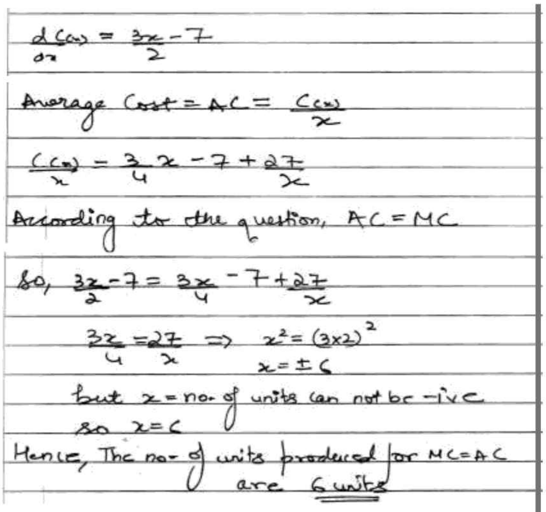
OR



Total Cost function $(C(x)) = \frac{3}{4}x^2 - 7x + 27$

Marginal cost = MC = $\frac{dC(x)}{dx}$

So, for marginal cost (MC), we have to differentiate (C(x)) ~~b~~ in respect to x.



$\frac{dC(x)}{dx} = \frac{3x}{2} - 7$

Average Cost = AC = $\frac{C(x)}{x}$

$\frac{C(x)}{x} = \frac{3}{4}x - 7 + \frac{27}{x}$

According to the question, AC = MC

So, $\frac{3x}{2} - 7 = \frac{3x}{4} - 7 + \frac{27}{x}$

$\frac{3x}{4} = \frac{27}{x} \Rightarrow x^2 = (3\times2)^2$

$x = \pm 6$

but x = no. of units can not be −ive

so $x = 6$

Hence, The no. of units produced for MC = AC are 6 units

[Topper's Answer, 2019]

Q. 3. The selling price of a commodity is fixed at ₹60 and its cost function is : $C(x) = 35x + 250$

(i) Determine the profit function.

(ii) Find the break even points. [ISC, 2020]

Sol. Let the number of articles be x

Total sell price = ₹$60x$

Total cost price = $35x + 250$

(i) Profit function $P(x)$

$= \text{S.P.} - \text{C.P.}$

$= 60x - 35x - 250$

$= 25x - 250$

Hence, profit function is $25x - 250$.

(ii) Break-even point i.e. $P(x) = 0$

$25x - 250 = 0 \Rightarrow x = 10$ units

Examiner's Comment

- Some candidate put $C(x) = 0$, to find break-even point, which is incorrect.

Answering Tip

- Always remember for break-even point put profit function, $P(x) = 0$.

AI Q. 4 The revenue function is given by $R(x) = 100x - x^2 - x^3$. Find

(i) The demand function.

(ii) Marginal revenue function. [ISC, 2020]

Sol.

(i) $R(x) = 100x - x^2 - x^3$

Demand function

$$= \frac{R(x)}{x} = 100 - x - x^2$$

(ii) Marginal revenue function (M.R.)

$$= \frac{dR(x)}{dx}$$

$$= \frac{d}{dx}(100x) - \frac{d}{dx}x^2 - \frac{d}{dx}x^3$$

Marginal revenue function

$= 100 - 2x - 3x^2$

Examiner's Comment

- This question was well attempted by most of the candidates. A few candidates made calculation errors in the process of simplification.

Answering Tip

- Understand the concept of demand and revenue function with the help of the examples.

Q. 5. Given that total cost function for x units of a commodity as :

$$C(x) = \frac{1}{3}x^3 + 3x^2 - 16x + 2$$

Find : (i) Marginal cost function

(ii) Average cost function **[ISC, 2018]**

Sol. Given, $C(x) = \frac{1}{3}x^3 + 3x^2 - 16x + 2$

(i) Marginal cost,
$$MC = \frac{d}{dx}[C(x)]$$
$$= \frac{d}{dx}\left[\frac{1}{3}x^3+3x^2-16x+2\right]$$
$$= x^2 + 6x - 16$$

(ii) Average cost function,
$$AC = \frac{C(x)}{x}$$
$$= \frac{\frac{1}{3}x^3+3x^2-16x+2}{x}$$
$$= \frac{1}{3}x^2+3x-16+\frac{2}{x}$$

[ISC Marking scheme, 2018]

Q. 6. The average cost function associated with producing and marketing x units of an item is given by $AC = 2x - 11 + \frac{50}{x}$. Find the range of values of the output x, for which AC is increasing. **[ISC, 2018]**

Sol. Given, average cost function, $AC = 2x - 11 + \frac{50}{x}$

Now, $\frac{d}{dx}(AC) = \frac{d}{dx}\left(2x-11+\frac{50}{x}\right)$

$$= 2-\frac{50}{x^2}$$

AC increases, $\frac{d}{dx}(AC) > 0$

$$\Rightarrow \quad 2-\frac{50}{x^2} > 0$$

$$\Rightarrow \quad x^2 - 25 > 0$$

$$\Rightarrow \quad (x-5)(x+5) > 0$$

$$\Rightarrow \quad x < -5 \text{ or } x > 5$$

$\because$ x is positive, AC increases, when $x > 5$.

[ISC Marking scheme, 2018]

Examiner's Comment

- Most of the candidates attempted the first part correctly but while calculating the value of x when AC was increasing, some candidates included the negative value of x.

Answering Tip

- Analyse in detail about conditions of increasing and decreasing functions and simplification of problems based on those concepts.

Q. 7. Find the cost of increasing from 100 to 200 units if the marginal cost in rupees per unit is given by the function, $MC = 0.003x^2 - 0.01x + 2.5$

[ISC Specimen Paper, 2018]

Sol. Given, $MC = 0.003x^2 - 0.01x + 2.5$

$\therefore$ Increase in cost (from 100 to 200 units)

$$C = \int_{100}^{200}(0.003x^2-0.01x+2.5)\,dx$$
$$= \left[\frac{0.003x^3}{3}-\frac{0.01x^2}{2}+2.5x\right]_{100}^{200}$$
$$= \left[0.001x^3-0.005x^2+2.5x\right]_{100}^{200}$$
$$= 0.001\,[(200)^3-(100)^3] - 0.005\,[(200)^2-(100)^2] + 2.5\,(200-100)$$
$$= 7000 - 150 + 250$$
$$= 7100$$

Q. 8. The total variable cost of manufacturing x units in a firm is ₹$\left(3x+\frac{x^5}{25}\right)$. Show that average variable cost increases with output x.

[ISC Specimen Paper, 2018]

Sol. Total variable cost $= 3x + \frac{x^5}{25}$

$$\text{Average variable cost} = \frac{\text{Total variable cost}}{\text{Output}}$$

$$\Rightarrow \text{Average variable cost} = \frac{3x+\frac{x^5}{25}}{x}$$
$$= 3+\frac{x^4}{25}$$

$\Rightarrow$ Average variable cost > 0 (as $x^4 > 0$)

Hence, average variable cost increases with output x.

Q. 9. A company produced a commodity with ₹ 20,000 fixed costs. The variable cost is estimated to 35% of the total revenue. When it is sold at a rate of ₹ 6 per unit. Find the total revenue, total cost and profit function of the revenue when it is sold.

Sol. Let x be the number of units produced and sold.

$\Rightarrow$ Revenue function $= R(x) = 6x$

Given that variable cost is estimated to 35% of $R(x)$

$$\Rightarrow \quad V(x) = 35\% \text{ of } R(x)$$

$$= \frac{35}{100} \times 6x$$

$$\Rightarrow \quad V(x) = \frac{21x}{10}$$

Hence, cost function $= C(x) =$ Fixed cost $+ V(x)$

$$= 20000 + \frac{21x}{10}$$

[Given, fixed cost = 20000]

Now, profit function $P(x) = R(x) - C(x)$

$$= 6x - 20000 - \frac{21x}{10}$$

$$\Rightarrow \quad P(x) = \frac{39x}{10} - 20000$$

Q. 10. The total cost function is given by $C = x + 2x^3 - 3.5x^2$, find the marginal average cost function (*MAC*). Also, find the points where the *MAC* curve cuts the x-axis and y-axis.

Sol. Given, total cost function,

$$C(x) = x + 2x^3 - 3.5x^2$$

Here, $$AC = \frac{C}{x}$$

$$= \frac{1}{x}(x + 2x^3 - 3.5x^2)$$

$$= 1 + 2x^2 - 3.5x$$

$$= 2x^2 - 3.5x + 1$$

We know that, marginal average cost (*MAC*) is

$$MAC = \frac{d}{dx}(AC)$$

$$= \frac{d}{dx}(2x^2 - 3.5x + 1)$$

$$\Rightarrow \quad MAC = 4x - 3.5$$

Since, *MAC* is linear, it is a straight line.

$\Rightarrow$ It will cut x-axis at (0.875, 0)

and y-axis at (0, – 3.5)

4 Marks Questions

Q. 1. The marginal cost of the production of the commodity is 30 + 2x, it is known that fixed costs are ₹200, find

(i) The total cost.

(ii) The cost of increasing output from 100 to 200 units. [ISC, 2020]

Sol.

(i) Marginal Cost (M.C.)

$$= 30 + 2x$$

$$\text{Total Cost} = \int \text{M.C.}\,dx$$

$$= \int (30 + 2x)dx$$

$$= 30x + 2\frac{x^2}{2} + C$$

Total Cost $= 30x + x^2 + 200$

(C is the fixed cost)

Total Cost $= x^2 + 30x + 200$

(ii) Cost of increasing output from 100 to 200 is

$$\int_{100}^{200} \text{M.C.}\,dx$$

$$= \int_{100}^{200} (30+2x)dx$$

$$= \left[30x + x^2\right]_{100}^{200}$$

$$= 30(200 - 100) + 200^2 - 100^2$$

$$= 3000 + (200 - 100)(200 + 100)$$

$$= 3000 + 100 \times 300$$

$$= ₹33000$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Some candidates did not to how to get total cost function from the given marginal cost function.

Answering Tip

- Remember when marginal cost function is given, then to obtain the total cost function just integrate marginal cost function.

Q. 2. The total cost function of a firm is given by $C(x) = \frac{1}{3}x^3 - 5x^2 + 30x - 15$ where the selling price per unit is given as ₹6. Find for what value of x will the profit be maximum. [ISC, 2020]

Sol. $$C(x) = \frac{1}{3}x^3 - 5x^2 + 30x - 15$$

$R(x) =$ ₹$6x$

(where x is the number of articles)

$$P(x) = R(x) - C(x)$$

$$= 6x - \frac{1}{3}x^3 + 5x^2 - 30x + 15$$

[where P(x) is profit function]

$$P(x) = -\frac{1}{3}x^3 + 5x^2 - 24x + 15$$

$$\frac{dP(x)}{dx} = -\frac{d}{dx}\frac{1}{3}x^3 + 5\frac{d}{dx}x^2 - 24\frac{d}{dx}x + \frac{d}{dx}15$$

$$\frac{dP(x)}{dx} = -x^2 + 10x - 24 + 0$$

$$\frac{dP(x)}{dx} = -x^2 + 10x - 24$$

For maximum or minimum $\frac{dP(x)}{dx} = 0$

$$-x^2 + 10x - 24 = 0$$
$$x^2 - 10x + 24 = 0$$
$$(x-6)(x-4) = 0$$
$$x = 6 \text{ or } x = 4$$
$$\frac{d^2P(x)}{dx^2} = -2x + 10$$

When $x = 6$ $\frac{d^2P(x)}{dx^2} = -2$

$$\left(\frac{d^2P(x)}{dx^2}\right)_{x=6} < 0$$

$\therefore$ Profit is maximum at $x = 6$

Hence profit is maximum at 6 units.

[ISC Marking Scheme, 2020]

Examiner's Comment

- Few candidates made mistake in framing revenue function and profit function. Some are confused whether they have to differentiate or integrate the function to get maximum profit.

Answering Tip

- Understand the difference between cost function and revenue function and the relationship between profit function, cost function and revenue function.

AI **Q. 3. The cost function of a product is given by**

$$C(x) = \frac{x^3}{3} - 45x^2 - 900x + 36$$

where x is the number of units produced. How many units should be produced to minimise the marginal cost? [ISC, 2019]

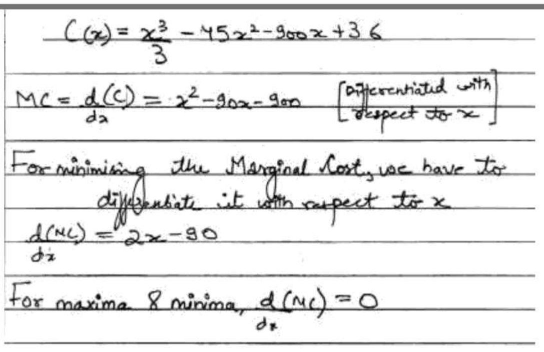
Sol.

$$C(x) = \frac{x^3}{3} - 45x^2 - 900x + 36$$
$$MC(x) = x^2 - 90x - 900$$
$$\frac{d}{dx}(MC) = 2x - 90 = 0$$
$$x = 45$$
$$\frac{d^2}{dx^2}(MC) = 2 > 0$$

at $x = 45$ MC is minimum

[ISC Marking Scheme, 2019]

OR



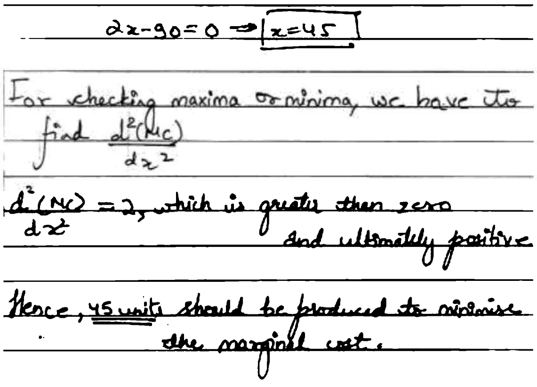
$C(x) = \frac{x^3}{3} - 45x^2 - 900x + 36$

$MC = \frac{d(C)}{dx} = x^2 - 90x - 900$ [Differentiated with respect to x]

For minimising the Marginal Cost, we have to differentiate it with respect to x

$\frac{d(MC)}{dx} = 2x - 90$

For maxima & minima, $\frac{d(MC)}{dx} = 0$



$2x - 90 = 0 \rightarrow \boxed{x = 45}$

For checking maxima or minima, we have to find $\frac{d^2(MC)}{dx^2}$

$\frac{d^2(MC)}{dx^2} = 2$, which is greater then zero and ultimately positive

Hence, 45 units should be produced to minimise the marginal cost.

[Topper's Answer, 2019]

Q. 4. The marginal cost function of x units of a product is given by $MC = 3x^2 - 10x + 3$. The cost of producing one unit is ₹7. Find the total cost function and average cost function. [ISC, 2019]

Sol. $MC = 3x^2 - 10x + 3$

$C(x) = \int MC dx + c$

$C(x) = x^3 - 5x^2 + 3x + c$

Cost of producing 1 unit is ₹7

$\therefore$ $7 = 1 - 5 + 3 + c$

$\Rightarrow$ $c = 8$

$\therefore$ Total cost function $= C(x) = x^3 - 5x^2 + 3x + 8$

Average cost function $= x^2 - 5x + 3 + \frac{8}{x}$

[ISC Marking Scheme, 2019]

AI **Q. 5. A product can be manufactured at a total cost $C(x) = \frac{x^2}{100} + 100x + 40$, where x is the number of units produced. The price at which each unit can be sold is given by $p = \left(200 - \frac{x}{400}\right)$. Determine the production level x at which the profit is maximum. What is the price per unit and total profit at the level of production? [ISC, 2018]**

Sol. Given, $C(x) = \frac{x^2}{100} + 100x + 40$

and $p(x) = 200 - \frac{x}{400}$

Revenue function, $R(x) = p.x$

$$= \left(200 - \frac{x}{400}\right).x$$
$$= 200x - \frac{x^2}{400}$$
$$P(x) = R(x) - C(x)$$
$$= \left(200x - \frac{x^2}{400}\right) - \left(\frac{x^2}{100} + 100x + 40\right)$$
$$= 100x - \frac{x^2}{80} - 40$$
$$\therefore \quad \frac{d}{dx}[P(x)] = 100 - \frac{2x}{80} = 100 - \frac{x}{40}$$

For max. or min,

$$\frac{d}{dx}[P(x)] = 0$$

$$\Rightarrow \qquad x = 4000$$

$$\text{Now,} \quad \frac{d^2}{dx^2}[P(x)] = \frac{-1}{40} < 0$$

$\therefore$ $P(x)$ is maximum at $x = 4000$

$$\text{At } x = 4000, p = \left(200 - \frac{x}{400}\right) = \left(200 - \frac{4000}{400}\right) = ₹\,190$$

$$\text{Maximum profit} = 100x - \frac{x^2}{80} - 40$$

$$= 100(4000) - \frac{(4000)^2}{80} - 40$$

$$= ₹\,1,99,960$$

[ISC Marking scheme, 2018]

Examiner's Comment

- Many candidates considered Price function as Revenue function which led to entire solution being incorrect.

Answering Tip

- Understand thoroughly the basic difference among Price function, cost function and Revenue function, etc.

Q. 6. A manufacture's marginal cost function is $\frac{500}{\sqrt{2x+25}}$. Find the cost involved to increase production from 100 units to 300 units.[ISC, 2018]

Sol. Given, $MC = \frac{500}{\sqrt{2x+25}}$

Total increased cost when x increases from 100 to 300 units is

$$C(300) - C(100) = \int_{100}^{300} MC(x)\,dx$$

$$= \int_{100}^{300} \frac{500}{\sqrt{2x+25}}\,dx$$

$$500\left[\sqrt{2x+25}\right]_{100}^{300} = 500\left[\sqrt{625} - \sqrt{225}\right]$$

$$= 500(25 - 15) = 5000$$

Required cost increase = ₹ 5000

[ISC Marking scheme, 2018]

Q. 7. If the demand function is given by $x = \frac{600-p}{8}$, where the price is ₹ p per unit and the manufacturer produces x unit per week at the total cost of ₹ $x^2 + 78x + 2500$, find the value of x for which the profit is maximum. [ISC Specimen Paper, 2018]

Sol. Demand function $x = \frac{600-p}{8}$

$$\text{Total cost} = x^2 + 78x + 2500$$

From demand function

$$8x = 600 - p$$

$$\Rightarrow \qquad p = 600 - 8x$$

$$\therefore \quad \text{Revenue function} = x(600 - 8x)$$

$$= 600x - 8x^2$$

$$\text{Profit function} \quad P(x) = 600x - 8x^2 - x^2 - 78x - 2500$$

$$P(x) = -9x^2 + 522x - 2500$$

$$\Rightarrow \qquad \frac{dP(x)}{dx} = -18x + 522$$

$$\text{For maximum profit } \frac{dP(x)}{dx} = 0$$

$$\Rightarrow \qquad -18x + 522 = 0$$

$$\Rightarrow \qquad 18x = 522$$

$$\Rightarrow \qquad x = \frac{522}{18} = 29$$

$$\text{Also} \qquad \frac{d^2P(x)}{dx^2} = -18 < 0$$

$\Rightarrow$ Maximum

$\therefore$ For output $x = 29$ profit is maximum

Q. 8. The fixed cost of new product is ₹ 35000 and the variable cost per unit is ₹ 500. If the demand function: $p = 5000 - 100\,x$, find the breakeven value (s) ?

[ISC Specimen Paper, 2018]

Sol. Given, Fixed cost = ₹ 35,000

Variable cost per unit = ₹ 500

and Demand function $p = 5000 - 100x$

$\therefore$ Total cost for x units is given by

$$35{,}000 + 500x$$

$$\text{Total revenue} = x(5000 - 100x)$$

$$= 5000x - 100x^2$$

For break even point,

$$35{,}000 + 500x = 5000x - 100x^2$$

$$\Rightarrow \qquad 100x^2 - 4500x + 35{,}000 = 0$$

$$x^2 - 45x + 350 = 0$$

$$\Rightarrow \qquad (x - 35)(x - 10) = 0$$

$$\Rightarrow \qquad x = 35, 10$$

AI Q. 9. The demand for a certain product is represented by the equation $p = 500 + 25x - \frac{x^2}{3}$ in ₹ where x is the number of units and p is the price per unit. Find : (i) Marginal revenue function

(ii) The marginal revenue when 10 units are sold.

[ISC, 2017]

Sol. Given, Demand function,

$$p = 500 + 25x - \frac{x^2}{3}$$

Revenue function, $R(x) = p \cdot x$

$$= \left(500 + 25x - \frac{x^2}{3}\right)x$$

$$= 500x + 25x^2 - \frac{x^3}{3}$$

(i) Marginal revenue function,

$$\frac{d}{dx}[R(x)] = \frac{d}{dx}\left(500x + 25x^2 - \frac{x^3}{3}\right)$$

$$= 500 + 50x - x^2$$

(iii) Marginal revenue, when $x = 10$

$$\left.\frac{d}{dx}[R(x)]\right|_{x=10} = 500 + 50(10) - (10)^2$$

$$= 500 + 500 - 100$$

$$= 900$$

[ISC Marking scheme, 2017]

Examiner's Comment

- Many candidates made errors in applying the concept of Revenue function. Some candidates made errors in differentiating the revenue function. In addition, calculation errors were made in finding the Marginal Revenue function at $x = 10$

Answering Tip

- Clarify each term like cost function, Break even point, profit function, Demand function, Revenue function and Marginal cost function, etc.
  Good practice of different type of questions should be done.

Q. 10. The average cost of producing x units of commodity is given by the equation $AC = \frac{x^2}{200} - \frac{x}{50} - 30 + \frac{5000}{x}$. Find the marginal cost function (MC) and verify that $\frac{d}{dx}(AC) = \frac{MC - AC}{x}$.

Sol. Given average cost, $AC = \frac{x^2}{200} - \frac{x}{50} - 30 + \frac{5000}{x}$

We know that cost, $C = x \times AC$

$$= x\left(\frac{x^2}{200} - \frac{x}{50} - 30 + \frac{5000}{x}\right)$$

$$= \frac{x^3}{200} - \frac{x^2}{50} - 30x + 5000$$

Now, marginal cost, $MC = \frac{dC}{dx}$

$$MC = \frac{d}{dx}\left(\frac{x^3}{200} - \frac{x^2}{50} - 30x + 5000\right)$$

$$= \frac{3x^2}{200} - \frac{2x}{50} - 30$$

$$\therefore \qquad MC = \frac{3x^2}{200} - \frac{x}{25} - 30$$

Since, $$AC = \frac{x^2}{200} - \frac{x}{50} - 30 + \frac{5000}{x}$$

$$\frac{d}{dx}(AC) = \frac{2x}{200} - \frac{1}{50} - \frac{5000}{x^2}$$

$$= \frac{x}{100} - \frac{1}{50} - \frac{5000}{x^2} \qquad \ldots\text{(i)}$$

and $$MC - AC = \left(\frac{3x^2}{200} - \frac{x}{25} - 30\right) - \left(\frac{x^2}{200} - \frac{x}{50} - 30 + \frac{5000}{x}\right)$$

$$= \frac{2x^2}{200} - \frac{x}{25} + \frac{x}{50} - 30 + 30 - \frac{5000}{x}$$

$$= \frac{x^2}{100} - \frac{x}{50} - \frac{5000}{x}$$

$$\therefore \qquad \frac{1}{x}(MC - AC) = \frac{1}{x}\left(\frac{x^2}{100} - \frac{x}{50} - \frac{5000}{x}\right)$$

$$= \frac{x}{100} - \frac{1}{50} - \frac{5000}{x^2}$$

$$\therefore \qquad \frac{MC - AC}{x} = \frac{d}{dx}(AC) \qquad \text{[From (i)]}$$

Q. 11. The total revenue function for a commodity is $R = 15x + \frac{1}{3}x^2 - \frac{1}{36}x^4$. Show that at the highest point of the average revenue function, is equal to the marginal revenue.

Sol. Given total revenue function,

$$R = 15x + \frac{1}{3}x^2 - \frac{1}{36}x^4$$

We know that, average revenue function

$$AR = \frac{R}{x} = \frac{1}{x}\left(15x + \frac{1}{3}x^2 - \frac{1}{36}x^4\right)$$

$$= 15 + \frac{1}{3}x - \frac{1}{36}x^3$$

To find the highest point of AR curve,

We take, $$\frac{d}{dx}(AR) = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(15 + \frac{1}{3}x - \frac{1}{36}x^3\right) = 0$$

$$\Rightarrow \quad \frac{1}{3} - \frac{1}{36}(3x^2) = 0$$

$$\Rightarrow \quad \frac{4 - x^2}{12} = 0$$

$$\Rightarrow \quad x^2 = 4$$

$$\Rightarrow \quad x = 2$$

(only positive value is taken)

Now, at $x = 2$,

$$\frac{d^2}{dx^2}(AR) = -\frac{2x}{12}$$

$$= \frac{-2(2)}{12} = -\frac{1}{3} < 0$$

$\therefore$ Highest point of AR occurs, when $x = 2$

$\therefore$ At $x = 2$, $$AR = 15 + \frac{1}{3}(2) - \frac{1}{36}(2)^3$$

$$= 15 + \frac{2}{3} - \frac{8}{36}$$

$$= 15\frac{4}{9}$$

and marginal revenue

$$MR = \left.\frac{dR}{dx}\right|_{x=2}$$

$$= \left(15 + \frac{1}{3}(2x) - \frac{1}{36}(4x^3)\right)$$

$$= 15 + \frac{4}{3} - \frac{8}{9}$$

$$= 15\frac{4}{9}$$

Therefore, AR = MR at a point where AR is maximum.

AI **Q. 12. A company is selling a certain product. The demand function of the product is linear. The company can sell 2000 units, when the price is ₹ 8 per unit and when the price is ₹ 4 per unit, it can sell 3000 units. Determine :**

(i) the demand function

(ii) the total revenue function

Sol. (i) Let demand function be

$$f(x) = ax + b \qquad \text{(given as linear)}$$

for ₹ x per unit price

Given that company sells 2000 units for ₹ 8 per unit

i.e., $2000 = 8a + b$ …(i)

and the company sells 3000 units when price is ₹ 4 per unit

i.e., $3000 = 4a + b$ …(ii)

Subtracting (i) from (ii), we get

$$-4a = 1000$$

$$\Rightarrow \quad a = -250$$

For $a = -250$,

$$3000 = 4(-250) + b \qquad \text{(From (ii))}$$

$$\Rightarrow \quad 3000 + 1000 = b$$

$$\Rightarrow \quad b = 4000$$

$\therefore$ The demand function $f(x) = -250x + 4000$

(ii) Total revenue function = Demand function × x

$$\Rightarrow \quad R(x) = (-250x + 4000) \times x$$

$$\Rightarrow \quad R(x) = -250x^2 + 4000x$$

6 Marks Questions

Q. 1. A television manufacturer finds that the total cost for the production and marketing of x number of television sets is $C(x) = 300x^2 + 4200x + 13500$. Each product is sold for ₹ 8400. Determine the breakeven points. **[ISC, 2016]**

Sol. Total cost function for x number of television sets is given as

$$C(x) = 300x^2 + 4200x + 13500$$

Revenue function of x units product,

$$R(x) = 8400x$$

($\because$ Each product is sold at ₹8400)

For breakeven points = $4000x - 150x^2$

$$C(x) = R(x)$$

$$\therefore \quad 300x^2 + 4200x + 13500 = 8400x$$

$$\Rightarrow \quad 300x^2 - 4200x + 13500 = 0$$

$$\Rightarrow \quad 3x^2 - 42x + 135 = 0$$

$$\Rightarrow \quad 3x^2 - 27x - 15x + 135 = 0$$

$$\Rightarrow \quad 3x(x-9) - 15(x-9) = 0$$

$$\Rightarrow \quad (3x-15)(x-9) = 0$$

$$\Rightarrow \quad x = \frac{15}{3} \text{ or } x = 9$$

i.e., $x = 5$ or $x = 9$

$\therefore$ The breakeven points are 9 or 5.

AI **Q. 2. The demand function is $x = \frac{24-2p}{3}$ where x is the number of units demanded and p is the price per unit. Find :**

(i) The revenue function R in terms of p

(ii) The price and the number of units demanded for which the revenue is maximum. **[ISC, 2016]**

Sol. Total revenue function = px

$$\text{Revenue function } = \frac{p(24-2p)}{3}$$

$$= \frac{24p - 2p^2}{3}$$

$$\frac{dR}{dp} = \frac{24-4p}{3}$$

For maximum or minimum, $\frac{dR}{dp} = 0$

$$\frac{24-4p}{3} = 0$$

$$\Rightarrow \quad p = 6$$

Now, $\frac{d^2R}{dp^2} = -\frac{4}{3} < 0$

$\therefore$ R is maximum

Number of units, at $p = 6$ $x = \frac{(24-12)}{3}$

$$\Rightarrow \quad x = 4$$

Hence, R is maximum when 4 units are demanded at the price of ₹ 6 per unit.

[ISC Marking Scheme, 2016]

Examiner's Comment

- Some of the candidates made mistakes in obtaining Revenue function in terms of p. Several candidates forgot to apply the condition for the maxima $\frac{d^2R}{dp^2} < 0$.

Answering Tip

- Question should be answered as asked in the paper and Revenue function must be found in terms of p. Application of Maxima and Minima should be practised by taking up more examples.
- All relevant terms and formulae should be learnt for complete understanding.

Q. 3. The average cost function, AC for a commodity is given by $AC = x + 5 + \frac{36}{x}$, in terms of output x. Find :

(a) The total cost, C and marginal cost, MC as a function of x.

(b) The outputs for which AC increases.

[ISC 2010, 2015].

Sol. (a) Cost function, $C = AC \times x$

$$= \left(x + 5 + \frac{36}{x}\right)x$$

$$= x^2 + 5x + 36$$

Marginal Cost, $MC = \frac{dC}{dx} = \frac{d}{dx}(x^2 + 5x + 36)$

$$= 2x + 5$$

Also, $\frac{d(AC)}{dx} = \frac{d}{dx}\left(x + 5 + \frac{36}{x}\right)$

$$= 1 - \frac{36}{x^2}$$

(b) For AC to be increasing, $\frac{d(AC)}{dx} > 0$

$$\Rightarrow \quad 1 - \frac{36}{x^2} > 0$$

$$\Rightarrow \quad x^2 - 36 > 0$$

$$\Rightarrow \quad (x - 6)(x + 6) > 0$$

$$x < -6 \text{ or } x > 6$$

Hence, average cost increases if the output x is > 6.

[ISC Marking Scheme, 2015]

Examiner's Comment

- Some candidates wrote incorrect formula of cost function, so their marginal cost was incorrect. Some wrote incorrect differentiation of the expression. Many candidates were not able to answer the second part of the question. They were confused with the maximum, minimum condition. They found the derivative and put it equal to zero.

Answering Tip

- The concepts of Marginal Cost, Total Cost and Average Cost should be learned in depth for increasing and decreasing functions by giving sufficient examples.

AI Q. 4. A firm has the cost function, $C = \frac{x^3}{3} - 7x^2 + 111x + 50$ and demand function, $x = 100 - p$.

(i) Write the total revenue function in terms of x.

(ii) Formulate the total profit function P in terms of x.

(iii) Find the profit maximising level of output x.

[ISC, 2014]

Sol. Total revenue function = $x(100 - x)$

Profit function = Revenue function – Cost function

$$= 100x - x^2 - \left(\frac{x^3}{3} - 7x^2 + 111x + 50\right)$$

$$= -\frac{x^3}{3} + 6x^2 - 11x - 50$$

$$\frac{dP}{dx} = -x^2 + 12x - 11$$

For maximum or minimum, $\frac{dP}{dx} = 0$

$$-x^2 + 12x - 11 = 0$$

$$\Rightarrow \quad x = 1, 11$$

$$\therefore \quad \frac{d^2P}{dx^2} = -2x + 12$$

$$= -2 \times 11 + 12$$

$$= -10 < 0$$

Profit is maximum when $x = 11$

[ISC Marking Scheme, 2014]

Examiner's Comment

- Revenue and profit functions were incorrectly stated by some candidates. Profit function was not correctly differentiated by a few. The concept of the maxima and minima for profit maximisation was not known to few candidates.

Answering Tip

- Learn terms like Cost and Demand functions. Revenue and profit functions too require revision.

Detailed Solution :

Given, Cost function

$$C = \frac{x^3}{3} - 7x^2 + 11x + 50$$

and demand function

$$x = 100 - p$$

i.e., $p = 100 - x$

(i) Revenue function,

$$R = px$$

$$R = (100 - x)x$$

$$= 100x - x^2$$

(ii) Profit function is given as

$$P(x) = \text{Revenue function} - \text{Cost function}$$

$$= (100x - x^2) - \left(\frac{x^3}{3} - 7x^2 + 111x + 50\right)$$

$$= 100x - x^2 - \frac{x^3}{3} + 7x^2 - 111x - 50$$

$$\therefore \quad P(x) = -\frac{x^3}{3} + 6x^2 - 11x - 50$$

(iii) Since, $P(x) = -\frac{x^3}{3} + 6x^2 - 11x - 50$

$$\therefore \quad \frac{dP}{dx} = -\frac{3x^2}{3} + 12x - 11$$

Now, $$\frac{dP}{dx}=0$$

$\Rightarrow \quad -x^2+12x-11=0$

$\Rightarrow \quad x^2-12x+11=0$

$\Rightarrow \quad (x-11)(x-1)=0$

$\Rightarrow \quad x=1 \text{ or } x=11$

Now, $$\frac{d^2P}{dx^2}=\frac{d}{dx}(-x^2+12x-11)$$
$$=-2x+12$$

At $x = 1$, $$\frac{d^2P}{dx^2}=-2+12=10>0$$

At $x = 11$, $$\frac{d^2P}{dx^2}=-22+12=-10<0$$

$\therefore$ At $x = 1$, profit will be maximum.

Q. 5. A company produces a commodity with ₹ 24,000 fixed cost. The variable cost is estimated to be 25% of the total revenue recovered on selling the product at a rate of ₹ 8 per unit. Find the following :

(i) Cost function

(ii) Revenue function

(iii) Breakeven point. [ISC, 2013]

Sol. Suppose that x number of units are produced and sold.

(i) As each unit's variable cost is 25% of revenue.

$\therefore$ The variable cost of x units

$$= 25\% \text{ of ₹ } 8x$$
$$=\frac{25}{100}\times 8x$$
$$=\frac{1}{4}\times 8x= \text{ ₹ } 2x$$

$\therefore$ Total cost of producing x units

$C(x)$ = Fixed cost of the commodity + variable cost of the commodity

$= \text{₹ } (24000 + 2x)$

(ii) Price of one unit = ₹ 8

$\therefore$ Total revenue on selling x units = $R(x)$ = ₹ $8x$

(iii) At breakeven values

$$C(x) = R(x)$$
$$24000 + 2x = 8x$$

$\Rightarrow \quad 24000 = 6x$

$\Rightarrow \quad x = 4000$

Hence, the breakeven point is 4000.

Q. 6. If total cost function is given by $C = a + bx + cx^2$, where x is the quantity of output. Show that :

$$\frac{d}{dx}(AC)=\frac{1}{x}(MC-AC),$$

where MC is the marginal cost and AC is the average cost. [ISC, 2012]

Sol. Here, $C = a + bx + cx^2$

Quantity $= x$

$\therefore$ We know that, $$AC=\frac{C}{x}=\frac{a+bx+cx^2}{x}$$
$$=\frac{a}{x}+b+cx$$

and $$MC=\frac{d}{dx}(C)=\frac{d}{dx}(a+bx+cx^2)$$
$$= b + 2cx$$

$\therefore$ $$\frac{1}{x}(MC-AC)=\frac{1}{x}\left(b+2cx-\frac{a}{x}-b-cx\right)$$
$$=\frac{1}{x}\left(cx-\frac{a}{x}\right)$$

$\therefore$ $$\frac{1}{x}(MC-AC)=c-\frac{a}{x^2} \quad \text{...(i)}$$

$$\frac{d}{dx}(AC)=\frac{d}{dx}\left(\frac{a}{x}+b+cx\right)$$
$$\frac{d}{dx}(AC)=-\frac{a}{x^2}+c \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{d}{dx}(AC)=\frac{1}{x}(MC-AC)$$ **Hence Proved.**

AI **Q. 7. Given that the total cost function for x units of a commodity is :**

$$C(x)=\frac{x^3}{3}+3x^2-7x+16$$

(i) Find the Marginal Cost (MC)

(ii) Find the Average Cost (AC)

(iii) Prove that :

Marginal Average Cost (MAC) $=\dfrac{x(MC)-C(x)}{x^2}$

[ISC, 2011]

Sol. Given, $$C(x)=\frac{x^3}{3}+3x^2-7x+16$$

(i) Marginal cost $$(MC)=\frac{d}{dx}\left[\frac{x^3}{3}+3x^2-7x+16\right]$$
$$=\frac{3x^2}{3}+6x-7$$
$$=x^2+6x-7$$

(ii) Average cost $$(AC)=\frac{\text{Total cost}}{\text{Output}}$$
$$=\frac{\frac{x^3}{3}+3x^2-7x+16}{x}$$
$$=\frac{x^2}{3}+3x-7+\frac{16}{x}$$

(iii) Marginal Average Cost (MAC)

$$=\frac{d}{dx}(AC)=\frac{d}{dx}\left[\frac{x^2}{3}+3x-7+\frac{16}{x}\right]$$
$$=\frac{2x}{3}+3-\frac{16}{x^2}$$

Again, $$\frac{x(MC)-C(x)}{x^2}$$

$$=\frac{x(x^2+6x-7)-\frac{x^3}{3}-3x^2+7x-16}{x^2}$$

$$=\frac{1}{x^2}\left[x^3+6x^2-7x-\frac{x^3}{3}-3x^2+7x-16\right]$$

$$=\frac{1}{x^2}\left[x^3-\frac{x^3}{3}+3x^2-16\right]$$

$$=\frac{2x^3}{3x^2}+3-\frac{16}{x^2}=\frac{2}{3}x+3-\frac{16}{x^2}$$

$\Rightarrow$ Marginal average cost (MAC)

$$=\frac{x(MC)-C(x)}{x^2}$$ **Hence Proved.**

Q. 8. The cost of manufacturing of certain items consists of ₹ 1600 as overheads ₹ 30 per item as the cost of the material and the labour cost ₹ $\frac{x^2}{100}$ for x items produced. How many items must be produced to have a minimum average cost ? [ISC, 2009]

Sol. Total cost of manufacturing x items

= cost of materials + labour cost + overheads

$\therefore$ Cost function $(C)=\frac{x^2}{100}+30x+1600$

We know that,

$$\text{Average cost } (AC)=\frac{C}{x}=\frac{x}{100}+30+\frac{1600}{x}$$

$$\therefore \quad \frac{d}{dx}(AC)=\frac{1}{100}+0-\frac{1600}{x^2}$$

For maximum and minimum cost, $\frac{d}{dx}(AC)=0$

$$\therefore \quad \frac{1}{100}-\frac{1600}{x^2}=0$$

$$\Rightarrow \quad \frac{1600}{x^2}=\frac{1}{100}$$

$$\Rightarrow \quad x^2 = 160000$$

$$\Rightarrow \quad x = \pm 400$$

Now, $\frac{d^2(AC)}{dx^2}=\frac{3200}{x^3}=$ + ve, at $x = 400$

$\therefore$ AC is minimum at $x = 400$

Hence, 400 items must be produced to have a minimum average cost.

AI **Q. 9. The average cost function associated with producing and marketing x units of an item is given by,**

$$AC = 2x - 11 + \frac{50}{x}$$

Find :

(i) the total cost function and marginal cost function

(ii) the range of values of the output x, for which AC is increasing [ISC, 2008]

Sol. Given, $AC=2x-11+\frac{50}{x}$ as average cost function

(i) Since, Average cost $=\frac{\text{Total cost}}{x}$

We have

Total cost = $AC \times x$

$$=\left(2x-11+\frac{50}{x}\right)x$$

Total cost function = $2x^2 - 11x + 50$

Marginal cost function

$$(MC)=\frac{d}{dx}(C)$$

$$=\frac{d}{dx}(2x^2-11x+50)$$

$$\therefore \quad MC = 4x - 11$$

(ii) Now, for increasing AC, the condition is

$$\frac{d}{dx}(AC)>0$$

$$\Rightarrow \quad \frac{d}{dx}\left(2x-11+\frac{50}{x}\right)>0$$

$$\Rightarrow \quad 2-\frac{50}{x^2}>0$$

$$\Rightarrow \quad 2x^2 - 50 > 0$$

$$\Rightarrow \quad (x + 5)(x - 5) > 0$$

$$\Rightarrow \quad x > 5$$

(as $x < -5$ is not possible)

$\therefore$ AC is increasing for $x > 5$

Q. 10. A profit making company wants to launch a new product. It observes that the fixed cost of the new product is ₹ 35,000 and the variable cost per unit is ₹ 500. The revenue received on the sale of x units is given by $5000x - 100x^2$. Find :

(i) profit function

(ii) breakeven point

(iii) the values of x that results in a loss.

Sol. Given that fixed cost is ₹ 35000

and variable cost is ₹ 500 per unit

$\therefore$ Total cost = fixed cost + variable cost per unit

$\therefore$ $C(x) = 35000 + 500x$

Also, given that $R(x) = 5000x - 100x^2$

(i) Profit function $P(x) = R(x) - C(x)$

$= (5000x - 100x^2) - (35000 + 500x)$

$\therefore$ $P(x) = 4500x - 100x^2 - 35000$

(ii) For breakeven points,

$P(x) = 0$ (or $R(x) = C(x)$)

$\Rightarrow 4500x - 100x^2 - 3500= 0$

$\Rightarrow \quad 45x - x^2 - 350 = 0$

$\Rightarrow \quad x^2 - 45x + 350 = 0$

$\Rightarrow \quad (x - 35)(x - 10) = 0$

$\Rightarrow x = 35$ or $x = 10$

$\therefore$ Breakeven values are 10 or 35.

(iii) Company will incure a loss when $P(x) < 0$

$\Rightarrow \quad x^2 - 45x + 350 < 0$

$\Rightarrow \quad (x - 35)(x - 10) < 0$

$\Rightarrow \quad 10 < x < 35$

Q. 11. A firm starts producing pocket calculators. During the first year, the cost for setting up the unit is 3 lakh. The additional cost of producing a calculator is ₹ 70. This cost is directly related to the production and is a variable cost. The firm expects the revenue from the sales of the calculators to be ₹ 270 per calculator. Assuming all the produced calculators are sold, find the number x of the calculators for which

(i) the firm will breakeven

(ii) will make profit

(iii) will suffer a loss.

Sol. Given that revenue function $= R(x)$

$$= 270x$$

where x is the number of calculators and ₹ 270 is the price per calculator.

Cost function, $C(x) = 70x + 300000$

$\therefore$ Profit function,

$$P(x) = R(x) - C(x)$$
$$= 270x - 70x - 300000$$
$$= 200x - 300000$$

(i) For breakeven points

$$R(x) = C(x)$$

or $\quad P(x) = 0$

$\therefore \quad 200x - 300000 = 0$

$\Rightarrow \quad 200(x - 1500) = 0$

$\Rightarrow \quad x = 1500$

(ii) Firm earns profit when $P(x) > 0$

$\Rightarrow \quad 200(x - 1500) > 0$

$\Rightarrow \quad x > 1500$

(iii) Firm suffers loss when $P(x) < 0$

$\Rightarrow \quad 200(x - 1500) < 0$

$\Rightarrow \quad x < 1500$

Q. 12. The cost function $C(x)$, for x breads is given by $C(x) =$ ₹$(3.5x + 1200)$. Each bread is put to a special levy of 20 paise for Orissa cyclone victims. Then,

(i) if each bread is sold for ₹ 6 determine the minimum number of breads that should be produced and sold to ensure no loss.

(ii) if the selling price is increased by 70 paise per bread, what would be the break even point ?

(iii) if 6000 breads are sold, what price per bread should be charged to guarantee no loss ?

Sol. Given that ₹ 0.2 is the levy per bread

$\therefore$ Cost function $C(x)$ becomes

$$C(x) = 3.5x + 12000 + 0.2x$$
$$= 3.7x + 12000$$

(i) Here, revenue function $R(x) = 6x$

$\therefore$ For minimum number of breads that should be produced and sold to ensure no loss,

$$R(x) = C(x)$$

$\Rightarrow \quad 6x = 3.7x + 12000$

$\Rightarrow \quad x = 5217.4$

Thus, 5218 breads should be produced and sold for no loss.

(ii) The new selling price = ₹ (6 + 0.70)

= ₹ 6.70 per bread

$\therefore$ New revenue function,

$$R(x) = 6.7x$$

For breakeven point,

$$R(x) = C(x)$$

$\Rightarrow \quad 6.7x = 3.7x + 12000$

$\Rightarrow \quad x = 4000$

(iii) Let ₹p be the price per bread.

When 6000 breads are sold

$$6000p = 3.7 \times 6000 + 12000$$

$\Rightarrow \quad p = 3.7 + 2$

$\Rightarrow \quad p = 5.7$

$\therefore$ To guarantee no loss per bread ₹ 5.70 should be charged.

OO

SELF ASSESSMENT TEST

UNIT–VIII

Maximum Marks : 24 **Timing : 1 Hour**

Q. 1. **The marginal cost function of a product is given by $dC/dx = 100 - 10x + 0.1x^2$ where x is the output. Obtain the total and the average cost function of the firm under the assumption, that its fixed cost is ₹500.**

Q. 2. **Find the revenue function and the demand function if the marginal revenue for x units is $MR = 10 + 3x - x^2$**

Q. 3. **A monopolist demand function is $p = 300 - 5x$. Find**

(i) the marginal revenue function

(ii) the relationship between the slopes of AR and MR.

Q. 4. **If the total cost function for a manufacturer is given by $C = \dfrac{5x^2}{\sqrt{x^2+3}} + 5000,$ find the marginal cost.**

Q. 5. **The demand law of a commodity is $p = a\sqrt{x} + b$. If the price is one unit, the demand is 100 units and if the demand is 16 units, the total revenue is 144 units. Find the constants a and b. If $p = 2$ units, what is the total revenue ?**

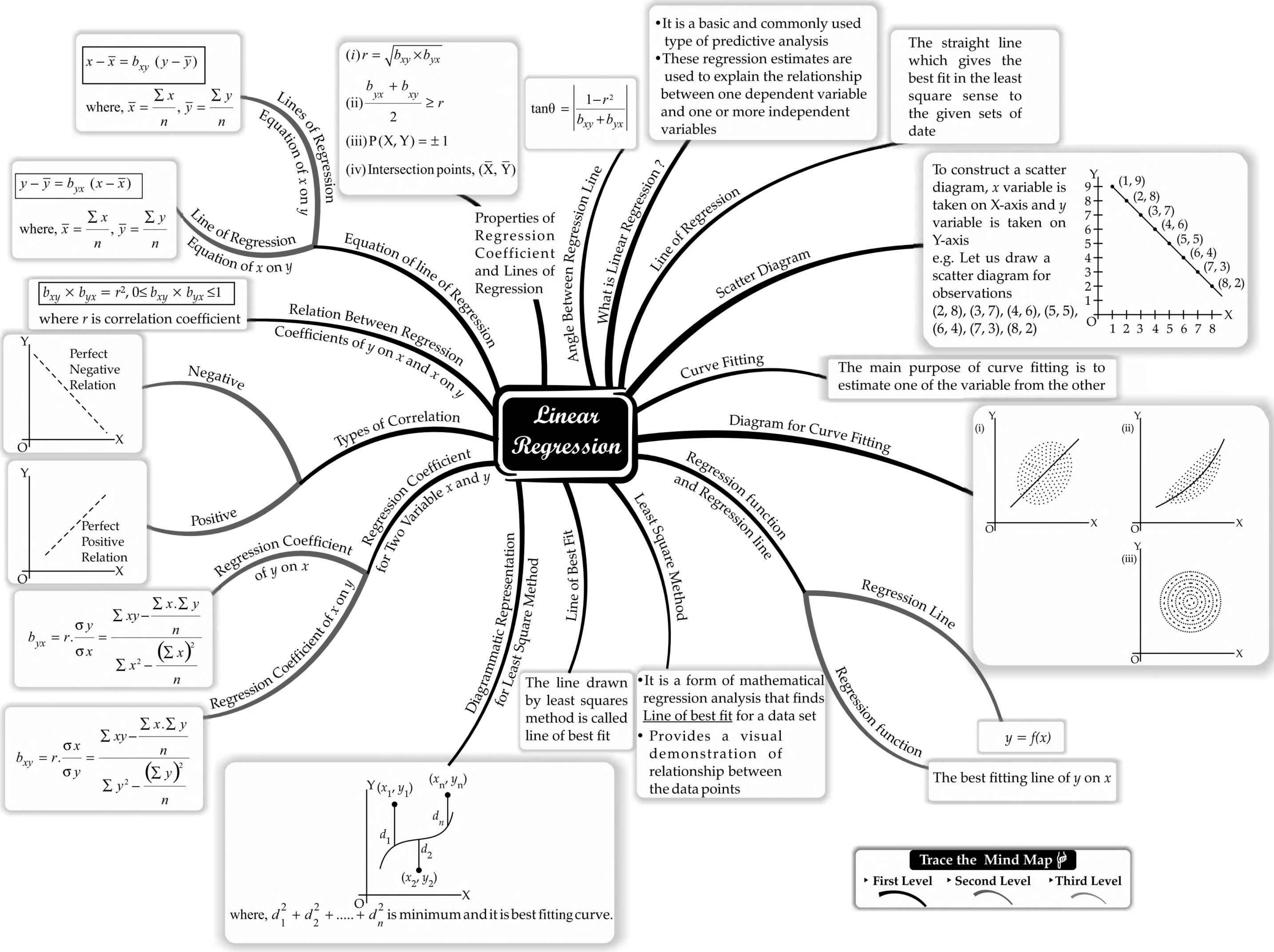
Q. 6. **The demand function for a manufacturer's product is $p = \dfrac{80 - x}{4}$, where x is the number of units and ₹ p is the price per unit. At what value of x will there be maximum revenue ? What is the maximum revenue ?**

OSWAAL EXAM TOOLS

OSWAAL PREPARATION TOOLS

Solutions for Self-assessment Test

Visit : *http://qr.w69b.com/g/pi8gdC0qk* **OR Scan the Code**



Linear Regression

- **Line of Regression**
  - The straight line which gives the best fit in the least square sense to the given sets of date
- **Scatter Diagram**
  - To construct a scatter diagram, x variable is taken on X-axis and y variable is taken on Y-axis
  - e.g. Let us draw a scatter diagram for observations (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2)
- **Curve Fitting**
  - The main purpose of curve fitting is to estimate one of the variable from the other
- **Diagram for Curve Fitting**
  - (i) (ii) (iii)
- **Regression function and Regression line**
  - Regression Line
    - $y = f(x)$
  - Regression function
    - The best fitting line of y on x
- **Least Square Method**
  - It is a form of mathematical regression analysis that finds Line of best fit for a data set
  - Provides a visual demonstration of relationship between the data points
- **Line of Best Fit**
  - The line drawn by least squares method is called line of best fit
- **Diagrammatic Representation for Least Square Method**
  - where, $d_1^2 + d_2^2 + ... + d_n^2$ is minimum and it is best fitting curve.
- **Regression Coefficient for Two Variable x and y**
  - Regression Coefficient of y on x
    - $$b_{yx} = r.\frac{\sigma y}{\sigma x} = \frac{\Sigma xy - \frac{\Sigma x.\Sigma y}{n}}{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}$$
  - Regression Coefficient of x on y
    - $$b_{xy} = r.\frac{\sigma x}{\sigma y} = \frac{\Sigma xy - \frac{\Sigma x.\Sigma y}{n}}{\Sigma y^2 - \frac{(\Sigma y)^2}{n}}$$
- **Types of Correlation**
  - Negative
    - Perfect Negative Relation
  - Positive
    - Perfect Positive Relation
- **Relation Between Regression Coefficients of y on x and x on y**
  - $\boxed{b_{xy} \times b_{yx} = r^2, 0 \le b_{xy} \times b_{yx} \le 1}$
  - where r is correlation coefficient
- **Equation of line of Regression**
  - Lines of Regression Equation of x on y
    - $\boxed{x - \bar{x} = b_{xy}\,(y - \bar{y})}$
    - where, $\bar{x} = \frac{\Sigma x}{n}$, $\bar{y} = \frac{\Sigma y}{n}$
  - Line of Regression Equation of x on y
    - $\boxed{y - \bar{y} = b_{yx}\,(x - \bar{x})}$
    - where, $\bar{x} = \frac{\Sigma x}{n}$, $\bar{y} = \frac{\Sigma y}{n}$
- **Properties of Regression Coefficient and Lines of Regression**
  - (i) $r = \sqrt{b_{xy} \times b_{yx}}$
  - (ii) $\frac{b_{yx} + b_{xy}}{2} \ge r$
  - (iii) $P(X, Y) = \pm 1$
  - (iv) Intersection points, $(\bar{X}, \bar{Y})$
- **Angle Between Regression Line**
  - $\tan\theta = \left|\frac{1 - r^2}{b_{xy} + b_{yx}}\right|$
- **What is Linear Regression ?**
  - It is a basic and commonly used type of predictive analysis
  - These regression estimates are used to explain the relationship between one dependent variable and one or more independent variables

UNIT-IX

Linear Regression

CHAPTER

14

LINEAR REGRESSION

SYLLABUS

- ***Lines of regression of x on y and y on x.***
- ***Scatter diagrams***
- ***The method of least squares.***
- ***Lines of best fit.***
- ***Regression coefficient of x on y and y on x.***
- $b_{xy} \times b_{yx} = r^2, 0 \le b_{xy} \times b_{yx} \le 1$
- ***Identification of regression equations.***
- ***Angle between regression line and properties of regression lines.***
- ***Estimation of the value of one variable using the value of other variable from appropriate line of regression.***

  Self-explanatory

Trend Analysis

List of Concept	2016	2017	2018	2019	2020
Linear Regression	1 Q (6 marks)	1 Q (6 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (2 marks) 1 Q (4 marks)	1 Q (2 marks) 1 Q (4 marks)

Revision Notes

Scan to know more about this topic

Linear Regression and Correlation

- **Regression :** Regression is an estimate or prediction of unknown values of one variable from known values of another variable. Regression measures the extent of correlation and the nature of correlation.
- **Linear Regression :** If there are only two variables under consideration, then the regression is called as simple regression. This simple regression is called linear regression.
- **Regression Lines :** If two variables given in a bivariate frequency distribution are correlated, then the dots in the scatter diagram of the distribution cluster around a straight line, called the line of regression.
- **Two methods of drawing regression lines :**

  (i) Curve Fitting Method :

  In this method, if the data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is plotted on a scatter diagram and then visualize a smooth curve which is approximating the data is called curve fitting method.

  (ii) Method of least squares :

  In this method, a regression line is fitted through different points in such a way that the sum of squares of the deviations of the observed value from the fitted line shall be least. The line drawn by this method is called **line of best fit**.
- **Regression Equation :** Regression equations are the algebraic formulation of regression lines. They represent regression lines.

  Regression equations can be fixed in following ways :

  1. Regression Equation of y on x : This equation is used to estimate the probable values of y on the basis of the given values of x.

i.e., $y = a + bx$

where, a and b are constants.

Regression equation of y on x can also be written as

$$y - \bar{y} = r \cdot \frac{\sigma_y}{\sigma_x}(x - \bar{x}) \qquad \left[\text{where } \bar{x} = \frac{\Sigma x}{n},\ \bar{y} = \frac{\Sigma y}{n}\right]$$

or $$y - \bar{y} = b_{yx}(x - \bar{x})$$

where, yx = Regression coefficient of y on x.

2. **Regression Equation of x on y :** This equation is used to estimate the probable values of x on the basis of the given values of y.

i.e., $x = a_0 + b_0 y$

where, a_0 and b_0 are constants.

Regression equation of x on y can also be written as

$$x - \bar{x} = r \cdot \frac{\sigma_x}{\sigma_y}(y - \bar{y}) \qquad \left[\text{where, } \bar{x} = \frac{\Sigma x}{n},\ \bar{y} = \frac{\Sigma y}{n}\right]$$

[$\because$ σ_x = standard deviation for x, σ_y = standard deviation for y]

or $$x - \bar{x} = b_{xy}(y - \bar{y})$$

where, b_{xy} = Regression coefficient of x on y.

- **Method to Identify the Regression :**

Suppose two equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given and we have to identify the regression line of y on x or x on y, we use the following steps :

Step 1 : Firstly, we consider any one of the equations say first, *i.e.,* $a_1x + b_1y + c_1 = 0$ as y on x and other equation say second, *i.e.,* $a_2x + b_2y + c_2 = 0$ as x on y.

Step 2 : Adjust the first equation y on x such that y should be on left side with coefficient 1 and rest of the terms be on right side (*i.e.,* $y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1}$). The coefficient of x will considered as regression as regression coefficient y on x, *i.e.,* $b_{yx} = -\frac{a_1}{b_1}$. Similarly, for another equation x on y, adjust equation like as $x = -\frac{b_2}{a_2}y - \frac{c_2}{a_2}$. The coefficient of y will give regression coefficient of x on y, *i.e.,* $b_{xy} = -\frac{b_2}{a_2}$.

Step 3 : If b_{xy} and b_{yx} are not of same sign, then we do not determine the regression line. If they have the same sign, then determine coefficient by using the formula.

$$r = \sqrt{b_{yx} \times b_{xy}}$$

Step 4 : If the value of r lies 0 to 1, then our consideration is true, otherwise we have to change the considering from equations.

- **Angle between the two lines of Regression :**

If θ is the angle between the two lines of regression, then $\tan\theta = \pm\left(\frac{1 - r^2}{r}\right)\frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}$

(i) If $r = 0$, $\tan\theta = \infty$ or $\theta = \frac{\pi}{2}$, *i.e.,* the lines of regression are perpendicular.

(ii) If $r = \pm 1$, $\theta = 0$, *i.e.,* the lines of regression are coincident or parallel.

- **Properties of Regression Coefficient and Lines of Regression :**

(i) Coefficient of correlation is the geometric mean between the regression coefficient,

i.e., $$r = \sqrt{b_{xy} \times b_{yx}}$$

(ii) Both the regression coefficients must have the same algebraic signs.

(iii) The coefficient of correlation will have the same sign as that of regression coefficients.

(iv) Both the regression coefficients cannot be greater than unity.

(v) If one regression coefficient is greater than unity, then other regression coefficient must be less than unity.

(vi) If $r = 0$, the variables are uncorrelated, the lines of regression becomes perpendicular to each other.

(vii) If $r = \pm 1$, the two lines of regression either coincide or parallel to each other.

(viii) Arithmetic mean of two regression coefficients is either equal to or greater than the correlation coefficient.

i.e., $$\frac{b_{yx} + b_{xy}}{2} \geq r$$

(ix) The two regression lines coincide if and only if there is perfect linear relation between X and Y.

i.e., $\rho(X, Y) = \pm 1$

(x) The intersection points of two regression lines is, $(\bar{x}, \bar{y})$, were $\bar{x}$ is mean of x-series and $\bar{y}$ is mean of y-series.

(xi) The angle between regression lines indicates the degree of dependence between the variables.

Know the Formulae

- **Regression coefficient of y on x :**

$$b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x} = \frac{\Sigma xy - \frac{\Sigma x \cdot \Sigma y}{n}}{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}$$

- **Regression coefficient of x on y :**

$$b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y} = \frac{\Sigma xy - \frac{\Sigma x \cdot \Sigma y}{n}}{\Sigma y^2 - \frac{(\Sigma y)^2}{n}}$$

- **Regression equation of y on x :**

$$(y - \bar{y}) = b_{yx}(x - \bar{x})$$

or

$$(y - \bar{y}) = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

- **Regression equation of x on y :**

$$(x - \bar{x}) = b_{xy}(y - \bar{y})$$

or

$$(x - \bar{x}) = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

- **Correlation coefficient :**

$$r = \sqrt{b_{yx} \times b_{xy}}$$

$0 \le r \le 1$ and r, b_{yx} and b_{xy} have same sign.

- **Angle between the lines of regression :**

$$\tan\theta = \pm\left(\frac{1 - r^2}{r}\right)\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

Two lines coincide if $\theta = 0$, $r = \pm 1$

Two lines are perpendicular if $\theta = \frac{\pi}{2}$, $r = 0$

Mnemonics

Correlation coefficient : $r = \sqrt{b_{yx} \times b_{xy}}$ $0 \le r \le 1$ and r, b_{yx} and b_{xy} have same sign.

Mnemonics : Rohit Sometimes Builds Building

Interpretation :

Rohit		Sometimes	Builds	Building
↓		↓	↓	↓
r	=	***Square rood***	b_{xy}	b_{yx}

1 Mark Questions

AI Q. 1 If the lines of regression are parallel to coordinate axes, then the coefficient of correlation is :

(a) 1 **(b)** 0

(c) –1 **(d)** $\frac{1}{2}$

[ISC Specimen Paper, 2021]

Sol. Correct Option : (b)

Explanation:

When $r = 0, \tan\theta = \infty$

Hence the lines of regression are perpendicular.

The lines of regression in this cases are

$$x = \bar{x}$$

and

$$y = \bar{y}$$

Q. 2. The purpose of regression is to

(a) maximize the r^2.

(b) explain the mean of the independent variable.

(c) explain the variation of the dependent variable.

(d) explain the variation of the independent variable.

Sol. Correct Option : (c)

Explanation : The purpose is to explain the variation of the dependent variable.

Q.3. The slope of the regression line of Y on X is also called the :

(a) Correlation coefficient of X on Y (b) Correlation coefficient of Y on X

(c) Regression coefficient of X on Y (d) Regression coefficient of Y on X

Sol. Correct Option : (d)

Explanation : The slope of the regression line of Y on X is also called the Regression coefficient of Y on X.

Q. 4. The straight line graph of the linear equation $Y = a + bX$, slope will be upward if :

(a) $b = 0$ **(b) $b < 0$**

(c) $b > 0$ **(d) $b \neq 0$**

Sol. Correct Option : (c)

Explanation : The straight line graph of the linear equation $Y = a + bX$, slope will be upward if $b > 0$.

2 Marks Questions

Q. 1. For the lines of regression $4x - 2y = 4$ and $2x - 3y + 6 = 0$, find the mean of 'x' and the mean of 'y'. **[ISC, 2020]**

Sol.

$$4x - 2y - 4 = 0$$

$$2x - 3y + 6 =$$

$$\frac{x}{-12-12} = \frac{y}{-8-24}\,\frac{1}{-12+4}$$

$$\frac{x}{24} = \frac{y}{-32} = \frac{1}{-8}$$

$$x = \frac{-24}{-8} = 3$$

$$y = \frac{-32}{-8} = 4$$

$$\therefore \qquad \bar{x} = 3 \text{ and } \bar{y} = 4$$

[ISC Marking Scheme, 2020]

AI Q. 2. If the regression coefficients for the variable x and y are $b_{yx} = 1$ and $b_{xy} = \frac{1}{2}$, then find the angle between the regression lines.

[ISC Specimen Paper, 2021]

Sol.

$$b_{yx} = 1 \;\&\; b_{xy} = \frac{1}{2}$$

$\therefore$ Slope of regression equation of dy on

$$x = b_{yx} = 1 = \mathrm{m}$$

Let angle between the lines is θ.

$$\therefore \qquad \tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

$$= \pm\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right) = \pm\frac{1}{3}$$

$$\therefore \qquad \theta = \tan^{-1}\left(\pm\frac{1}{3}\right)$$

AI Q. 3. If $\bar{x} = 18, \bar{y} = 100, \sigma_x = 14, \sigma_y = 20$ and correlation coefficient $r_{xy} = 0{\cdot}8$, find the regression equation of y on x. **[ISC, 2019]**

Sol. Given $\bar{x} = 18, \bar{y} = 100, \sigma_x = 14, \sigma_y = 20, r = 0.8$

$$b_{xy} = r.\frac{\sigma_y}{\sigma_x} = 0.8 \times \frac{20}{14} = \frac{8}{7}$$

Regression equation y on x

$$y - \bar{y} = b_{yx}.(x - \bar{x})$$

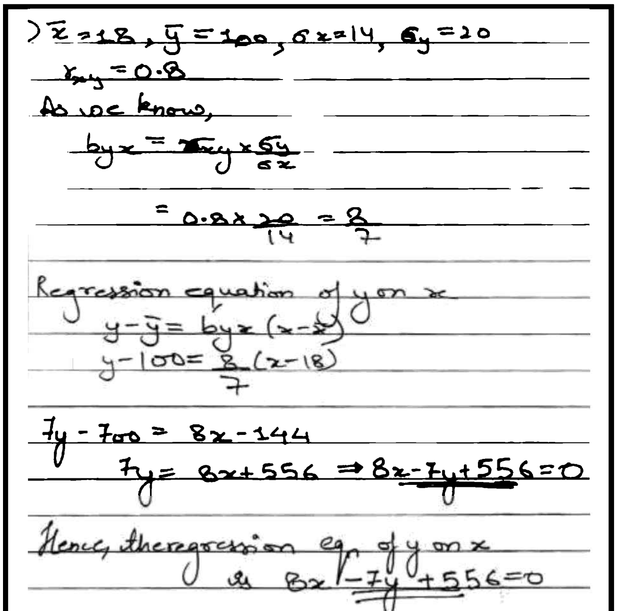
$$y - 100 = \frac{8}{7}(x - 18)$$

$$7y - 700 = 8x - 144$$

$$8x - 7y + 556 = 0$$

[ISC Marking Scheme, 2019]

OR



$\bar{x} = 18, \bar{y} = 100, \sigma x = 14, \sigma_y = 20$

$r_{xy} = 0{\cdot}8$

As we know,

$b_{yx} = r_{xy} \times \frac{\sigma_y}{\sigma x}$

$= 0{\cdot}8 \times \frac{20}{14} = \frac{8}{7}$

Regression equation of y on x

$y - \bar{y} = b_{yx}(x - \bar{x})$

$y - 100 = \frac{8}{7}(x - 18)$

$7y - 700 = 8x - 144$

$7y = 8x + 556 \Rightarrow 8x - 7y + 556 = 0$

Hence, the regression eq^n of y on x is $8x - 7y + 556 = 0$

[Topper's Answer 2019]

Examiner's Comment

- Several candidates wrote incorrect formula for by x due to lack of understanding of regression coefficients and their formulae.

Answering Tip

- Students are advised to maintain highest level of accuracy while solving problems of this topic.

Q. 4. Find the coefficient of correlation from the regression lines :
$x - 2y + 3 = 0$ and $4x - 5y + 1 = 0$. [ISC, 2018]

Sol. Let the line of regression of y on x be

$$x - 2y + 3 = 0$$

$$\Rightarrow \quad 2y = x + 3$$

$$\Rightarrow \quad y = \frac{x}{2} + \frac{3}{2}$$

$$\therefore \quad b_{yx} = \frac{1}{2}$$

Let the line of regression of x on y be

$$\Rightarrow \quad 4x - 5y + 1 = 0$$

$$\Rightarrow \quad 4x = 5y - 1$$

$$\therefore \quad x = \frac{5}{4}x - \frac{1}{4}$$

$$b_{xy} = \frac{5}{4}$$

$$\therefore \quad r^2 = b_{yx} \times b_{xy}$$

$$= \left(\frac{1}{2}\right)\left(\frac{5}{4}\right) = \frac{5}{8} < 1$$

Hence, our assumption of regression equation is correct.

$$\therefore \quad r = \sqrt{\frac{5}{8}} = 0.79$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Many candidates made errors while applying condition to identify correct regression coefficients of the given lines thereby, calculated incorrect correlation coefficient.

Answering Tip

- Understand the method of identifying correct regression coefficients of given lines and do enough practice of solving problems.

AI Q. 5. The two lines of regressions are $4x + 2y - 3 = 0$ and $3x + 6y + 5 = 0$. Find the correlation coefficient between x and y. [ISC Specimen, 2018, ISC, 2015]

Sol. Let the line of regression of x on y be

$$4x + 2y - 3 = 0$$

$$\Rightarrow \quad x = -\frac{1}{2}y + \frac{3}{4}$$

$$\Rightarrow \quad b_{xy} = -\frac{1}{2}$$

Let the line of regression of y on x be

$$3x + 6y + 5 = 0$$

$$\Rightarrow \quad y = -\frac{1}{2}x - \frac{5}{6}$$

$$\Rightarrow \quad b_{xy} = -\frac{1}{2}$$

$$\therefore \quad r^2 = b_{yx} \times b_{xy}$$

$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{1}{4}$$

$$\therefore \quad r = -\frac{1}{2}$$

[Since b_{xy} and b_{yx} are negative.]

[ISC Marking Scheme, 2015]

Examiner's Comment

- Many candidates answered this question correctly. However, a few candidates wrote incorrect regression coefficient.

Answering Tip

- Always remember that value of 'r' should be less than 1; b_{xy} and b_{yx} both positive, 'r' will be positive otherwise negative.

Q. 6. Two regression lines are represented by $4y + 10x = 9$ and $6x + 3y = 4$. Find the line of regression of y on x. [ISC, 2013]

Sol. Assume that $4x + 10y = 9$ is the line of regression of y on x.

$$\therefore \quad 10y = -4x + 9$$

$$y = -\frac{4}{10}x + \frac{9}{10}$$

$$\Rightarrow \quad b_{yx} = -\frac{2}{5}$$

Now, $6x + 3y = 4$ is the line of regression of x on y.

$$\therefore \quad 6x = -3y + 4$$

$$\Rightarrow \quad x = -\frac{3}{6}y + \frac{4}{6}$$

$$\Rightarrow \quad b_{xy} = -\frac{1}{2}$$

Here,

$$b_{xy} \times b_{yx} = \left(-\frac{2}{5}\right) \times \left(-\frac{1}{2}\right)$$

$$= \frac{1}{5} < 1$$

Which is true. Hence our assumption is correct and line of regression of y on x is $4x + 10y = 9$.

Q. 7. Two regression lines are represented by $2x + 3y - 10 = 0$ and $4x + y - 5 = 0$. Find the line of regression of y on x. [ISC, 2011]

Sol. Let the line of regression of y on x be

$$2x + 3y - 10 = 0$$

$$\Rightarrow \quad 3y = -2x + 10$$

$$\Rightarrow \quad y = -\frac{2}{3}x - \frac{10}{3}$$

$$\therefore \quad b_{yx} = -\frac{2}{3}$$

Let the line of regression of x on y be

$$4x + y - 5 = 0$$

$$\Rightarrow \quad 4x = -y + 5$$

$$\Rightarrow \quad x = -\frac{1}{4}y + \frac{5}{4}$$

$$\therefore \quad b_{xy} = -\frac{1}{4}$$

Here,

$$b_{yx} \times b_{xy} = \left(-\frac{2}{3}\right)\left(-\frac{1}{4}\right) = \frac{1}{6} < 1$$

Which is true. Hence, our assumption is correct and line of regression of y on x is $2x + 3y - 10 = 0$

4 Marks Questions

AI **Q. 1. Find the equation of the regression line of y on x, if the observations (x, y) are as follows :**
(1, 4), (2, 8), (3, 2), (4, 12), (5, 10), (6, 14), (7, 16), (8, 6), (9, 18)
Also, find the estimated value of y when $x = 14$.
[ISC, 2011]

Sol.

x_i	y_i	$x_i y_i$	x_i^2
1	2	4	1
2	8	16	4
3	2	6	9
4	12	48	16
5	10	50	25
6	14	84	36
7	16	84	36
8	6	48	64
9	18	162	81
$\sum x_i = 45$	$\sum y_i = 90$	$\sum x_i y_i = 530$	$\sum x^2_i = 892$

$$\therefore \quad b_{yx} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

$$= \frac{9 \times 530 - 45 \times 90}{9 \times 285 - (45)^2} = \frac{4}{3}$$

$\therefore$ Regression equation of y on x

$$y - 10 = \frac{4}{3}(x - 5)$$

$$y = \frac{4}{3}x + \frac{10}{3}$$

Now, when $x = 14$

$$y = \frac{4}{3} \times 14 + \frac{10}{3} = 22$$

Examiner's Comment

- Many candidates wrote incorrect formula for regression coefficients which caused error in finding the regression equation y on x. Several candidate were unable to interpret the result of y for $x = 14$.

Answering Tip

- Practice a variety of problems based on regression.

Q. 2. The correlation coefficient between x and y is 0.6. If the variance of x is 225, the variance of y is 400, mean of x is 10 and mean of y is 20, find
(i) the equations of two regression lines.
(ii) the expected value of y when x = 2. [ISC, 2020]

Sol. Given : Correlation coefficient

$$= 0.6$$

Variance of x = 225
Variance of y = 400
Mean of x = 10
Mean of y = 20

$$\sigma_x = \sqrt{\text{Variance}}$$

$$= \sqrt{225 \pm 15}$$

$$\sigma_y = \sqrt{\text{Variance}}$$

$$= \sqrt{400 \pm 20}$$

(i) $$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$= 0.6 \times \frac{20}{15} = 0.8$$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

$$= 0.6 \times \frac{15}{20} = 0.45$$

Regression line y on x

$$y - \bar{y} = b_{yx}(x - \bar{x})$$
$$y - 20 = 0.8(x - 10)$$
$$y - 20 = 0.8x - 8$$
$$y = 0.8x + 12 \quad \text{...(1)}$$

Regression line x on y

$$x - \bar{x} = b_{xy}(y - \bar{y})$$
$$x - 10 = 0.45(y - 20)$$
$$x = 0.45y - 9 + 10$$
$$x = 0.45y + 1 \quad \text{...(2)}$$

(ii) When $x = 2$

$$y = 0.8 . 2 + 12 = 13.6$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- Many candidates applied incorrect formulae in the process of calculating b_{yx} and b_{xy}. Thereby made errors in calculating other two values also.

Answering Tip

- Learn formulae for calculating regression coefficients b_{yx} and b_{xy} correctly.

Q. 3. Find the regression coefficients b_{yx}, b_{xy} and correlation coefficient 'r' for the following data :
(2, 8), (6, 8), (4, 5), (7, 6), (5, 2) [ISC, 2020]

Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
2	8	04	64	16
6	8	36	64	48
4	5	16	25	20
7	6	49	36	42
5	2	25	04	10
$\sum x_i = 24$	$\sum y_i = 29$	$\sum x_i^2 = 130$	$\sum y_i^2 = 193$	$\sum x_i y_i = 136$

$n = 5$

$$by_x = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{n}}{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}$$

$$= \frac{136 - \frac{24 \times 29}{5}}{130 - \frac{(24)^2}{5}}$$

$$= \frac{680 - 696}{5(130 - 115.2)}$$

$$\therefore \quad b_{yx} = \frac{-16}{5(14.8)} = \frac{-16}{74}$$

$$= \frac{-8}{37} = -0.2162 \text{ (approx.)}$$

$$b_{xy} = \frac{\Sigma xy - \frac{1}{n}\Sigma x \Sigma y}{\Sigma y^1 \frac{1}{n}(\Sigma y)^2}$$

$$= \frac{36 - \frac{1}{5}24 \times 29}{193 - \frac{1}{5}(29)^2}$$

$$= \frac{680 - 696}{5(193 - 168.2)} = \frac{-16}{5 \times 24.8}$$

$$= \frac{-16}{124} = \frac{-4}{31}$$

$$b_{xy} = 0.129 \text{ (approx.)}$$

$$r = \pm\sqrt{b_{xy} \times b_{yx}}$$

$$= -\sqrt{\frac{8}{37} \times \frac{4}{31}} = -\sqrt{\frac{32}{1147}}$$

$$= -\sqrt{0.02790}$$

$$r = -0.1670 \text{ (approx.)}$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- The concept of regression coefficient (b_{yx} and b_{xy}) was not clear to a number of candidates. Many candidates solved the given equations for x and y unnecessarily. Some forgot that the product of the regression coefficients yields a position fraction less than unity.

Answering Tip

- Learn how to identify the regression lines as "y on x" and "x on y" separately, using the fact $b_{yx} \cdot b_{xy} = r^2$ and $r^2 < 1$. Also the fact that r, b_{xy}, b_{yx} are either all positive or all negative.

AI Q. 4. The following results were obtained with respect to two variables x and y :

$\sum x = 15$, $\sum y = 25$, $\sum xy = 83$, $\sum x^2 = 55$, $\sum y^2 = 135$ and $n = 5$

(i) Find the regression coefficient b_{xy}.

(ii) Find the regression equation of x on y.

[ISC, 2019]

Sol.

(i) $\sum x = 15$, $\sum y = 25$, $\sum xy = 83$, $\sum x^2 = 55$, $\sum y^2 = 135$, $n = 5$

$$b_{xy} = \frac{\sum xy - \frac{\sum x . \sum y}{n}}{\sum y^2 - \frac{(\sum y)^2}{n}} = \frac{83 - \frac{15.25}{5}}{135 - \left(\frac{625}{5}\right)} = \frac{4}{5} = 0.8$$

(ii) $\bar{x} = \frac{15}{5} = 3$, $\bar{y} = \frac{25}{5} = 5$

Regression equation x on y

$$x = \bar{x} = b_{xy}(y - \bar{y})$$

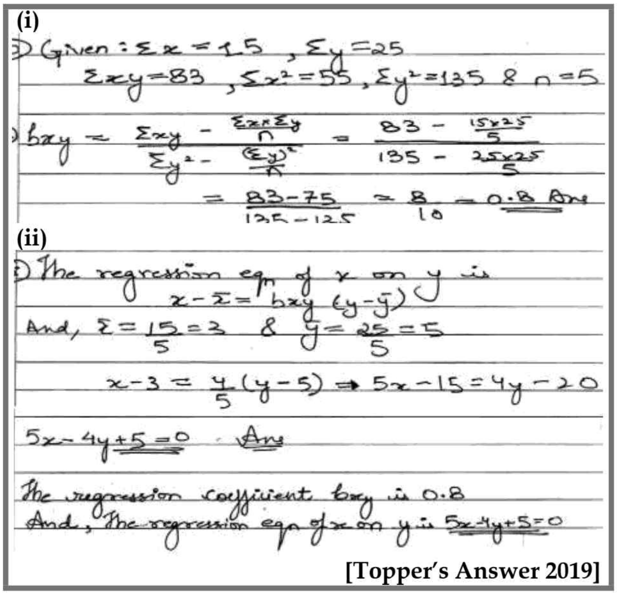
$$x - 3 = \frac{4}{5}(y - 5)$$

$$5x - 15 = 4y - 20$$

$$5x - 4y + 5 = 0$$

[ISC Marking Scheme, 2019]

OR



(i)

Given : $\Sigma x = 15$, $\Sigma y = 25$

$\Sigma xy = 83$, $\Sigma x^2 = 55$, $\Sigma y^2 = 135$ & $n = 5$

$$bxy = \frac{\Sigma xy - \frac{\Sigma x \times \Sigma y}{n}}{\Sigma y^2 - \frac{(\Sigma y)^2}{n}} = \frac{83 - \frac{15 \times 25}{5}}{135 - \frac{25 \times 25}{5}}$$

$$= \frac{83 - 75}{135 - 125} = \frac{8}{10} = 0.8 \text{ Ans}$$

(ii)

The regression eqn of x on y is

$$x - \bar{x} = bxy\ (y - \bar{y})$$

And, $\bar{x} = \frac{15}{5} = 3$ & $\bar{y} = \frac{25}{5} = 5$

$$x - 3 = \frac{4}{5}(y - 5) \Rightarrow 5x - 15 = 4y - 20$$

$5x - 4y + 5 = 0$ Ans

The regression coefficient bxy is 0.8

And, The regression eqn of x on y is $5x - 4y + 5 = 0$

[Topper's Answer 2019]

Q. 5. Find the line of regression of y on x from the following table :

x	1	2	3	4	5
y	7	6	5	4	3

Hence, estimate the value of y when x = 6.

[ISC, 2018]

Sol.

x	y	xy	x^2
1	7	7	1
2	6	12	4
3	5	15	9
4	4	16	16
5	3	15	25
$\Sigma x = 15$	$\Sigma y = 25$	$\Sigma xy = 65$	$\Sigma x^2 = 55$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{15}{5} = 3$$

and $$\bar{y} = \frac{\Sigma y}{n} = \frac{25}{5} = 5$$

$$b_{yx} = \frac{\Sigma xy - \frac{\Sigma x . \Sigma y}{n}}{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}$$

$$= \frac{65 - \frac{15.25}{5}}{55 - \frac{(15)^2}{5}}$$

$$= \frac{65-75}{55-45} = \frac{-10}{10} = -1$$

Regression line y on x is given by

$$(y - \bar{y}) = b_{yx}\,(x - \bar{x})$$

$$\therefore \quad y - 5 = -1(x-3)$$

$$\Rightarrow \quad y - 5 = -x + 3$$

$$\Rightarrow \quad x + y = 8 \qquad \text{...(i)}$$

When $x = 6$,

From (i), we get

$$y = 8 - 6$$

$$\Rightarrow \quad y = 2$$

[ISC Marking Scheme, 2018]

Examiner's Comment

- Some candidates used incorrect formula to calculate regression coefficient like b_{yx} which resulted incorrect regression equation.

Answering Tip

- Do practice a number of problems based on regression.

Q. 6. From the given data:

Variable	x	y
Mean	6	8
Standard Deviation	4	6

and correlation coefficient $\frac{2}{3}$. Find :

(i) Regression coefficients b_{yx} and b_{xy}

(ii) Regression line x on y

(iii) Most likely value of x when $y = 14$ [ISC, 2018]

Sol. Given, $\bar{x} = 6$, $\bar{y} = 8$, $\sigma_x = 4$, $\sigma_y = 6$ and $r = \frac{2}{3}$

(i)

$$b_{yx} = r.\frac{\sigma_y}{\sigma_x} = \frac{2}{3}.\frac{6}{4} = 1$$

$$b_{xy} = r.\frac{\sigma_x}{\sigma_y} = \frac{2}{3}.\frac{4}{6} = \frac{4}{9}$$

(ii) Regression line x on y is given by

$$(x - \bar{x}) = b_{xy}(y - \bar{y})$$

$$x - 6 = \frac{4}{9}(y-8)$$

$$\Rightarrow \quad 9x - 54 = 4y - 32$$

$$\Rightarrow \quad 9x - 4y = 22 \qquad \text{...(1)}$$

(iii) From eq (1),

When $y = 14$,

$$9x - 4(14) = 22$$

$$\Rightarrow \quad 9x = 78$$

$$\Rightarrow \quad x = \frac{78}{9} = 8.67$$

[ISC Marking Scheme, 2018]

AI **Q. 7. By using the data $\bar{x} = 25$, $\bar{y} = 30$, $b_{yx} = 1{\cdot}6$ and $b_{xy} = 0{\cdot}4$, find :**

(i) The regression equation y on x.

(ii) What is the most likely value of y when $x = 60$?

(iii) What is the coefficient of correlation between x and y? [ISC, 2017]

Sol. Given, $\bar{x} = 25$, $\bar{y} = 30$, $b_{yx} = 1.6$ and $b_{xy} = 0.4$

(i) Regression equation y on x :

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$\Rightarrow \quad y - 30 = 1.6(x - 25)$$

$$\Rightarrow \quad y - 30 = 1.6x - 40$$

$$\Rightarrow \quad y = 1.6x - 10 \qquad \text{...(i)}$$

(ii) From eq. (i)

When $x = 60$, $\quad y = (1.6) \times (60) - 10$

$$= 96 - 10 = 86$$

(iii) Coefficient of correlation :

$$r = \pm\sqrt{b_{yx}.b_{xy}}$$

$$= \pm\sqrt{(1.6)(0.4)}$$

$$= \pm\sqrt{0.64}$$

$$= 0.8$$

($\because$ both regression coefficients are $+ve$)

[ISC Marking Scheme, 2017]

Examiner's Comment

- Some candidates considered incorrect regression equations due to lack of understanding of the regression concept.

Answering Tip

- Understand the basic difference between the regression equations x on y and y on x.

Q. 8. The two lines of regression are $x + 2y - 5 = 0$ and $2x + 3y - 8 = 0$ and the variance of x is 12. Find the variance of y and the coefficient of correlation.

[ISC, 2016]

Sol.

$$b_{xy} = \frac{-3}{2},\ b_{yx} = \frac{-1}{2}$$

$$\therefore \quad r = \sqrt{b_{yx} \times b_{xy}} = -\sqrt{\frac{3}{4}}$$

$$= -0.866, \qquad -1 \le r \le 1$$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} \text{ and } \sigma_y^2 = 4$$

[ISC Marking Scheme, 2016]

Detailed Solution :

Given lines of regression are

$$x + 2y - 5 = 0$$

and $2x + 3y - 8 = 0$

i.e., $$y = -\frac{x}{2} + \frac{5}{2}$$

and $$x = -\frac{3}{2}y + \frac{8}{2}$$

$$\therefore \quad b_{yx} = r\frac{\sigma_y}{\sigma_x} = -0.5$$

(regression line of y on x)

and $$b_{xy} = r\frac{\sigma_x}{\sigma_y} = -1.5$$

(regression line of x on y)

$$r^2 = b_{xy} \times b_{yx} = (-1.5) \times (-0.5)$$
$$= 0.75$$
$$\therefore \quad r = \pm\sqrt{0.75} = \pm 0.866$$

As b_{xy} and b_{yx} are both negative, r is also – ve.

$\therefore$ Correlation coefficient, $r = -0.866$

Given, $\sigma_x^2 = 12$

We have, $$r\frac{\sigma_y}{\sigma_x} = -0.5$$

$$\Rightarrow \quad \sigma_y = \frac{(-0.5)\times\sigma_x}{r}$$
$$\sigma_y^2 = \frac{0.25\times\sigma_x^2}{r^2}$$
$$= \frac{0.25\times 12}{0.75} = 4$$

$\therefore$ Variance of $y = \sigma_y^2 = 4$

Q. 9. For the given lines of regression, $3x - 2y = 5$ and $x - 4y = 7$, find

(i) regression coefficients b_{yx} and b_{xy}

(ii) coefficient of correlation r (x, y). [ISC, 2014]

Sol. (i) Let the line of regression of x on y be
$$3x - 2y = 5$$
and the line of regression of y on x be
$$x - 4y = 7$$
Writing the first equation in the form
$$x = \frac{2}{3}y + \frac{5}{3}$$
we get, $$b_{xy} = \frac{2}{3}$$
Writing the second equation in the form
$$y = \frac{1}{4}x - \frac{7}{4}$$
we get, $$b_{yx} = \frac{1}{4}$$
Now, $$r^2 = b_{yx} \times b_{xy}$$
$$= \frac{1}{4}\times\frac{2}{3} = \frac{1}{6}$$
$$\therefore \quad r = \frac{1}{\sqrt{6}},$$
Since r, b_{yx} and b_{xy} all have same sign.

[ISC Marking Scheme, 2014]

Examiner's Comment

- The concept of regression coefficient (b_{yx} and b_{xy}) was not clear to a number of candidates. Many candidates solved the given equations for x and y unnecessarily. Some forgot that the product of the regression coefficients yields a position fraction less than unity.

Answering Tip

- Learn how to identify the regression lines as "y on x" and "x on y" separately, using the fact b_{yx}. $b_{xy} = r^2$ and $r^2 < 1$. Also the fact that r, b_{xy}, b_{yx} are either all positive or all negative.

AI Q. 10. The following table shows the mean and standard deviation of the marks of Mathematics and Physics scored by the students in a school :

	Mathematics	Physics
Mean	84	81
Standard Deviation	7	4

The correlation coefficient between the given marks is 0.86. Estimate the likely marks in Physics if the marks in Mathematics are 92. [ISC, 2013]

Sol. Given, $\bar{x} = 84, \bar{y} = 81$
$$\sigma_x = 7, \sigma_y = 4$$
and $r = 0.86$

$$\therefore \quad b_{yx} = r\cdot\frac{\sigma_x}{\sigma_y}$$
$$= 0.86\times\frac{4}{7} = 0.49$$

$\therefore$ Regression equation of y on x
$$y - \bar{y} = b_{yx}(x - \bar{x})$$
$$\Rightarrow \quad y - 81 = 0.49(x - 84)$$
$$\Rightarrow \quad y - 81 = 0.49x - 41.16$$
$$\Rightarrow \quad y = 0.49x + 39.84$$
Putting $x = 92$,
$$y = 45.08 + 39.84 = 84.92$$
Hence, the likely marks in Physics are 84.92.

Q. 11. From the equation of the two regression lines, $4x + 3y + 7 = 0$ and $3x + 4y + 8 = 0$, find :

(i) Mean of x and y

(ii) Regression coefficient

(iii) Coefficient of correlation. [ISC, 2012]

Sol. (i) Given lines of regression are
$$4x + 3y + 7 = 0 \quad \ldots\text{(i)}$$
and $$3x + 4y + 8 = 0 \quad \ldots\text{(ii)}$$
Solve (i) and (ii) simultaneously, we get
$$x = -\frac{4}{7} \text{ and } y = -\frac{11}{7}$$
Point of intersection of the two lines gives the required mean :
$$\bar{x} = -\frac{4}{7}, \bar{y} = -\frac{11}{7}$$

(ii) Let equation (i) be the line of regression of y on x and (ii) be the line of regression of x on y.

$$4x + 3y + 7 = 0$$

$$\Rightarrow \quad 3y = -4x - 7$$

$$\Rightarrow \quad y = -\frac{4}{3}x - \frac{7}{3}$$

$$\Rightarrow \quad b_{yx} = -\frac{4}{3}$$

From (ii),

$$3x + 4y + 8 = 0$$

$$3x = -4y - 8$$

$$\Rightarrow \quad x = -\frac{4}{3}y - \frac{8}{3}$$

$$\Rightarrow \quad b_{xy} = -\frac{4}{3}$$

but $\quad r^2 = b_{yx} . b_{xy}$

$$= \left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right) > 1$$

which is not possible.

$\therefore$ Eq. (ii) is the line of regression of y on x and eq. (i) is the line of regression of x on y.

$$3x + 4y + 8 = 0$$

$$\Rightarrow \quad 4y = -3x - 8$$

$$\Rightarrow \quad y = -\frac{3}{4}x - \frac{8}{4}$$

$$\Rightarrow \quad b_{yx} = -\frac{3}{4}$$

and $4x + 3y + 7 = 0$

$$\Rightarrow \quad 4x = -3y - 7$$

$$\Rightarrow \quad x = -\frac{3}{4}y - \frac{7}{4}$$

$$\Rightarrow \quad b_{xy} = -\frac{3}{4}$$

(iii) Coefficient of correlation,

$$|r| = \sqrt{b_{yx} \times b_{xy}} = \sqrt{\left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)}$$

$$r = \pm\frac{3}{4}$$

$\therefore \quad r = -\frac{3}{4}$, as r, b_{xy} and b_{yx} all have same signs.

Q.12. If the regression equation of *x* on *y* is given by *mx* – *y* + 10 = 0 and the equation of *y* on *x* is given by – 2*x* + 5*y* = 14, determine the value of '*m*' if the coefficient of correlation between *x* and *y* is $\frac{1}{\sqrt{10}}$ **[ISC, 2010]**

Sol. Given, regression equation of x on y is

$$mx - y + 10 = 0$$

$$\Rightarrow \quad b_{xy} = \frac{1}{m}$$

Also, regression equation of y on x is given as

$$-2x + 5y = 14$$

$$\Rightarrow \quad b_{yx} = \frac{2}{5}$$

We have, $\quad r^2 = b_{yx} \times b_{xy}$

$$\left(\frac{1}{\sqrt{10}}\right)^2 = \frac{2}{5} \times \frac{1}{m}$$

$$\Rightarrow \quad \frac{1}{10} = \frac{2}{5} \times \frac{1}{m}$$

$$\Rightarrow \quad m = \frac{10 \times 2}{5}$$

$$\Rightarrow \quad m = 4$$

AI **Q. 13. Given two regression lines 4*x* + 3*y* + 7 = 0 and 3*x* + 4*y* + 8 = 0, determine :**

(i) The regression line of *y* on *x*.

(ii) The regression line of *x* on *y*.

(iii) The coefficient of correlation. **[ISC, 2007]**

Sol. Given lines :

$$4x + 3y + 7 = 0 \quad \ldots(1)$$

and $\quad 3x + 4y + 8 = 0 \quad \ldots(2)$

Let $4x + 3y + 7 = 0$ be the line of regression of x on y and $3x + 4y + 8$ be the line of regression of y on x.

$\therefore$ From (1),

$$4x + 3y + 7 = 0$$

$$\Rightarrow \quad x = -\frac{3}{4}y - \frac{7}{4} \quad \ldots(i)$$

$$\therefore \quad b_{xy} = -\frac{3}{4}$$

And from (2),

$$3x + 4y + 8 = 0$$

$$\Rightarrow \quad y = -\frac{3}{4}x - \frac{8}{4}$$

$$\therefore \quad b_{yx} = -\frac{3}{4}$$

$$\therefore \quad r = -\sqrt{b_{yx} \cdot b_{xy}}$$

$$= -\sqrt{\frac{-3}{4} \times \frac{-3}{4}} = -\frac{3}{4} = -0.75$$

(i) $3x+4y + 8 = 0$ is the line of regression of y on x.

(ii) $4x+3y + 7 = 0$ is the line of regression of x on y.

(iii) Correlation coefficient = – 0.75.

6 Marks Questions

AI **Q. 1. Find the line of best fit for the following data, treating *x* as dependent variable (Regression equation *x* on *y*) :**

x	14	12	13	14	16	10	13	12
y	14	23	17	24	18	25	23	24

Hence, estimate the value of *x* when *y* = 16.

[ISC, 2007]

Sol.

x	y	d_x	d_y	d^2_x	d^2_y	d_xd_y
14	14	1	– 7	1	49	–7
12	23	–1	2	1	4	–2
13	17	0	– 4	0	16	0
14	24	1	3	1	9	3
16	18	3	– 3	9	9	–9
10	25	–3	4	9	16	–12
13	23	0	2	0	4	0
12	24	–1	3	1	9	–3
Σx = 104	Σy = 168	Σd_x = 0	Σd_y = 0	Σd_x^2 = 22	Σd_y^2 =116	Σd_xd_y = – 30

Here, $\Sigma x = 104$, $\Sigma y = 168$, $\Sigma d_x = 0$, $\Sigma d_y = 0$, $\Sigma d_x^2 = 22$, $\Sigma d_y^2 = 116$ and $\Sigma d_xd_y = -30$

$$\therefore \; \bar{x} = \frac{\Sigma x}{n} = \frac{104}{8} = 13 \text{ and } \bar{y} = \frac{\Sigma y}{x} = \frac{168}{8} = 21$$

$$b_{xy} = \frac{\Sigma d_xd_y}{\Sigma d_y^2} = \frac{-30}{116}$$

Required regression line :

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 13 = \frac{-30}{116}(y - 21)$$

$$116x + 30y - 2138 = 0$$

When $y = 16$,
$$x = \frac{(-30)\times 16 + 2138}{116}$$

$$= \frac{-48 + 2138}{116} = 14.29$$

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates made errors in finding the regression equation x on y. Candidates were not clear about the formulae used for b_{yx} and b_{xy}. Many candidates failed to get the correct value of x for y = 16. Some candidates started solving with the help of normal equations but did not get the result.

Answering Tip

- Use the appropriate formula as per the requirement of the question. Accuracy levels are to be maintained at highest level as such simplification involve simple calculations only.

Q. 2. The marks obtained by 10 candidates in English and Mathematics are given below :

Marks in English	Marks in Mathematics
20	17
13	12
18	23
21	25
11	14
12	8
17	19
14	21
19	22
15	19

Estimate the probable score for Mathematics if the marks obtained in English are 24. [ISC, 2016]

Sol.

Eng (x)	Maths (y)	$d_x = x - \bar{x}$	$d_y = x - \bar{y}$	$(d_x)^2$	$(d_y)^2$	d_xd_y
20	17	4	–1	16	1	–4
13	12	–3	–6	9	36	18
18	23	2	5	4	25	10
21	25	5	7	25	49	35
11	14	–5	–4	25	16	20
12 8	–4	–10	16	100	40	
17	19	1	1	1	1	1
14	21	–2	3	4	9	–6
19	22	3	4	9	16	12
15	19	–1	1	1	1	–1
Σx = 160	Σy = 180			Σd_x^2 = 110	Σd_y^2 =254	Σd_xd_y = 125

$\Sigma x = 160$, $\Sigma y = 180$, $\Sigma d_xd_y = 125$,
$\Sigma d^2{}_x = 110$, $\Sigma d^2{}_y = 254$

$$b_{yx} = \frac{\Sigma(d_xd_y)}{d^2y}$$

$$= \frac{125}{110} = \frac{25}{22} = 1.13$$

Regression equation y on x is $y - 18 = \frac{25}{22}(x - 16)$

Probable score $y - 18 = \frac{25}{22}(24 - 16)$

$y = 9.0909 + 18 = 27.0909 = 27.1$

[ISC Marking Scheme, 2016]

Examiner's Comment

- Candidates were not clear about the formulae used for b_{xy} and b_{yx}. In most of the cases, candidates applied wrong formula to evaluate b_{xy}. Some candidates used the equation for the regression line of 'x on y' instead of 'y on x' to find y when x is given.

Answering Tip

- The students should understand the need for the appropriate formula at appropriate situation. Accuracy levels should be maintained at highest level as such simplifications involve simple calculations only.

Q. 3. Given that the observations are :

(9, –4), (10, –3), (11,–1), (12, 0), (13, 1), (14, 3) (15, 5), (16, 8).

Find the two lines of regression and estimate the value of y when x = 13.5. [ISC, 2015]

Sol.

x	y	xy	x^2	y^2
9	– 4	– 36	81	16
10	– 3	– 30	100	9
11	– 1	– 11	121	1
12	0	0	144	0
13	1	13	169	1
14	3	42	196	9
15	5	75	225	25
16	8	128	256	64
Σx = 100	Σy = 9	Σxy = 181	Σx^2 = 1292	Σy^2 = 125

$$\bar{x} = \frac{100}{8} = 12.5$$

$$\bar{y} = \frac{9}{8} = 1.125$$

$$b_{yx} = \frac{\Sigma xy - \frac{1}{n}\Sigma x\Sigma y}{\Sigma x^2 - \frac{1}{n}(\Sigma x)^2} = \frac{181 - \frac{1}{8}\times 100\times 9}{1292 - \frac{1}{8}(100)^2}$$

$$= \frac{68.5}{42} = 1.63$$

$$b_{xy} = \frac{\Sigma xy - \frac{1}{n}\Sigma x\Sigma y}{\Sigma y^2 - \frac{1}{n}(\Sigma y)^2} = \frac{181 - \frac{1}{8}\times 100\times 9}{125 - \frac{1}{8}(9)^2}$$

$$= \frac{68.5}{114.875} = 0.596$$

Line of regression of y on x, $y - \frac{9}{8} = 1.63(x - 12.5)$

Line of regression of x on y, $x - 12.5 = 0.596\left(y - \frac{9}{8}\right)$

$x = 0.596y + 11.83$

Now, $y = 1.63x - 19.25$

when $x = 13.5$, $y = 1.63 \times 13.5 - 19.25$

$= 2.755 = 2.76$

[ISC Marking Scheme, 2015]

Examiner's Comment

- Many candidates found b_{yx} and b_{xy} incorrectly, as a result, the two regression lines were incorrect. Some candidates found the value of y from given value of x by using regression equation of x on y instead of y on x. Several candidates were unable to calculate the correct values of Σxy, Σx^2, Σy^2, b_{yx} and b_{xy} which led to wrong results.

Answering Tip

- Various methods of finding b_{yx} and b_{xy} should be learned with examples. Students should be careful about the formulae for b_{yx} and b_{xy} as well as the regression equation of x on y and that of y on x.

AI Q. 4. The coefficient of correlation between the values denoted by x and y is 0.5. The mean of x is 3 and that of y is 5. Their standard deviations are 5 and 4 respectively. Find :

(i) The two lines of regression

(ii) the expected value of y, when x is given 14

(iii) the expected value of x, when y is given 9.

[ISC, 2014]

Sol. (i) Given, $\bar{x} = 3, \bar{y} = 5, \sigma_x = 5, \sigma_y = 4, r = 0.5$

The line of regression of y on x is given by

$$y - \bar{y} = r\frac{\sigma_Y}{\sigma_X}(x - \bar{x})$$

$$y - 5 = 0.5 \times \frac{4}{5}(x - 3)$$

i.e., $5y - 25 = 2x - 6$

$\Rightarrow$ $2x - 5y + 19 = 0$ ...(1)

The line of regression of x on y is given by

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 3 = 0.5 \times \frac{5}{4}(y - 5)$$

i.e., $8x - 24 = 5y - 25$

$8x - 5y + 1 = 0$ ...(2)

(ii) Expected value of y when $x = 14$

$2 \times 14 - 5y + 19 = 0 \Rightarrow y = 9.4$ [from (1)]

(iii) Expected value of x when $y = 9$

$8x - 5y + 1 = 0 \Rightarrow x = 5.5$ [from (2)]

[ISC Marking Scheme, 2014]

Q. 5. The following result were obtained with respect to two variables x and y :

$\Sigma x = 30$, $\Sigma y = 42$, $\Sigma xy = 199$, $\Sigma x^2 = 184$,

$\Sigma y^2 = 318$, $n = 6$.

Find the following :

(i) The regression coefficients.

(ii) Correlation coefficient between x and y.

(iii) Regression equation of y on x.

(iv) The likely value of y when $x = 10$. **[ISC, 2012]**

Sol. (i) We know that

$$b_{yx} = \frac{\Sigma xy - \frac{1}{n}\Sigma x \cdot \Sigma y}{\Sigma x^2 - \frac{1}{n}(\Sigma x)^2}$$

$$= \frac{199 - \frac{1}{6}(30)(42)}{184 - \frac{1}{6}(30)^2}$$

$$= \frac{199 - 210}{184 - 150} = \frac{-11}{34} = -0.323$$

and

$$b_{xy} = \frac{\Sigma xy - \frac{1}{n}\Sigma x \cdot \Sigma y}{\Sigma y^2 - \frac{1}{n}(\Sigma y)^2}$$

$$= \frac{199 - \frac{1}{6}(30)(42)}{318 - \frac{1}{6}(42)^2}$$

$$= \frac{199-210}{318-294} = \frac{-11}{24} = -0.458$$

(ii) Correlation coefficient between x and y.

$$r^2 = b_{yx} \cdot b_{xy}$$

$$= (-0.323) \times (-0.458) = 0.1479$$

$$\therefore \quad r = \sqrt{0.1479} = -0.384$$

Since, b_{xy} and b_{yx} have the negative sign, therefore, we use negative sign for r.

(iii)

$$\bar{x} = \frac{\Sigma x}{6} = \frac{30}{6} = 5$$

$$\bar{y} = \frac{\Sigma y}{6} = \frac{42}{6} = 7$$

Regression line of y on x.

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$\Rightarrow \quad y - 7 = \frac{-11}{34}(x-5)$$

$$\Rightarrow \quad 34y - 238 = -11x + 55$$

$$\Rightarrow \quad 11x + 34y - 293 = 0$$

(iv) when $x = 10$

$$11 \times 10 + 34y - 293 = 0$$

$$\Rightarrow \quad 34y = 183$$

$$\Rightarrow \quad y = \frac{183}{34}$$

$$\Rightarrow \quad y = 5.382$$

Q. 6. The following observations are given :

(1, 4), (2, 8), (3, 2), (4, 12), (5, 10), (6, 14), (7, 16), (8, 6), (9, 18)

Estimate the value of y when the value of x is 10 and also estimate the value of x when the value of $y = 5$. [ISC, 2011]

Sol. Here we have

x	x^2	y	y^2	xy
1	1	4	16	4
2	4	8	64	16
3	9	2	4	6
4	16	12	144	48
5	25	10	100	50
6	36	14	196	84
7	49	16	256	112
8	64	6	36	48
9	81	18	324	162
$\Sigma x = 45$	$\Sigma x^2 =$ 285	$\Sigma y = 90$	$\Sigma y^2 =$ 1140	$\Sigma xy =$ 530

$$\bar{x} = \frac{45}{9} = 5, \bar{y} = \frac{90}{9} = 10$$

$$b_{yx} = \frac{\Sigma xy - \frac{1}{n}(\Sigma x \Sigma y)}{\Sigma x^2 - \frac{1}{n}(\Sigma x)^2}$$

$$= \frac{530 - \frac{1}{9} \times 45 \times 90}{285 - \frac{1}{9} \times 45 \times 45}$$

$$= \frac{80}{60} = \frac{4}{3}$$

$$b_{xy} = \frac{\Sigma xy - \frac{1}{n}(\Sigma x \Sigma y)}{\Sigma y^2 - \frac{1}{n}(\Sigma y)^2}$$

$$= \frac{530 - \frac{1}{9} \times 45 \times 90}{1140 - \frac{1}{9} \times 90 \times 90}$$

$$= \frac{80}{240} = \frac{1}{3}$$

$\therefore$ Regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 10 = \frac{4}{3}(x-5)$$

$$\Rightarrow \quad 3y - 30 = 4x - 20$$

$$\Rightarrow \quad 3y = 4x + 10$$

Given, $x = 10$, we get

$$3y = 4 \times 10 + 10 = 50$$

$$\Rightarrow \quad y = \frac{50}{3} = 16.67$$

Again, Regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$\text{or} \quad x - 5 = \frac{1}{3}(y - 10)$$

$$\text{or} \quad 3x - 15 = y - 10$$

$$3x = y + 5$$

Given, $y = 5$, we get

$$3x = 5 + 5$$

$$\Rightarrow \quad x = \frac{10}{3} = 3.33$$

Q. 7. Find the equations of the two lines of regression for the following observations :

(3, 6), (4, 5), (5, 4), (6, 3), (7, 2).

Find the estimate of y for $x = 2.5$. [ISC, 2010]

Sol. We have $\Sigma x_i = 3 + 4 + 5 + 6 + 7 = 25$

$$\Sigma y_i = 6 + 5 + 4 + 3 + 2 = 20$$

$$\Sigma x_i^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 135$$

$$\Sigma y_i^2 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 = 90$$

$$\Sigma x_i y_i = 18 + 20 + 20 + 18 + 14 = 90$$

$$n = 5, \bar{x} = \frac{\Sigma x_i}{n} = \frac{25}{5} = 5, \bar{y} = \frac{\Sigma y_i}{n} = \frac{20}{5} = 4$$

$$b_{yx} = \frac{\Sigma x_i \Sigma y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}}{\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}}$$

$$= \frac{90 - \frac{25 \times 20}{5}}{135 - \frac{625}{5}} = \frac{90 - 100}{10} = -1$$

$$b_{xy} = \frac{\Sigma x_i \Sigma y_i - \dfrac{(\Sigma x_i)(\Sigma y_i)}{n}}{\Sigma y_i^2 - \dfrac{(\Sigma y_i)^2}{n}}$$

$$= \frac{90 - \dfrac{25 \times 20}{5}}{90 - \dfrac{400}{5}}$$

$$= \frac{90 - 100}{90 - 80} = -1$$

The regression lines are

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 4 = -x + 5$$

$$x + y = 9$$

and $$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 5 = -y + 4$$

$$x + y = 9$$

At $x = 2.5$,

$$y = 9 - 2.5 = 6.5$$

Q. 8. If the two regression lines of bivariate distribution are $4x - 5y + 33 = 0$ and $20x - 9y - 107 = 0$.

(i) calculate $\bar{x}$ and $\bar{y}$, the arithmetic means of x and y respectively.

(ii) estimate the value of x when $y = 7$.

(iii) find the variance of y when $\sigma_x = 3$. **[ISC, 2009]**

Sol. The given regression lines are,

$$4x - 5y + 33 = 0$$

and $$20x - 9y - 107 = 0$$

We know that two regression lines intersect at the point $(\bar{x}, \bar{y})$

(i) Equations are :

$$4x - 5y = -33 \qquad \ldots(1)$$

$$20x - 9y = 107 \qquad \ldots(2)$$

Multiply by 5 in equation (1) and solving with equation (2), we get

$$\begin{array}{r} 20x - 25y = -165 \\ \underset{-}{20x} \underset{+}{-9y} = \underset{-}{107} \\ \hline -16y = -272 \end{array}$$

$$\Rightarrow \quad y = 17$$

Put the value of y in equation (1), we get

$$4x - 5y = -33$$

$$\Rightarrow \quad 4x - 5 \times 17 = -33$$

$$\Rightarrow \quad 4x - 85 = -33$$

$$\Rightarrow \quad 4x = -33 + 85$$

$$\Rightarrow \quad 4x = 52$$

$$\Rightarrow \quad x = 13$$

$$\bar{x} = 13, \bar{y} = 17$$

(ii) The line of regression of x on y gives the value of y Assume that (1) is the regression line of y on x. Then line (2) is the regression line of x on y. Let us check wither our assumption is correct or not.

Rewriting eqs. (1) and (2), we get

$$y = \frac{4}{5}x + \frac{33}{5} \text{ and } x = \frac{9}{20}y + \frac{107}{20}$$

$$\Rightarrow b_{yx} = \frac{4}{5} \text{ and } b_{xy} = \frac{9}{20}$$

Now, $b_{yx}.b_{xy} = \frac{4}{5} \times \frac{9}{20} = \frac{9}{25} < 1$

i.e., our assumption is correct and line (2), is the regression line of x on y.

Putting $y = 7$ in the equation (2), we get

$$20x - 9y - 107 = 0$$

$$20x - 63 - 107 = 0$$

$$20x - 170 = 0$$

$$20x = 170$$

$$x = \frac{170}{20}$$

$$x = 8.5$$

(iii) Given, $\sigma_x = 3$

Now, $$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$$

$$= \pm\sqrt{\frac{4}{5} \times \frac{9}{20}}$$

$$= \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$$\therefore \quad r = \frac{3}{5}$$

('r' should be of same sign as b_{xy} & b_{yx})

$$\because \quad r\frac{\sigma_y}{\sigma_x} = b_{yx}$$

$$\frac{3}{5} \times \frac{\sigma_y}{3} = \frac{4}{5}$$

$$\Rightarrow \quad \sigma_y = 4$$

$$\therefore \quad \text{Variance } (\sigma_y^2) = 16$$

Q. 9. The data for marks in Physics and History obtained by ten students are given below :

Marks in Physics	Marks in History
15	10
12	25
8	17
8	11
7	13
7	17
7	20
6	13
5	9
3	15

Using this data :

(i) Find the line of regression in which Physics is taken as the independent variable.

(ii) A candidate had scored 10 marks in Physics but was absent from the History test.

Estimate his probable score for the later test.

[ISC, 2008]

Sol. Given marks are :

Physics (x)	**History (y)**	xy	x^2	y^2
15	10	150	225	100
12	25	300	144	625
8	17	136	64	289
8	11	88	64	121
7	13	91	49	169
7	17	119	49	289
7	20	140	49	400
6	13	78	36	169
5	9	45	25	81
3	15	45	9	225
$\Sigma x = 78$	**$\Sigma y =$ 150**	**$\Sigma xy =$ 1192**	**$\Sigma x^2 =$ 714**	**$\Sigma y^2 =$ 2468**

(i) Regression line of y on x

$$y-\bar{y}=b_{yx}(x-\bar{x})$$

Here,
$$b_{yx}=\frac{\Sigma xy-\dfrac{\Sigma x\Sigma y}{n}}{\Sigma x^2-\dfrac{(\Sigma x)^2}{n}}$$

$$=\frac{1192-\dfrac{78\times150}{10}}{714-\dfrac{78\times78}{10}}$$

$$=\frac{22}{105.6}=\frac{5}{24}$$

$$\bar{x}=\frac{\Sigma x}{n}=\frac{78}{10}=7.8$$

$$\bar{y}=\frac{\Sigma y}{n}=\frac{150}{10}=15$$

Equation of line of regression is

$$y-15=\frac{5}{24}(x-7.8)$$

$$24y-360=5x-39$$

$$\Rightarrow \qquad 24y=5x+321$$

(ii) At $x = 10$,

$$y-15=\frac{5}{24}(2.2)$$

$$\therefore \qquad y=\frac{11}{24}+15$$

$$=\frac{11+360}{24}$$

$$=\frac{371}{24}=15.4 \text{ (Approx.)}$$

SELF ASSESSMENT TEST

UNIT–IX

Maximum Marks : 18 **Timing : 1 Hour**

Q. 1. The following results were obtained with respect to two variable x and y :

$\Sigma x = 30$, $\Sigma y = 42$, $\Sigma xy = 199$,

$\Sigma x^2 = 184$, $\Sigma y^2 = 318$, $n = 6$.

Find the regression coefficients.

Q. 2. Let x and y be two variables with same mean. If the line of regression of y on x and x on y are respectively $y = ax + b$ and $x = \alpha y + \beta$, then find value of common mean.

Q. 3. The following table shows the sales and advertisement expenditure of a firm :

	Sales (in ₹ crore)	Advertisement expenditure (in ₹ crore)
Mean	40	6
Standard Deviation	10	1.5

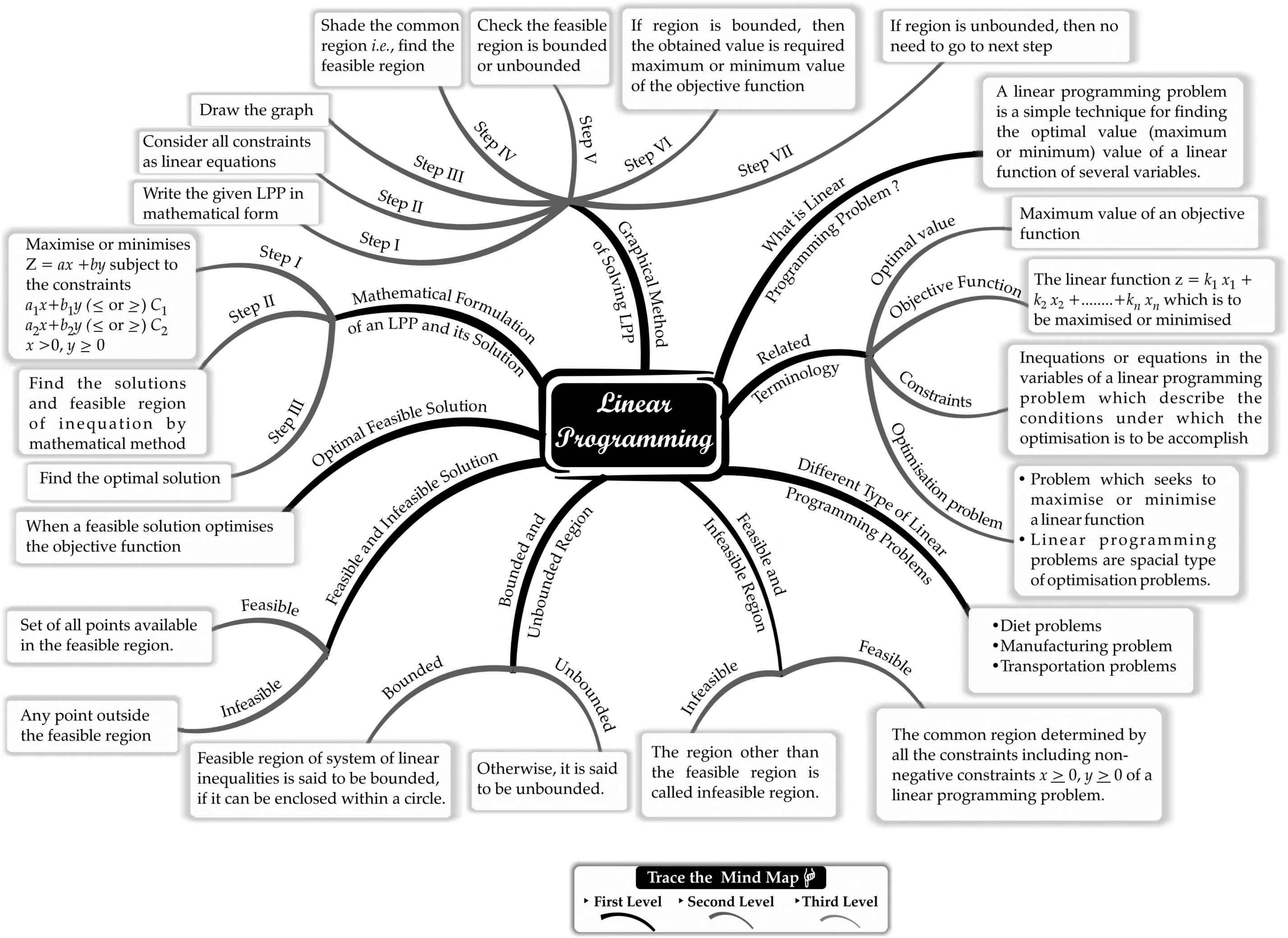
Coefficient of correlation, $r = 0.9$

Estimate the likely sales for a proposed advertisement expenditure of ₹ 10 crore.

Q. 4. If the regression line of y on x is $ax + by + c = 0$ and the regression line of x on y is $a_1x + b_1y + c_1 = 0$, then prove that : $ab_1 \leq a_1b$.

Q. 5. There are two series of index numbers : P for price index and S to stock of a commodity. The mean and standard deviation of P are 100 and 8 and of S are 103 and 4 respectively. The correlation coefficient between the two series is 0.4. With these data, obtain the regression lines of P on S and S on P.

OSWAAL EXAM TOOLS
OSWAAL PREPARATION TOOLS
Solutions for Self-assessment Test
Visit : *http://qr.w69b.com/g/pi8gdC0qk* OR Scan the Code



Linear Programming

- What is Linear Programming Problem ?
  - A linear programming problem is a simple technique for finding the optimal value (maximum or minimum) value of a linear function of several variables.
- Related Terminology
  - Optimal value
    - Maximum value of an objective function
  - Objective Function
    - The linear function $z = k_1 x_1 + k_2 x_2 + \ldots\ldots\ldots + k_n x_n$ which is to be maximised or minimised
  - Constraints
    - Inequations or equations in the variables of a linear programming problem which describe the conditions under which the optimisation is to be accomplish
  - Optimisation problem
    - Problem which seeks to maximise or minimise a linear function
    - Linear programming problems are spacial type of optimisation problems.
- Different Type of Linear Programming Problems
  - Diet problems
  - Manufacturing problem
  - Transportation problems
- Feasible and Infeasible Region
  - Feasible
    - The common region determined by all the constraints including non-negative constraints $x \geq 0, y \geq 0$ of a linear programming problem.
  - Infeasible
    - The region other than the feasible region is called infeasible region.
- Bounded and Unbounded Region
  - Bounded
    - Feasible region of system of linear inequalities is said to be bounded, if it can be enclosed within a circle.
  - Unbounded
    - Otherwise, it is said to be unbounded.
- Feasible and Infeasible Solution
  - Feasible
    - Set of all points available in the feasible region.
  - Infeasible
    - Any point outside the feasible region
- Optimal Feasible Solution
  - When a feasible solution optimises the objective function
- Mathematical Formulation of an LPP and its Solution
  - Step I
    - Maximise or minimises $Z = ax + by$ subject to the constraints $a_1x + b_1y\ (\leq \text{ or } \geq)\ C_1$ $a_2x + b_2y\ (\leq \text{ or } \geq)\ C_2$ $x > 0, y \geq 0$
  - Step II
    - Find the solutions and feasible region of inequation by mathematical method
  - Step III
    - Find the optimal solution
- Graphical Method of Solving LPP
  - Step I
    - Write the given LPP in mathematical form
  - Step II
    - Consider all constraints as linear equations
  - Step III
    - Draw the graph
  - Step IV
    - Shade the common region *i.e.*, find the feasible region
  - Step V
    - Check the feasible region is bounded or unbounded
  - Step VI
    - If region is bounded, then the obtained value is required maximum or minimum value of the objective function
  - Step VII
    - If region is unbounded, then no need to go to next step

Trace the Mind Map

UNIT-X
Linear Programming

CHAPTER
15
LINEAR PROGRAMMING

SYLLABUS

Introduction, Related terminology such as constraints, Objective function, Optimization, Different types of linear programming (L.P.) problems, Mathematical formulation of L.P. problems, Graphical method of solution for problems in two variables, Feasible and infeasible regions, (bounded and unbounded) feasible and infeasible solutions, Optimal feasible solutions (upto three non-trivial constraints).

Introduction, definition of related terminology such as constraints, objective function, optimization, advantages of linear programming limitation of linear programming; application areas of linear programming; different types of linear programming (LP) problems, mathematical formulation of LP problems, graphical method of solution for problems in two variables, feasible and infeasible regions, feasible and infeasible solutions, optimum feasible solution.

Trend Analysis

List of Concept	2016	2017	2018	2019	2020
Linear Programming	1 Q (6 marks)	1 Q (6 marks)	1 Q (6 marks)	1 Q (6 marks)	1 Q (6 marks)

Revision Notes

Scan to know more about this topic

LPP with Bounded Region

Scan to know more about this topic

LPP with Unbounded Region

- **Linear programming problems :** Problems which minimize or maximize a linear function Z subject to certain conditions determined by a set of linear inequalities with non-negative variables are known as linear programming problems.
- **Objective function :** A linear function $Z = ax + by$, where a and b are constants which has to be maximized or minimized according to a set of given conditions, is called as linear objective function.
- **Decision variables :** In the objective function $Z = ax + by$, the variables x, y are said to be decision variables.
- **Constraints :** The restrictions in the form of inequalities on the variables of a linear programming problem are called constraints. The condition $x \geq 0$, $y \geq 0$ are known as non-negative restrictions.
- **Feasible region :** The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of linear programming problem is known as the feasible region.
- **Feasible solution :** Points within and on the boundary of the feasible region represents feasible solutions of constraints. In the feasible region, there are infinitely many points (solutions) which satisfy the given conditions.
- **Theorem 1:** Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner p into (vertex) of the feasible region.
- **Theorem 2 :** Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function R has both maximum and minimum values of R and each of these occurs at a corner point (vertex) of R.

  However, if the feasible region is unbounded, the optimal value obtained may not be maximum or minimum.

CORNER OF EXTREME POINT METHOD

Step 1 : Formulate the linear programming problem in x and y with given conditions.

Step 2 : Convert the inequality constraints into equality constraints and plot each line on the graph paper.

Step 3 : Find the feasible region and check if the feasible region is bounded or unbounded.

Step 4 : Evaluate the value of the objective function Z at each corner point. Let M be the greatest and m be the smallest value of the objective function Z.

(i) When the feasible region is bounded : M and m are the maximum and minimum values of the objective function Z, respectively.

(ii) When the feasible region is unbounded :

(a) M is the maximum value of the objective function Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region.

Otherwise the objective function has no maximum value.

(b) m is the minimum value of the objective function Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region.

Otherwise the objective function has no minimum value.

Important Facts

- If two corner points of the feasible region are both optimal solutions of the same type (max or min), then any point on the line segment joining these two points is also an optimal of the same type.

➢ **Different types of linear programming problems :** A few important linear programming problems are as follows:

(i) Manufacturing problem : In such problem, we determine :

(a) Number of units of different products to be produced and sold.

(b) Manpower required, machines hours needed, warehouse space available, etc. Objective function is to maximize profit.

(ii) Diet problem : Here, we determine the amount of different types of constituent or nutrients which should be included in the diet.

Objective function is to minimize the cost of production.

(iii) Transportation problem : These problems deal with the cost of transportation which is to be minimized under given constraints.

(iv) Blending problem : To determine the optimum amount of several constituents used in producing a set of products while determining the optimum quantity of each product to be produced.

(v) Investment problem : To determine the amount of investment in fixed income securities to maximize the return on these investment.

➢ **Limitation of Linear Programming :**

(i) To specify an objective function in mathematical form is not an easy task.

(ii) Even if objective function is determined, it is difficult to determine social, institutional, financial and other constraints.

(iii) It is also possible that the objective function of constraints may not be directly specified by linear in equality equations.

Mnemonics

LPP Parameter

Mnemonics : NOC

Interpretation :

N	O	C
↓	↓	↓
Non-negative Variables	Objective Function	Constraints

4 Marks Questions

AI Q. 1. A carpenter has 90, 80 and 50 running feet respectively of teak wood, plywood and rosewood which is used to produce product *A* and product *B*. Each unit of product *A* required 2, 1 and 1 running feet and each unit of product *B* requires 1, 2 and 1 running feet of teak wood, plywood and rosewood respectively. If product *A* is sold for ₹48 per unit and product *B* is sold for ₹40 per unit, how many units of product *A* and product *B* should be produced and sold by the carpenter, in order to obtain the maximum gross income, Formulate the above as a Linear Programming Problem and solve it, indicating clearly the feasible region in the graph.

[ISC Specimen Paper, 2021] [ISC, 2019]

Sol. Let the number of units of product *A* and product *B* to be made are x and y respectively. Therefore, the problem can be written as:

Product	No. of units	Wood used per unit			Income per unit
		Teak	Plywood	Rosewood	
A	x	2	1	1	48
B	y	1	2	1	40
Available stock		90	80	50	

Maximise $Z = 48x + 40y$ subject to the constraints

$2x + y \leq 90, x + 2y \leq 80, x + y \leq 50$

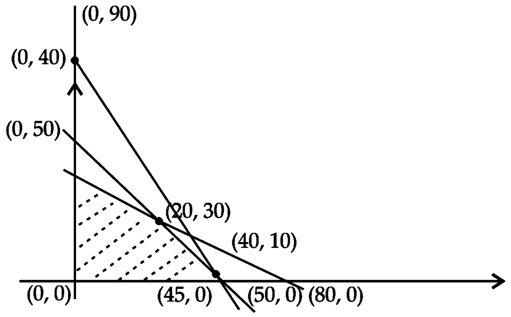
$x \geq 0, y \geq 0$

Its graphical solution is :

A at (0,0)	$Z = 0$
(45, 0)	$Z = 48 \times 45 + 40 \times 0 = 2160$
(40, 10)	$Z = 48 \times 40 + 40 \times 10 = 2320$
(20, 30)	$Z = 48 \times 20 + 40 \times 30 = 2160$
(0, 40)	$Z = 48 \times 0 + 40 \times 40 = 1600$

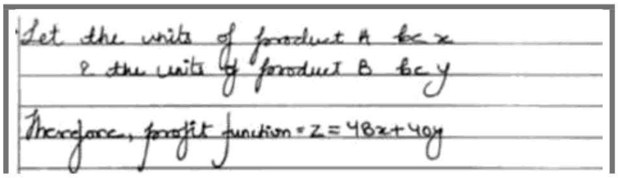
Hence, Z is maximum, 2320 at (40, 10)

$\therefore x = 40, y = 10$



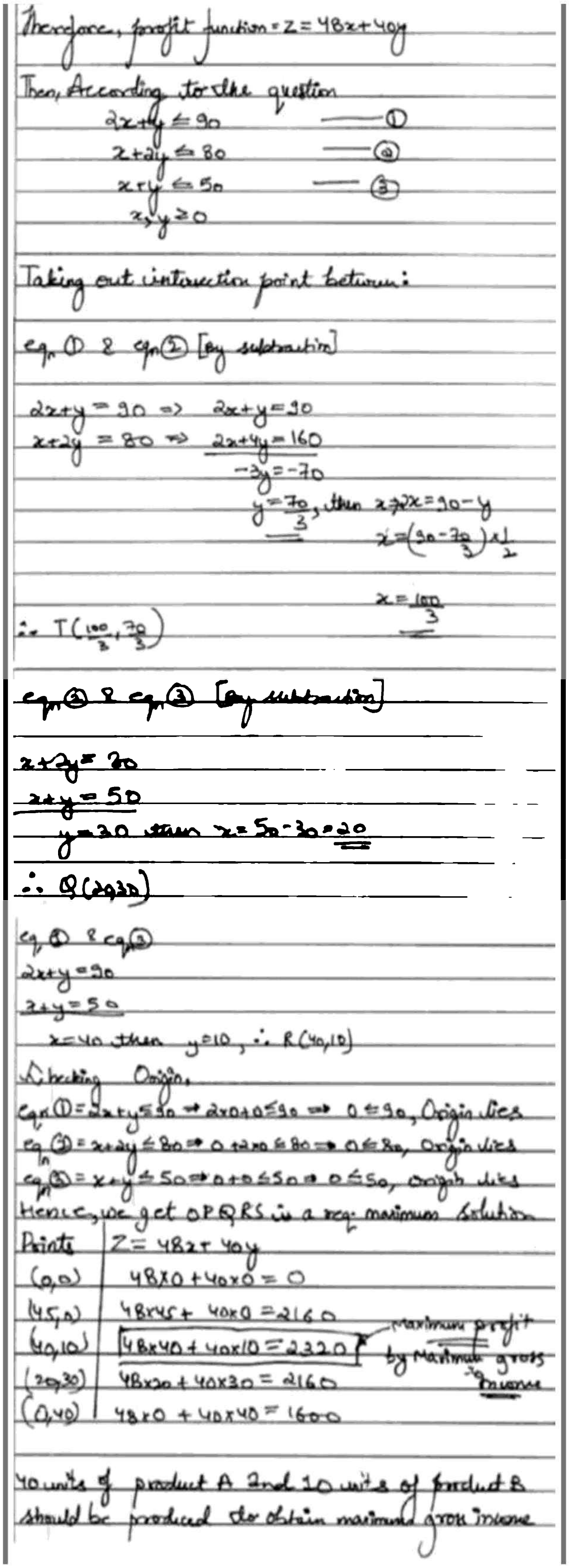
[ISC Marking Scheme, 2019]

OR



Let the units of product A be x
& the units of product B be y

Therefore, profit function = Z = 48x+40y



Therefore, profit function = Z = 48x+40y

Then, According to the question

$2x+y \leq 90$ —— ①
$x+2y \leq 80$ —— ②
$x+y \leq 50$ —— ③
$x, y \geq 0$

Taking out intersection point between:

eqn ① & eqn ② [by substraction]

$2x+y = 90 \Rightarrow 2x+y=90$
$x+2y = 80 \Rightarrow \underline{2x+4y=160}$
$-3y=-70$
$\underline{\underline{y=\frac{70}{3}}}$, then $2x=90-y$
$x=\left(90-\frac{70}{3}\right)\times\frac{1}{2}$
$\underline{\underline{x=\frac{100}{3}}}$

$\therefore T\left(\frac{100}{3}, \frac{70}{3}\right)$

eqn ② & eqn ③ [by substraction]

$x+2y= 80$
$\underline{x+y= 50}$
$y=30$ then $x= 50-30=\underline{\underline{20}}$

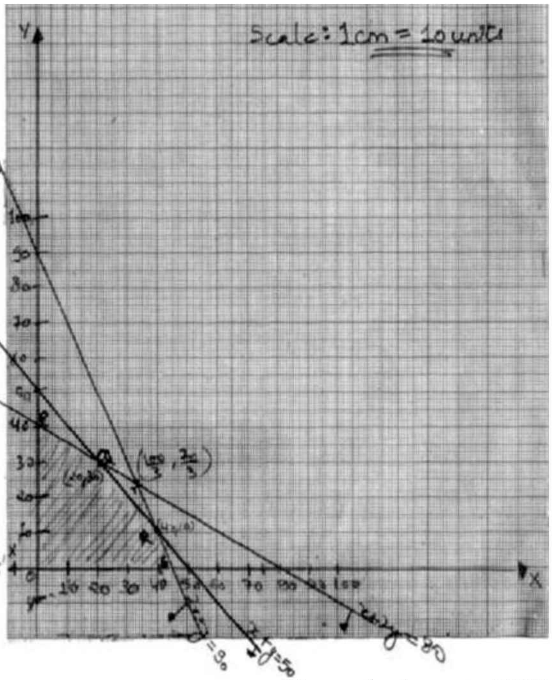
$\therefore Q(20,30)$

eq ① & eq ③
$2x+y=90$
$\underline{x+y=50}$
$x=40$ then $y=10$, $\therefore R(40,10)$

Checking Origin.
eqn ① = $2x+y\leq 90 \Rightarrow 2\times0+0\leq 90 \Rightarrow 0\leq 90$, Origin lies
eqn ② = $x+2y\leq 80 \Rightarrow 0+2\times0\leq 80 \Rightarrow 0\leq 80$, Origin lies
eqn ③ = $x+y\leq 50 \Rightarrow 0+0\leq 50 \Rightarrow 0\leq 50$, Origin lies
Hence, we get OPQRS is a req. maximum Solution

Points	$Z= 48x+40y$
(0,0)	$48\times0+40\times0=0$
(45,0)	$48\times45+40\times0=2160$
(40,10)	$\boxed{48\times40+40\times10=2320}$ maximum profit by Maximum gross income
(20,30)	$48\times20+40\times30=2160$
(0,40)	$48\times0+40\times40=1600$

40 units of product A and 10 units of product B should be produced to obtain maximum gross income



[Topper's Answer, 2019]

Examiner's Comment

- Many candidates could not express the given constraints in the form of linear inequalities correctly. Several candidates did not write non-negative constraints x ≥ 0 and $y \geq 0$. A few candidates did not indicate the feasible region on the graph correctly.

Answering Tip

- Revise the concept of solving linear inequalities
- Do adequate practice of solving different types of Linear Programming problems.

Q. 2. A farm is engaged in breeding hens. The hens are fed products *A* and *B* grown in the farm which contains nutrients *P*, *Q* and *R*. One kilogram of product *A* contains 36 units, 3 units and 20 units of nutrients *P*, *Q* and *R* respectively, whereas one kilogram of product *B* contains 6 units, 12 units and 10 units of nutrients *P*, *Q* and *R* respectively. The minimum requirement of nutrients *P*, *Q* and *R* for a hen is 108, 36 and 100 units respectively. Product *A* costs ₹20 per kilogram and product *B* costs ₹40 per kilogram. Using linear programming, find the number of kilograms of products *A* and *B* to be produced to minimize the total cost. Identify the feasible region from the rough sketch.

[ISC Specimen Paper, 2021]

Sol. Let, x kg of product A be mined with y kg of product B.

$\therefore \quad z = 20x + 40y$

Subject to the constraints :

$36x + 6y \geq 108$

$\Rightarrow \quad 6x + y \geq 18$

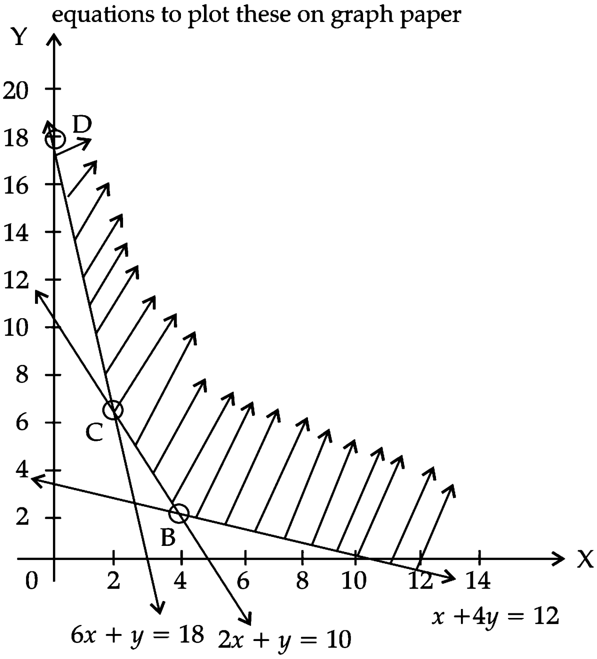
$3x + 12y \geq 36 \Rightarrow x + 4y \geq 12$

$20x + 10y \geq 100 \Rightarrow 2x + y \geq 10$

$x \geq 0$

$y \geq 0$

No considering these in equations in terms of equations to plot these on graph paper



$\therefore$ The corner points are $A(12, 0)$, $B(4, 2)$, $C(2,6)$ and $D(0, 18)$.

$\therefore$ At $A(12, 0)$; $z = 20 \times 12 = 240$

At $B(4, 2)$; $z = 20 \times 4 + 40 \times 2 = 160$

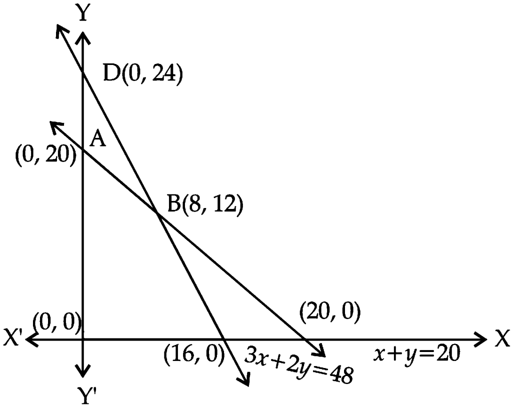
At $C(2, 6)$; $z = 20 \times 2 + 40 \times 6 = 280$

At $D(0, 18)$; $z = 40 \times 18 = 280$

$\therefore$ 4 kg of product A and 2 kg of product B be mixed together so that the cost will be minimum and minimum cost = ₹160

AI Q. 3. A dealer in rural area wishes to purchase a number of sewing machines. He has only ₹ 5760 to invest and has space for at most 20 items of storage. An electronic sewing machine cost him ₹ 360 and a manually operated sewing machine ₹ 240. He can sell an electronic sewing machine at a profit of ₹ 22 and a manually operated sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit ? Make it as a LPP and solve it graphically.

Sol. Let x and y be electronic and manually operated sewing machine purchased respectively.



$\therefore$ LPP is

Maximize $\quad Z = 22x + 18y$

Subject to,

$$360x + 240y \le 5760$$

$$3x + 2y \le 48$$

or $$x + y \le 20$$

$$x \ge 0, y \ge 0$$

For equation $3x + 2y = 48$

x	16	0
y	0	24

For equation $x + y = 20$

x	0	20
y	20	0

Vertices of feasible region are :

$A(0, 20)$, $B(8, 12)$, $C(16,0)$ and $O(0, 0)$, where $B(8, 12)$ is the point of intersection of equation $3x + 2y = 48$ and $x + y = 20$

Value of Z at corner points

Corner points	Value of objective function $Z = 22x + 18y$
$O(0, 0)$	$Z = 0$
$A(0, 20)$	$Z = 360$
$B(8, 12)$	$Z = 392 \rightarrow$ Max
$C(16, 0)$	$Z = 352$

$\therefore$ For maximum Z, Electronic machines = 8,
Manual machines = 12.

Q. 4. If a young man rides his motor-cycle at 25 km per hour, he had to spend ₹ 2 per km on petrol with very little pollution in the air. If he rides it at a faster speed of 40 km per hour, the petrol cost increases to ₹ 5 per km and rate of pollution also increase. He has ₹ 100 to spend on petrol and wishes to find what is the maximum distance he can travel within one hour. Express this problem as an L.P.P. Solve it graphically to find the distance to be covered with different speeds.

Sol. Let the young man drives x kms and y kms at 25 km/h and 40 km/h speed respectively, then the LPP is

Maximize Distance :

$$Z = x + y$$

Subject to

$$2x + 5y \le 100$$

$$\frac{x}{25} + \frac{y}{40} \le 1$$

or $$8x + 5y \le 200$$

$$x, y \ge 0$$

$$2x + 5y = 100 \qquad \text{...(i)}$$

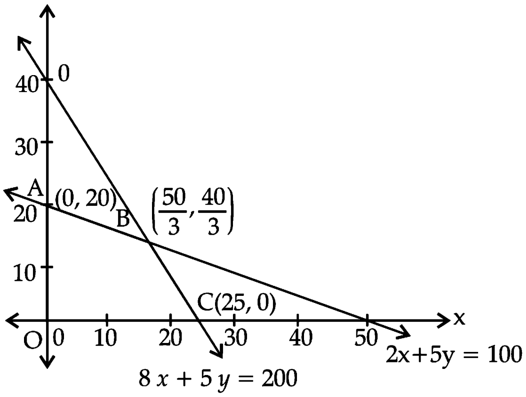
x	0	50
y	20	0

and $$8x + 5y = 200 \qquad \text{...(ii)}$$

x	0	25
y	40	0

Also, point of intersection of eqs. (i) and (ii) is obtained by solving simultaneously eq(i) and (ii), we get $B\left(\frac{50}{3}, \frac{40}{3}\right)$

We draw the two constrains on the graph paper and indicate the feasible region. The *OABC* represents the set of all feasible solutions.



Feasible region is *OABC* with,

Corner	Value of Z
(0, 20)	20
$\left(\frac{50}{3}, \frac{40}{3}\right)$	30 (Max.)
(25, 0)	25

$\therefore$ Max distance covered is 30 kms with $\frac{50}{3}$ km at 25 km/h and $\frac{40}{3}$ km at 40 km/h speed, respectively.

6 Marks Questions

Q. 1. A company uses three machines to manufacture two types of shirts, half sleeves and full sleeves. The number of hours required per week on machine M_1, M_2 and M_3 for one shirt of each type is given in the following table :

	M_1	M_1	M_3
Half sleeves	1	2	8/5
Full sleeves	2	1	8/5

None of the machines can be in operation for more than 40 hours per week. The profit on each half sleeve shirt is ₹1 and the profit on each full sleeve shirt is ₹1.50. How many of each type of shirts should be made per week to maximize the company's profit? **[ISC, 2020]**

Sol. Let number of half-sleeves shirts be x and full-sleeves shirts be y

Objective function $Z = 1x + 1.5y$

Constraints $x + 2y \le 40$

$$2x + y \le 40$$

$$\frac{8}{5}x + \frac{8}{5}y \le 40 \text{ or } x + y \le 25$$

$x + 2y = 40$

x	0	40	10
y	20	0	15

$2x + y = 40$

x	0	20	10
y	40	0	20

$x + y = 25$

x	0	25	10
y	25	0	15

Maximum profit at (0, 0)

$$Z = x + 1.5y$$
$$= 0 + 1.5 \times 0 = ₹0$$

Maximum profit at (0, 20)

$$Z = 0 + 1.5 \times 20$$
$$= 0 + 30 = ₹30$$

Maximum profit at (10, 15)

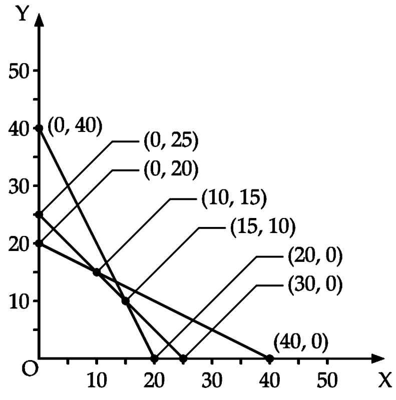
$$Z = 10 + 1.5 \times 15$$
$$= 10 + 22.5 = ₹32.5$$

Maximum profit at (15, 10)

$$Z = 15 + 1.5 \times 10$$
$$= 15 + 15 = ₹30$$

Maximum profit at (20, 0)

$$Z = 20 + 1.5 \times 0$$
$$= 20 + 0 = 0$$



Hence, Z is maximum at (10, 15)

$$Z = 32.5.$$

[ISC Marking Scheme, 2020]

Examiner's Comment

- A number of candidates could not form the correct inequalities subject to given constraints. Some noted incorrect corner points to find optimum value. A few candidates did not draw rough sketch of the feasible region.

Answering Tip

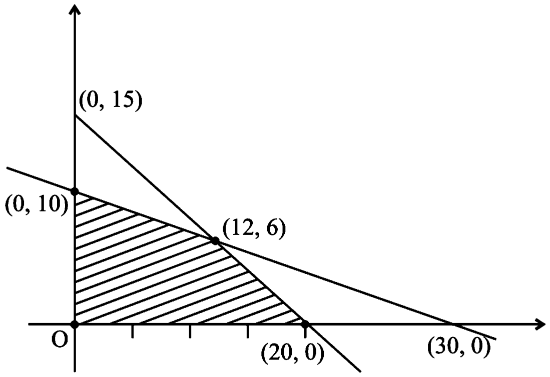
- Do adequate practice to frame inequalities using given constraints and finding corner points by solving inequalities.

AI **Q. 2. A manufacturing company makes two types of teaching aids A and B of Mathematics for class X. Each type of A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type A and ₹ 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Formulate this as Linear Programming Problem and solve it. Identify the feasible region from the rough sketch.**

[ISC, 2018]

Sol. Let x and y be the number of teaching aids of type A and type B respectively

$$z = 80x + 120y$$
$$9x + 12y \le 180$$
$$x + 3y \le 30$$
$$x \ge 0 \text{ and } y \ge 10$$



On solving equations,

$$9x + 12y \le 180$$
$$x + 3y \le 30$$

(0, 0), (20, 0), (0, 10) and (12, 6)

At	
(0, 0)	$z = 80.0 + 120.0 = 0$
(20, 0)	$z = 80.20 + 120.0 = 1600$
(0, 10)	$z = 80.0 + 120.10 = 1200$
(12, 6)	$z = 80.12 + 120.6 = 1680$

Hence z is maximum at (12, 6)

$\therefore x = 12$ and $y = 6$

Maximum z = ₹ 1680. **[ISC Marking Scheme, 2018]**

Detailed Answer :

Let the number of two types of teaching aids A and B be x and y, respectively.

Write the given data in tabular for mas following

Item	No of aids	Time on fabricating (in hrs)	Time on finishing (in hrs)	Profit (in ₹)
A	x	$9x$	x	$80x$
B	y	$12y$	$3y$	$120y$
Total	$x + y$	$9x + 12y$	$x + 3y$	$80x + 120y$
Availability		180	30	

The profit on type A is ₹ 80 and type B is ₹ 120. Thus, the required LPP is

$$\text{Max } Z = 80x + 120y \quad \text{...(i)}$$

Subject to constraints

$$9x + 12y \le 180 \text{ or } 3x + 4y = 60 \quad \text{...(ii)}$$
$$x + 3y \le 30 \quad \text{...(iii)}$$
$$x \ge 0, y \ge 0$$

Consider the equation,

$$3x + 4y = 60$$

x	0	20
y	15	0

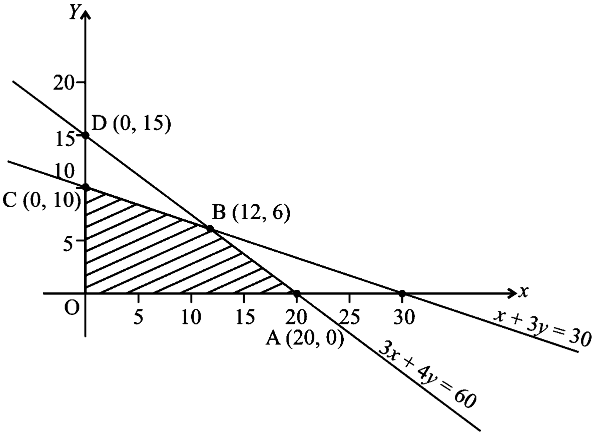
and $$x + 3y = 30$$

x	0	30
y	10	0

On solving eq, (ii) and (iii), we get

$x = 12$ and $y = 6$

The graphical representation of the system of inequations as shown below



From graph, feasible region is

OABCO, whose corner points are (0, 0), *A*(20, 0), *B*(12, 6) and *C*(0, 10).

The values of *Z* at corner points are as follows

Corner Points	Value of $Z = 80x + 120y$
$O(0, 0)$	$Z = 80 \times 0 + 120 \times 0 = 0$
$A(20, 0)$	$Z = 80 \times 20 + 120 \times 0 = 1600$
$B(12, 6)$	$Z = 80 \times 12 + 120 \times 6 = 1680$
$C(0, 10)$	$Z = 80 \times 0 + 120 \times 10 = 1200$

From the table, the maximum value of *Z* is ₹ 1680. Hence, 12 pieces of type *A* and 6 pieces of type *B* should be manufactured per week to get maximum profit of ₹ 1680 per week.

Q. 3. A toy company manufactures two types of dolls *A* and *B*. Market test and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demands for the dolls of type *B* is almost half of that for dolls of type *A*. Further, the production level of type *A* can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of ₹ 12 and ₹ 16 per doll respectively on dolls *A* and *B*, how many of each type of dolls should be produced weekly, in order to maximise the profit ? [ISC Specimen Paper, 2018]

Sol. Let number of dolls $A = x$

and number of dolls $B = y$

According to question

Profit function $\quad Z = 12x + 16y$

Subject to constraints :

$$x + y \leq 1200$$
$$x - 3y \leq 600$$
$$y \leq \frac{x}{2}$$
$$x, y \geq 0$$

$$x + y = 1200 \qquad \ldots(i)$$

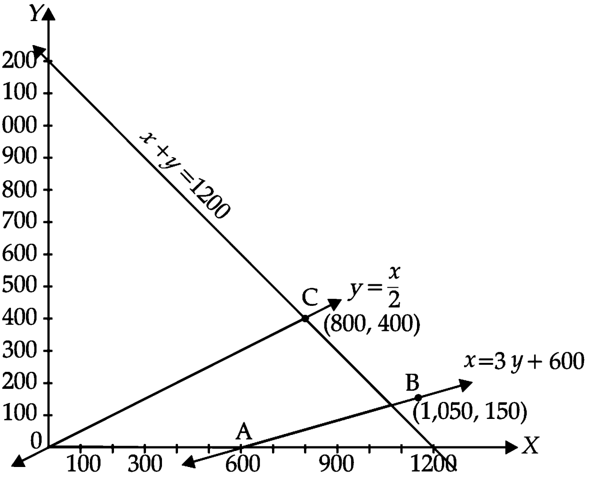
x	0	1200
y	1200	0

$$x \leq 3y + 600 \qquad \ldots(ii)$$

x	600	1200
y	0	200

$$y \leq \frac{x}{2} \qquad \ldots(iii)$$

x	0	600	1200
y	0	300	600



OABC is the feasible region

$$Z = 12x + 16y$$

Points	Z
$O(0,0)$	0
$A(600, 0)$	7200
$B(1050, 150)$	15,000
$C(800, 400)$	16,000

Maximum profit occurs at *C*(800, 400)

∴ No. of dolls *A* produced = 800

No. of dolls *B* produced = 400

Q. 4. A farmer has a supply of chemical fertilizer of type *A* which contains 10% nitrogen and 6% phosphoric acid and of type *B* which contains 5% nitrogen and 10% phosphoric acid. After soil test, it is found that atleast 7 kg of nitrogen and same quantity of phosphoric acid is required for a good crop. The fertilizer of type *A* costs ₹ 5.00 per kg and the type *B* costs ₹ 8.00 per kg. Using Linear programming, find how many kilograms of each type of the fertilizer should be bought to meet the requirement and for the cost to be minimum. Find the feasible region in the graph. [ISC, 2017]

Sol. Let x kg of type *A* fertilizer and y kg of type *B* fertilizer required.

Type A cost per kg = ₹ 5, type B cost per kg = ₹ 8

$C = 5x + 8y$ subject to the following constraints :

10% of x + 5% of $y \geq 7$ and 6% of x + 10% of $y \geq 7$, $x \geq 0$ and $y \geq 0$

$$10x + 5y \geq 700$$

$$2x + y \geq 140 \quad ...(i)$$

$$6x + 10y \geq 700$$

$$3x + 5y \geq 350 \quad ...(ii)$$

$$2x + y \geq 140 \Rightarrow x = 70, y = 0 \Rightarrow (70, 0)$$

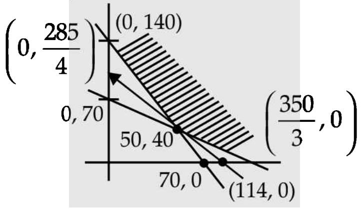
$$\Rightarrow x = 0, y = 140 \Rightarrow (0,140)$$

$$3x + 5y \geq 350 \Rightarrow x = \frac{350}{3}, y = 0$$

$$\Rightarrow \left(\frac{350}{3}, 0\right)$$

$$\Rightarrow x = 0, y = 70 \Rightarrow (0, 70)$$

Solving $2x + y = 140$ and $3x + 5y = 350$, we get $x = 50$ and $y = 40$



The shaded region is the feasible region.

$$C = 5x + 8y$$

At $\left(\frac{350}{3}, 0\right)$, $\quad C = 5 \cdot \frac{350}{3} + 0 = 583\frac{1}{3}$

At (50, 40), $\quad C = 5 \times 50 + 8 \times 40 = 570$

At (0, 140), $\quad C = 5 \times 0 + 8 \times 140 = 1120$

So, it is minimum at (50, 40)

∴Minimum cost is ₹ 570 for 50 kg of type *A* and 40 kg of type B fertilisers procured.

[ISC Marking Scheme, 2017]

Examiner's Comment

- Many candidates made errors in calculating the conditions using constraints. Some candidates were unable to represent feasible reason on the graph as such they may not have clarity of the concept of solving linear equations graphically. Some candidates solved it in terms of decimals and mostly went incorrect in simplifications.

Answering Tip

- The optimum function and all possible constraints in the form of inequations must be put down from what is stated in the problem.

Detailed Answer :

Let the fertilizer of type *A* be x kg and type *B* be y kg.

Fertilizer	Nitrogen	Phosphoric Acid	Quantity	Cost
A *B*	10% 5%	6% 10%	x kg y kg	₹ 5 ₹ 8
Total	7 kg	7 kg		

According to Question :

Nitrogen	Phosphoric Acid
Quantity of Nitrogen in *A* = 10% Quantity of Nitrogen in *B* = 5% Minimum available = 7 kg ∴ 10% of x + 5% of $y \geq 7$ $\frac{10}{100}x + \frac{5}{100}y \geq 7$ $2x + y \geq 140$ and $x \geq 0, y \geq 0$	Quantity of Phosphoric Acid in *A* = 6% Quantity of Phosphoric Acid in *B* = 10% Minimum availability = 7 kg ∴ 6% of x + 10% of $y \geq 7$ $\frac{60}{100}x + \frac{10}{100}y \geq 7$ $3x + 5y \geq 350$ $x \geq 0, y \geq 0$

∴ $\quad$ Min $Z = 5x + 8y$

Subject to constraints

$$2x + y \geq 140$$

$$3x + 5y \geq 350$$

$$x \geq 0, y \geq 0$$

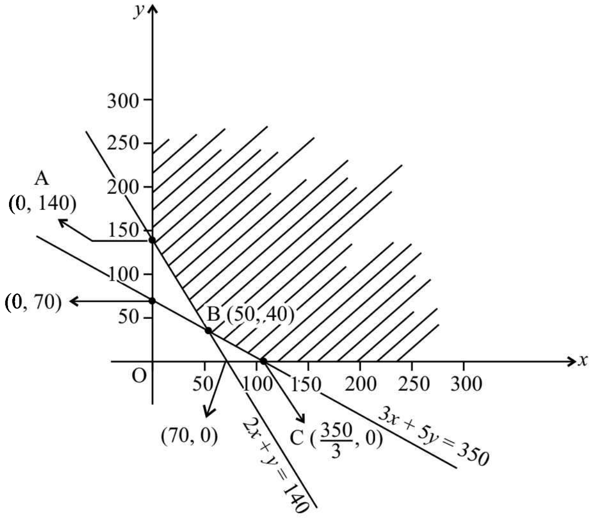
$$2x + y = 140 \quad ...(i)$$

x	70	0
y	0	140

$$3x + 5y = 350 \quad ...(ii)$$

x	0	$\frac{350}{3}$
y	70	0

On solving eq (i) and (ii), we get $x = 50$ and $y = 40$



We draw the lines $2x + y = 140$ and $3x + 5y = 350$ and obtain the feasible region (unbounded and convex) as shown in figure.

Thus, corner points are *A*(0, 140), *B*(50, 40), $C\left(\frac{350}{3}, 0\right)$.

The values of Z at these points are given in following table :

Corner Points	Objective function, $Z = 5x + 8y$
$A(0, 140)$	$Z = 5 \times 0 + 8 \times 140 = 1120$
$B(50, 40)$	$Z = 5 \times 50 + 8 \times 40 = 570 \rightarrow$ Minimum
$C\left(\frac{350}{3}, 0\right)$	$Z = 5\times\frac{350}{3} + 8 \times 0 = 583\frac{1}{3}$

As the feasible area is unbounded, Hence, 500 may or may not be the minimum value of Z.

For this, we need to graph inequality.

$$5x + 8y < 570$$

x	114	0
y	0	71.25

Since, there is not common point between the feasible region and $5x + 8y < 570$

Hence, the cost will be minimum, if

Fertilizer of type A used = 50 kg

Fertilizer of type B used = 40 kg

Minimum cost = ₹ 570.

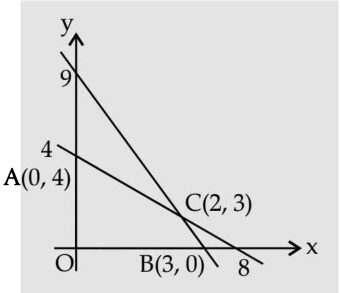
AI Q. 5. A company manufacturers two types of products *A* and *B*. Each unit of *A* requires 3 gram of nickel and 1 gram of chromium, while each unit of *B* requires 1 gram of nickel and 2 gram of chromium. The firm can produce 9 gram of nickel and 8 grams of chromium. The profit is ₹ 40 on each unit of product of type A and ₹ 50 on each unit of type *B*. How many units of each type should the company manufactures so as to earn maximum profit? Use linear programming to find the solution.

[ISC, 2016]

Sol. Let x units of product A and y units of product B.

Maximize $Z = 40x + 50y$

Subject to constraints



$$3x + y \leq 9$$
$$x + 2y \leq 8$$
$$x \geq 0, y \geq 0$$

On solving, we get

$A\ (0, 4), B(3, 0), C\ (2, 3)$

At A, $Z = 40 \times 0 + 50 \times 4 =$ ₹ 200

B, $Z = 40 \times 3 + 50 \times 0 =$ ₹ 120

C, $Z = 40 \times 2 + 50 \times 3 =$ ₹ 230

Maximum profit is ₹ 230, when 2 units of type A and 3 units of type B are produced.

[ISC Marking Scheme, 2016]

Examiner's Comment

- In some cases, all the constraints were not used and hence coordinates of all feasible points were not obtained.

Answering Tip

- The optimum function and all possible constraints in the form of inequations have to be put down from what is stated in the problem. Students should solve the constraints equations in pairs to obtain all feasible points leading to maximum or minimum value of desired function.

Detailed Answer :

Type of Product	Total items		Profit per unit
	9 gms Nickel (gm)	8 gms Chromium (gm)	
A	3	1	₹ 40
B	1	2	₹ 50

Let x = Number of units of type A

y = Number of units of type B.

Maximize $Z = 40x + 50y$

Subject to the constraints,

$$3x + y \leq 9$$
$$x + 2y \leq 8$$

and $x >. 0, y \geq 0$

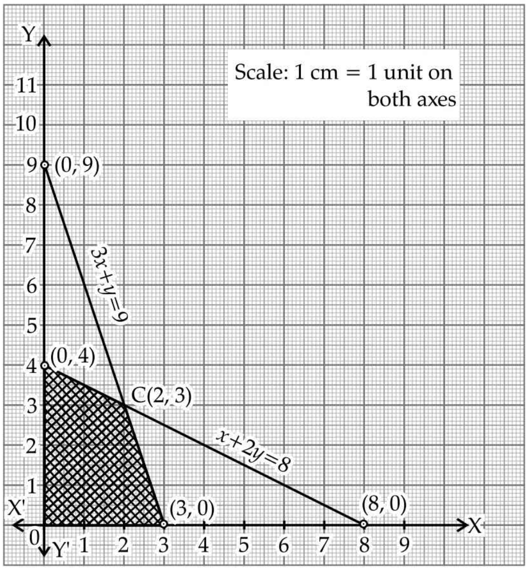
Consider the equation,

$$3x + y = 9$$

x	0	3	2
y	9	0	3

and $x + 2y = 8$

x	0	8	2
y	4	0	3



Scale: 1 cm = 1 unit on both axes

Shaded region in the diagram represents the feasible solution.

Now, we can determine the maximum value of Z by evaluating the value of Z at the four points (vertices) as shown below.

Vertices	$Z = 40x + 50y$
(0, 0)	$Z = 40 \times 0 + 50 \times 0 =$ ₹ 0
(3, 0)	$Z = 40 \times 3 + 50 \times 0 =$ ₹ 120
(0, 4)	$Z = 40 \times 0 + 50 \times 4 =$ ₹ 200
(2, 3)	$Z = 40 \times 2 + 50 \times 3 =$ ₹ 230

From graph,

Maximum profit, Z = ₹ 230

$\therefore$ Number of units of type A is 2 and number of units of type B is 3.

Q. 6. A dietician wishes to mix two kinds of food X and Y in such a way that the mixture contains atleast 10 units of vitamin A, 12 units B and 8 units of vitamin C. The vitamin contents of one kg food is given below :

Food	Vitamin A	Vitamin B	Vitamin C
X	1 units	2 units	3 units
Y	2 units	2 units	1 units

One kg of food X costs ₹ 24 and one kg of food Y costs ₹ 36. Using Linear Programming, find the least cost of the total mixture which will contain the required vitamins. [ISC, 2015]

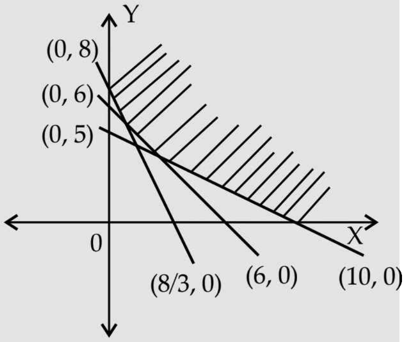
Sol. Let there be x units of food X and y units of food Y.

Min $Z = 24x + 36y$

Subject to the constraints

$$x + 2y \geq 10$$
$$2x + 2y \geq 12$$
$$3x + y \geq 8$$
$$x \geq 0, y \geq 0$$

x	y	Z (cost)
10	0	240
0	8	288
1	5	204
2	4	192 Min cost



[ISC Marking Scheme, 2015]

Examiner's Comment

- Many candidates took incorrect inequality sign, hence they got incorrect feasible region and their corner points were also incorrect. Some candidates did not show any graphical representation of the inequalities. In some cases, the representation of the problem was not up to the mark, and the work was not systematic resulting in candidates missing the point of minimum cost.

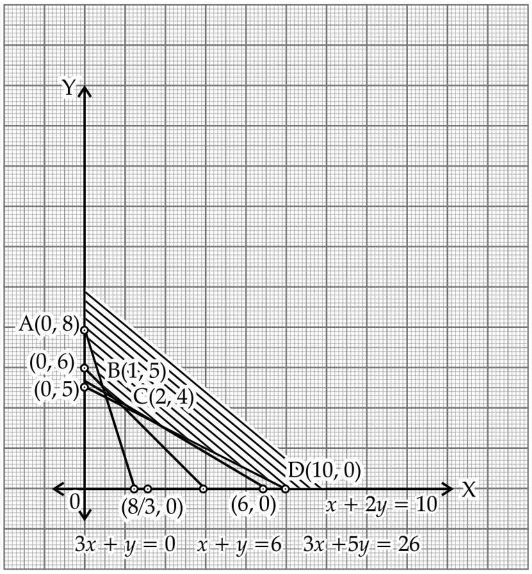
Answering Tip

- Students should practice the sketching of lines. They should practice to express the line in intercept from *i.e.* $\frac{x}{a} + \frac{y}{b} = 1$, so that sketching will be easy. Correct feasible region and its plotting is important.

Detailed Answer :

Let x be the number of units of food X and y be the number of units of food Y be mixed to obtain the desire diet. Then LPP of the given problem is :

Minimise, $Z = 24x + 36y$



Subject to the constraints,

$x + 2y \geq 10, 2x + 2y \geq 12$ *i.e.*, $x + y \geq 6$ and

$3x + y \geq 8$

$x \geq 0, y \geq 0$

We draw the lines $x+2y=10$, $x+y = 6$, $3x + y = 8$ and obtain the feasible region (unbounded and convex) shown in the figure.

Thus, corner points are $A(0, 8)$, $B(1, 5)$, $C(2, 4)$ and $D(10, 0)$.

The values of Z (in *rs.*) at these points are given in the following table :

Corner Point	Objective Function $Z = 24x + 36y$
A (0, 8)	$Z = 24 \times 0 + 36 \times 8 = 288$
B (1, 5)	$Z = 24 \times 1 + 36 \times 5 = 204$
C (2, 4)	$Z = 24 \times 2 + 36 \times 4 = 192$
D (10, 0)	$Z = 24 \times 10 + 36 \times 0 = 240$

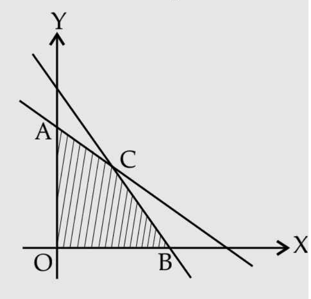
As the feasible region is unbounded, we draw the graph of the half plane $6x + 10y < 52$ *i.e.,* $3x + 5y < 26$ and note that there is no point common with the feasible region. It occurs at $C\ (2, 4)$.

i.e., when 2 kg of food X and 4 kg of food Y are mixed to get the desired diet.

Q. 7. A company manufactures two types of toys *A* and *B*. A toy of type *A* requires 5 minutes for cutting and 10 minutes for assembling. *A* toy of type *B* requires 8 minutes for cutting and 8 minutes for assembling. There are 3 hours available for cutting and 4 hours available for assembling the toys in a day. The profit is ₹ 50 each on a toy of type *A* and ₹ 60 each on a toy of type *B*. How many toys of each type should the company manufacture in a day to maximize the profit ? Use linear programming to find the solution. [ISC, 2014]

Sol. Let x units of type A toys and y units of type B toys.

Maximize $Z = 50x + 60y$



Subject to constraints

$$5x + 8y \leq 180$$
$$10x + 8y \leq 240$$
$$x \geq 0, y \geq 0$$

Solving graph, we get $A\ (0, 22.5)$, $B\ (24, 0)$, $C\ (12, 15)$

At A, $Z = 50 \times 0 + 60 \times 22.5 =$ ₹ 1350

At B, $Z = 50 \times 24 + 60 \times 0 =$ ₹ 1200

At C, $Z = 50 \times 12 + 60 \times 15 =$ ₹ 1500

Maximum profit is ₹ 1500, when 12 units of type A and 15 unit of type B is produced.

[ISC Marking Scheme, 2014]

Examiner's Comment

- Most mistakes happened while expressing the constraints in the form of equations with both sides expressed in the same units. Some candidates omitted the graphical representation of the in equations and erred in finding the feasible area and points.

Answering Tip

- Linear programming as a topic requires knowledge of line sketching. Hence, a little revision in this matter is essential. Correct depiction and shading of required area is a big advantage. Neat diagram help a lot. Casual approach is to be avoided and little extra practice is required.

Detailed Answer :

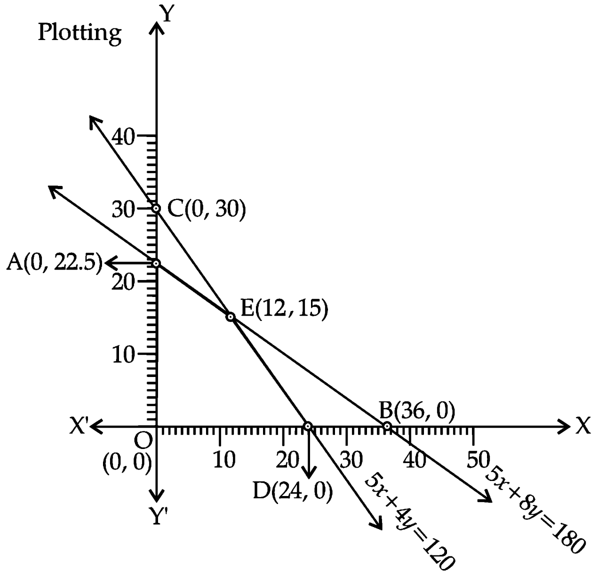
	Toy *A*	**Toy *B***	**Time in a day**
Cutting time	5 min	8 min	180 min
Assembling time	10 min	8 min	240 min
Profit	₹ 50	₹ 60	
Assumed quantity	x	y	

Objective function max. $Z = 50x + 60y$

$$10x + 8y \leq 240$$

or
$$5x + 4y \leq 120$$
$$x \geq 0, y \geq 0$$
$$5x + 8y \leq 180$$

Plotting



$5x + 8y = 180$

	A	*B*	*E*
x	0	36	12
y	22.5	0	15

$5x + 4y = 120$

	C	*D*	*E*
x	0	24	12
y	30	0	15

Corner points	**$Z = 50x + 60y$**
At $O(0, 0)$	0
At D (24, 0)	1200
At E (12, 15)	1500
At A (0, 22.5)	1350

Hence the maximum profit is ₹1500 at $E(12, 15)$. Therefore 12 units of type A and 15 units of types B is produced.

AI **Q. 8. A mill owner buys two types of machines *A* and *B* for his mill. Machines *A* occupies 1,000 sq.m of area**

and requires 12 men to operate it; while machine *B* occupies 1,200 sq.m of area and requires 8 men to operate it. The owner has 7,600 sq.m of area available and 72 men to operate the machines. If machine *A* produces 50 units and machine *B* produces 40 units daily, how many machines of each type should he buy to maximize the daily output ? Use Linear Programming to find the solution. [ISC, 2013]

Sol. The data given in the problem are as under :

	Machine *A*	Machine *B*	Maximum available
Area needed	1,000 sq.m	1,200 sq.m	7,600 sq.m
Labour force	12	8	72
Daily output	50 units	40 units	—

Let x and y be the number of machines A and B respectively

$1,000x + 1,200y \le 7,600$

or $10x + 12y \le 76$

or $5x + 6y \le 38$ ...(i)

$12x + 8y \le 72$

or $3x + 2y \le 18$ ...(ii)

$x \ge 0, y \ge 0$

Total output $Z = 50x + 40y$

Max $Z = 50x + 40y$ subject to constraints

$5x + 6y \le 38$

$3x + 2y \le 18$

$x \ge 0, y \ge 0$

$3x + 2y = 18$

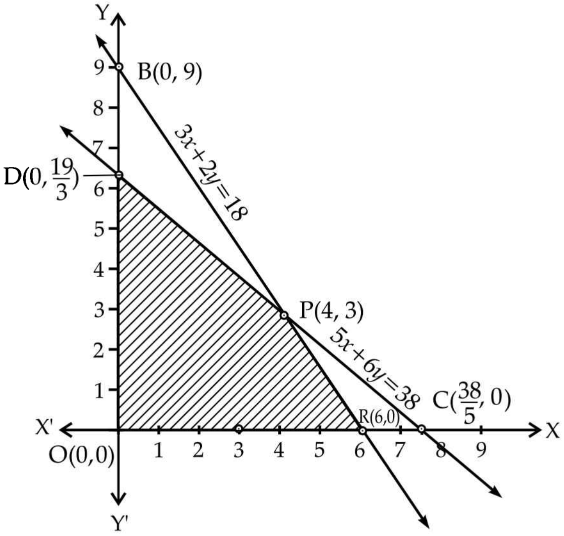
x	6	0	4
y	0	9	3

$5x + 6y = 38$

x	$\frac{38}{5}$	0	4
y	0	$\frac{19}{3}$	3

The vertices of the feasible region $ORPD$ are $O(0, 0)$, $R(6, 0)$, $P(4, 3)$ and $D\left(0, \frac{19}{3}\right)$.

The graph of these in equations is shown below :



Points	Value of $Z = 50x + 40y$
At $O(0, 0)$	$Z = 0$
At $R(6, 0)$	$Z = 50 \times 6 + 0 = 300$
At $P(4, 3)$	$Z = 50 \times 4 + 40 \times 3 = 320$
At $D\left(0, \frac{19}{3}\right)$	$Z = 50 \times 0 + 40 \times \frac{19}{3} = \frac{760}{3} = 253.33$

Thus, we see that Z is maximum at (4, 3)

Number of machine $A = 4$

Number of machine $B = 3$.

Q. 9. Two tailors *P* and *Q* earn ₹150 and ₹200 per day respectively. *P* can stitch 6 shirts and 4 trousers a day, while *Q* can stitch 10 shirts and 4 trousers per day. How many days should each work to produce at least 60 shirts and 32 trousers at minimum labour cost? [ISC, 2012]

Sol. Let the tailor P work for x days and the tailor Q work for y days respectively.

Here, the problem can be formulated as an L.P.P. as follows :

Minimize $Z = 150x + 200y$

Subject to the constraints

$6x + 10y \ge 60$

or $3x + 5y \ge 30$

$4x + 4y \ge 32$

or $x + y \ge 8$

and $x \ge 0, y \ge 0$

Converting them into equations we obtain the following equations :

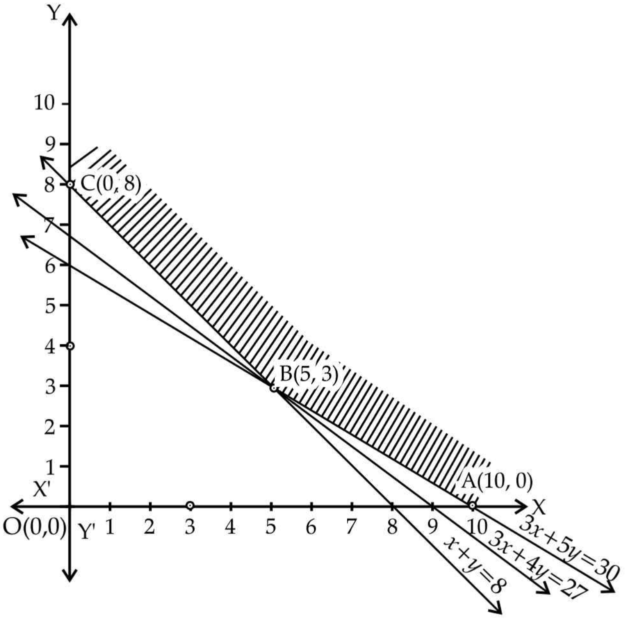
$3x + 5y = 30,$ $\quad x + y = 8$

$\Rightarrow \quad y = \frac{30 - 3x}{5}$ $\quad \Rightarrow \quad y = 8 - x$

x	0	10	5
y	6	0	3

x	0	8	5
y	8	0	3

The lines are shown on the graph paper and the feasible region is shown shaded. It is an unbounded region.



The corner points are

$A(10, 0)$, $B(5, 3)$ and $C(0, 8)$

At the corner point the value of $Z = 150x + 200y$

At $A(10, 0)$ $\quad Z = 1500$

At $B(5, 2)$ $\quad Z = 150 \times 5 + 200 \times 3$

$= 750 + 600 = 1350$

At $C(0,8)$ $\quad Z = 1600$

As the feasible region is unbounded, we draw the graph of the half-plane.

$$150x + 200y \leq 1350$$

$$3x + 4y \leq 27$$

There is no point common with the feasible region, therefore, Z has minimum value.

Minimum value of Z is ₹ 1350 and it occurs at the point $B(5, 3)$.

Hence, the labour cost is ₹ 1350 when P works for 5 days and Q works for 3 days.

Q. 10. A manufacturer manufactures two types of tea-cups, *A* and *B*. Three machines are needed for manufacturing the tea cups. The time in minutes required for manufacturing each cup on the machines is given below :

Type of Cup	Times in minutes		
	Machine I	Machine II	Machine III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of six hours per day. If the profit on each cup of type *A* is ₹ 1.50 and that on each cup of type *B* is ₹ 1.00, find the number of cups of each type that should be manufacturing in a day to get maximum profit. **[ISC, 2011]**

Sol. Let x be the number of A type tea cups and y be the number of B type tea cups.

Items	No.	Machine			Profit/ unit
		I	II	III	
Type A	x	12 min	18 min	6 min	₹ 1.5
Type B	y	6 min	0 min	9 min	₹ 1
Max Available Time		6 Hours $= 6 \times 60$ min $= 360$ min	6 Hours $= 6 \times 60$ min $= 360$ min	6 Hours $= 6 \times 60$ min $= 360$ min	

The problem can be formulated as

$$\text{Max } Z = 1.50x + 1.00y$$

$$\text{Max } Z = 1.50x + y$$

Subject to the constraints

$$12x + 6y \leq 360, \textit{ i.e.}, 2x + y \leq 60$$

$$18x + 0.y \leq 360, \textit{ i.e.}, x \leq 20$$

$$6x + 9y \leq 360, \textit{ i.e.}, 2x + 3y \leq 120$$

and $\quad x \geq 0, y \geq 0$

(Non-negative constraints)

We draw the straight lines. $2x + y = 60$

$x = 20$, $2x + 3y = 120$

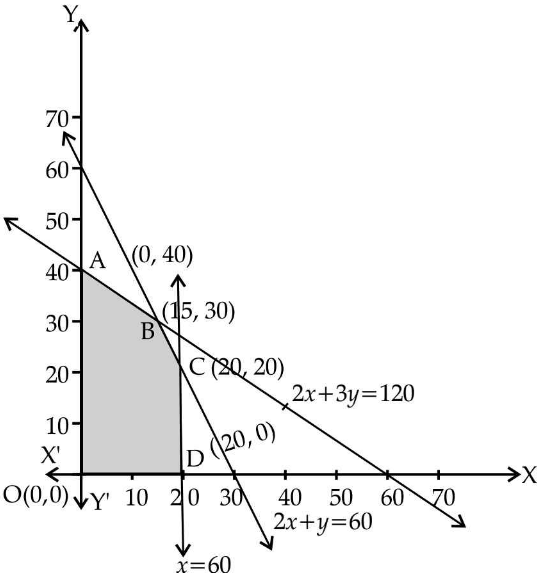
For $2x + y = 60$

x	30	0
y	0	60

For $2x + 3y = 120$

x	60	0
y	0	40

The shaded portion shows the feasible region. The points B and C are points of intersection of lines $2x + y = 60$, with $2x + 3y = 120$ and $x = 20$ with $2x + y = 60$ respectively.



Thus, the 5 corner points of the feasible region are $O(0,0)$, $A(0, 40)$, $B(15, 30)$, $C(20,20)$ and $D(20, 0)$.

$$Z = 1.5x + y$$

Corner Point	Value of Z
$O(0, 0)$	0
$A(0, 40)$	40
$B(15, 30)$	52.5 (Max)
$C(20, 20)$	50
$D(20, 0)$	30

Clearly, the maximum profit is at $x = 15$ and $y = 30$.

Thus, the manufacturer should manufacture 15 cups of type A and 30 cups of type B to get maximum profit in a day.

Q. 11. A company produces two types of items, *P* and *Q*. Manufacturing of both items requires the metals gold and copper. Each unit of item *P* requires 3 gms of gold and 1 gm of copper while that of item *Q* requires 1 gm of gold and 2 gm of copper. The company has 9 gm of gold and 8 gm of copper in its store. If each unit of item *P* makes a profit of ₹ 50 and each unit of item *Q* makes a profit of ₹ 60, determine the number of units of each item that the company should produce to maximize profit. What is the maximum profit ? **[ISC, 2010]**

Sol.

Items	Total items present in store		Profit per unit
	9 gm	8 gm	
	Gold (gm)	Copper (gm)	
P	3	1	₹ 50
Q	1	2	₹ 60

Let x_1 = number of units of item P.
x_2 = number of units of item Q.

$$\text{Max } Z = 50x_1 + 60x_2$$

Subject to the constraints

$$3x_1 + x_2 \le 9$$
$$x_1 + 2x_2 \le 8$$

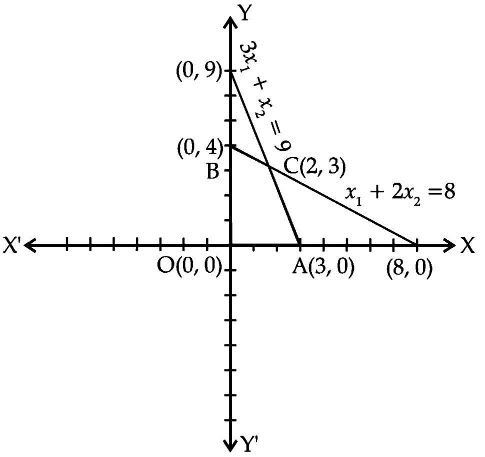
such that $x_1, x_2 \ge 0$

Now, $3x_1 + x_2 = 9$ ...(i)

x_1	0	3
x_2	9	0

and $x_1 + 2x_2 = 8$ ...(ii)

x_1	0	8
x_2	4	0



Now, consider the equation,

$$3x_1 + x_2 = 9 \times 2$$
$$x_1 + 2x_2 = 8$$
$$6x_1 + 2x_2 = 18$$
$$x_1 + 2x_2 = 8$$
$$\underline{(-)\ \ (-)\ \ \ (-)}$$ On subtracting
$$5x_1 = 10$$
$$\Rightarrow \quad x_1 = 2$$

From equation (i),

$$3 \times 2 + x_2 = 9$$
$$\Rightarrow \quad x_2 = 9 - 6 = 3$$

The corner points of the feasible region are $O(0, 0)$, $A(3, 0)$, $B(0, 4)$ and $C(2, 3)$

Corner points	$Z = 50x_1 + 60x_2$
at $O(0, 0)$,	$Z = 0$
at $A(3, 0)$,	$Z = 50 \times 3 + 60 \times 0 = 150$
at $C(2, 3)$,	$Z = 50 \times 2 + 60 \times 3 = 280$
at $B(0, 4)$,	$Z = 50 \times 0 + 60 \times 4 = 240$

∴ The company should produce 2 items of types P and 3 items of type Q.

∴ Maximum profit = ₹ 280

AI **Q. 12. A new cereal, formed of a mixture of bran and rice, contains at least 88 grams of protein and at least 36 milligram of iron. Knowing that bran contains 80 gram of protein and 40 milligram of iron per kilogram, and that of rice contains 100 gram of protein and 30 milligrams of iron per kilogram, find the minimum cost of producing a kilogram of this new cereal if bran costs ₹ 28 per kilogram and rice costs ₹ 25 per kilogram. [ISC, 2009]**

Sol. Let x kg of bran and y kg rice be produced.

Minimize, $Z = 28x + 25y$

Subject to constraints

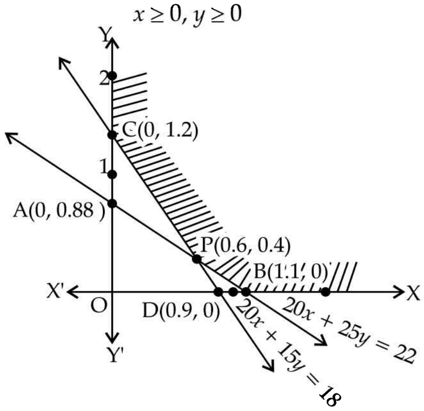
$$\frac{80}{1000}x + \frac{100}{1000}y \ge \frac{88}{1000}$$

$$\Rightarrow \quad 20x + 25y \ge 22 \quad \text{...(1)}$$

and $$\frac{0.040}{1000}x + \frac{0.030}{1000}y \ge \frac{0.036}{1000}$$

$$\Rightarrow \quad 20x + 15y \ge 18 \quad \text{...(2)}$$

$$x \ge 0, y \ge 0$$



First we draw the line AB and CD whose equations are $20x + 25y \ge 22$ and $20x + 15y \ge 18$, respectively.

Now,

$20x + 25y = 22$

	A	B
x	0	1.1
y	0.88	0

$20x + 15y = 18$

	C	D
x	0	0.9
y	1.2	0

The feasible region is $OBPCO$ which is shaded in the figure.

The vertices of the feasible region are $B(1.1, 0)$, P (0.6, 0.4) and $C(0, 1.2)$. P is the point of intersection of the lines $20x + 25y = 22$ and $20x + 15y = 18$. Solving those we get the point $P(0.6, 0.4)$

The minimum value of Z is :

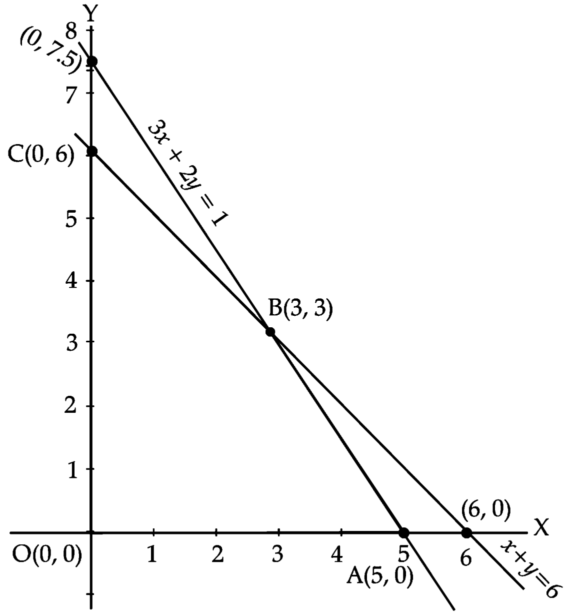
Corner points	$Z = 28x + 25y$
At $B(1.1, 0)$	$Z = 28 \times 1.1 + 25 \times 0 = 30.8$
At $P(0.6, 0.4)$	$Z = 28 \times 0.6 + 25 \times 0.4 = 26.8$
At $C(0, 1.2)$	$Z = 0 + 25 \times 1.2 = 30$

Hence, the minimum value of Z is 26.8 at the point $P(0.6, 0.4)$

Q. 13. Manufacturer produces two types of steel trunks. He has two machines, *A* and *B*. The first type of trunk requires 3 hours on machine *A* and 3 hours on machine *B*. The second type requires 3 hours on machine *A* and 2 hours on machine *B*. Machines *A* and *B* can work at most for 18 hours and 15 hours per day respectively. He earns a profit of ₹ 30 per trunk on the first type of trunk and ₹ 25 per trunk on the second type. Formulate a Linear Programming Problem to find out how many trunks of each type he must make each day to maximize his profit. **[ISC, 2008]**

Sol. Let x units of A type and y units of B type be produced.

$\therefore$ Max $Z = 30x + 25y$



Under the following restrictions

$$3x + 3y \leq 18$$
$$3x + 2y \leq 15$$
$$x \geq 0, y \geq 0$$

Now $x + y = 6$ and $3x + 2y = 15$

x	0	6	3
y	6	0	3

x	0	5	3
y	7.5	0	3

We draw the two constraints on the graph paper and indicate the feasible region. The region $OABC$ represents the set of all feasible solutions.

The co-ordinates of the corner points of the feasible region are $O(0, 0)$, $A(5, 0)$, $B(3, 3)$ and $C(0, 6)$.

Now, $Z = 30x + 25y$

At $O(0, 0)$, $Z = 30(0) + 25(0) = 0$

At $A(5, 0)$, $Z = 30(5) + 25(0) = 150$

At $B(3, 3)$, $Z = 30(3) + 25(3) = 165$

At $C(0, 6)$, $Z = 30(0) + 25(6) = 150$

Maximum value of $Z = 165$, when $x = 3$ and $y = 3$

Hence, 3 trunks of first type and 3 trunks of second type should be produced each day in order to make the maximum profit.

SELF ASSESSMENT TEST

UNIT–X

Maximum Marks : 20 **Timing : 1 Hour**

Q. 1. **One kind of cake requires 200 g of flour and 25 g of fat, and another kind of cake requires 100 g of flour and 50 g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes. Formulate the above as an LPP and solve it graphically.**

Q. 2. **A retired person wants to invest an amount of ₹50,000. His broker recommends investing in two type of bonds '*A*' and '*B*' yielding 10% and 9% return respectively on the invested amount. He decides to invest at least ₹20,000 in bond '*A*' and at least ₹ 10,000 in bond '*B*'. He also wants to invest at least as much in bond '*A*' as in bond '*B*'. Solve this linear programming problem graphically to maximise his returns.**

Q. 3. **A company produces soft drinks that have a contract which requires that a minimum of 80 units of chemical *A* and 60 units of the chemical *B* go into each bottle of the drink. The chemicals are available in prepared mix packets from two different suppliers. Supplier *S* had a packet of mix of 4 units of *A* and 2 units of *B* that costs ₹ 10. The supplier *T* has a packet of mix of 1 unit of *A* and 1 unit of *B* that costs ₹ 4. How many packets of mixes from *S* and *T* should the company purchase to honour the contract requirement and yet maintain the minimum cost ? Make a LPP and solve graphically.**

Q. 4. **Two godowns *A* and *B* have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, *D*, *E* and *F*, whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table :**

Transportation Cost per quintal (in ₹)

From/To	*A*	*B*
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum cost?

OSWAAL EXAM TOOLS
OSWAAL PREPARATION TOOLS
Solutions for Self-assessment Test
Visit : *http://qr.w69b.com/g/pi8gdC0qk* OR Scan the Code

WRITING NOTES

1. ....
2. ....
3. ....
4. ....
5. ....
6. ....
7. ....
8. ....
9. ....
10. ....
11. ....
12. ....
13. ....
14. ....
15. ....
16. ....
17. ....
18. ....
19. ....
20. ....
21. ....
22. ....
23. ....
24. ....
25. ....

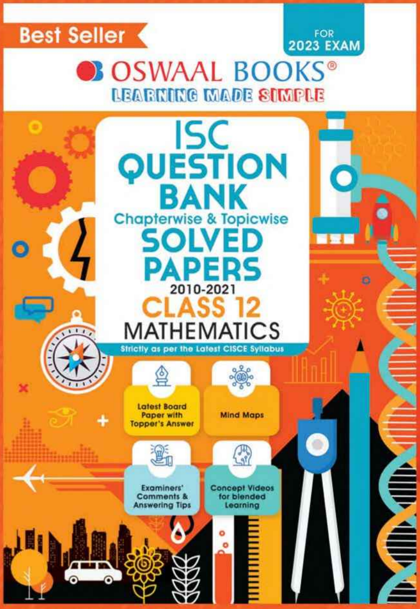
Highly recommended titles from
OSWAAL BOOKS®
LEARNING MADE SIMPLE

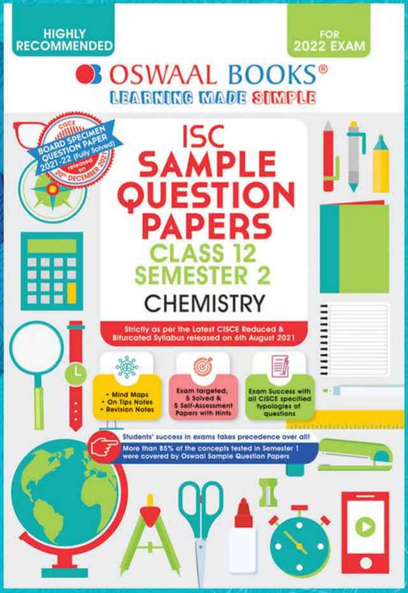
ISC
QUESTION BANK
FOR CLASS XII

BENEFITS

- Latest Board Examination Paper
- Latest Solved Board Specimen Paper
- Strictly as per the Latest Syllabus
- 'On Tips Notes' & Mind Map
- Topics Found Difficult & Suggestions for Students
- Examiners'Comments & Answering Tips
- Concept Videos for blended Learning

ALL SUBJECTS ARE AVAILABLE IN THIS SERIES


Best Seller
FOR 2023 EXAM
OSWAAL BOOKS®
LEARNING MADE SIMPLE
ISC QUESTION BANK
Chapterwise & Topicwise
SOLVED PAPERS
2010-2021
CLASS 12
MATHEMATICS
Strictly as per the Latest CISCE Syllabus
Latest Board Paper with Topper's Answer
Mind Maps
Examiners' Comments & Answering Tips
Concept Videos for blended Learning


HIGHLY RECOMMENDED
FOR 2022 EXAM
OSWAAL BOOKS®
LEARNING MADE SIMPLE
ISC
SAMPLE QUESTION PAPERS
CLASS 12
SEMESTER 2
CHEMISTRY
Strictly as per the Latest CISCE Reduced & Bifurcated Syllabus released on 6th August 2021
• Mind Maps • On Tips Notes • Revision Notes
Exam targeted, 5 Solved & 5 Self-Assessment Papers with Hints
Exam Success with all CISCE specified typologies of questions
Students' success in exams takes precedence over all!
More than 85% of the concepts tested in Semester 1 were covered by Oswaal Sample Question Papers

ISC
SAMPLE QUESTION PAPER
FOR CLASS XII

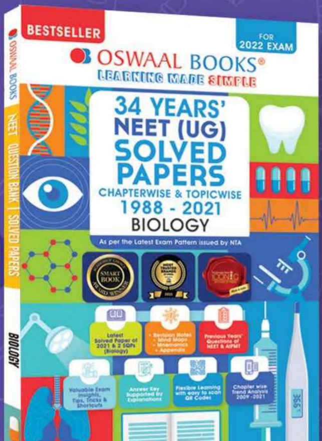
BENEFITS

- Updated Typologies of Questions as per Latest ISC Specimen Paper
- Strictly as per the latest syllabus
- 'On Tips Notes' & Mind Map
- Topics found difficult & suggestions for students
- Answer key with Explanations for every question Academically Important Questions (AI)

ALL SUBJECTS ARE AVAILABLE IN THIS SERIES

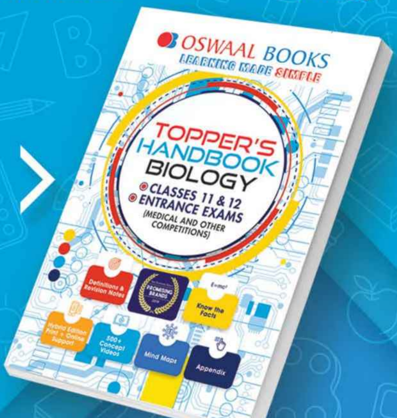
Scan this QR code on your mobile for instant access to all these books and more.

Get future ready
with comprehensive titles from
OSWAAL BOOKS®
LEARNING MADE SIMPLE


BESTSELLER
OSWAAL BOOKS®
LEARNING MADE SIMPLE
FOR 2022 EXAM
34 YEARS' NEET (UG) SOLVED PAPERS
CHAPTERWISE & TOPICWISE
1988 - 2021
BIOLOGY

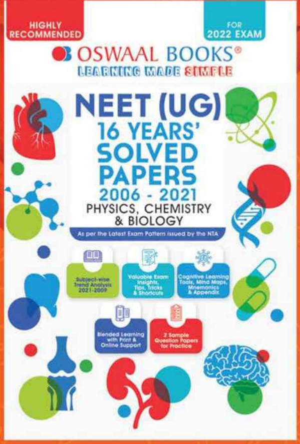
TOPPER'S HANDBOOK
BENEFITS
- Definitions and Revision Notes
- Mind Maps
- 500+ Concept Videos
- Hybrid Edition with Print & Online Support,
- Useful for Class 12 Board Examination and medical and other competitive examinations

SUBJECTS AVAILABLE IN THIS SERIES
- Physics
- Chemistry
- Mathematics
- Biology


OSWAAL BOOKS
LEARNING MADE SIMPLE
TOPPER'S HANDBOOK
BIOLOGY
CLASSES 11 & 12
ENTRANCE EXAMS
(MEDICAL AND OTHER COMPETITIONS)

34 YEARS NEET (UG) SOLVED PAPERS
BENEFITS
- Latest Sample Question Papers
- Previous 34 Years' Questions of NEET & AIPMT
- Chapterwise and Topicwise Presentation
- Hybrid Edition with Print & Online Support

SUBJECTS AVAILABLE IN THIS SERIES
- Physics
- Chemistry
- Biology


HIGHLY RECOMMENDED
FOR 2022 EXAM
OSWAAL BOOKS®
LEARNING MADE SIMPLE
NEET (UG) 16 YEARS' SOLVED PAPERS
2006 - 2021
PHYSICS, CHEMISTRY & BIOLOGY

NEET (UG)
16 YEARS' SOLVED PAPERS
2006-2021
BENEFITS
- Subjectwise Trend Analysis 2021-2009
- Valuable Exam Insights, Tips, Tricks & Shortcuts
- Blended Learning with Print & Online Support
- 2 Sample Question Papers for Practice
- Cognitive Learning Tools, Mind Maps, Mnemonics & Appendix

JEE (MAIN) MOCK TEST
15 SAMPLE QUESTION PAPERS &
JEE (MAIN) SOLVED PAPERS

- Latest Solved Papers (all 4 Shifts) of 2021 Exam
- Concept based Revision Notes
- Valuable Exam insights, Tips, Tricks & Shortcuts
- Two Sample Practice Papers
- 'Strictly as per the scheme of examination for JEE (MAIN) 2022 issued by NTA
- Fully solved 15 Mock Papers with 90 Question each
- Get concpet clarity - Mind Maps

HIGHLY RECOMMENDED
FOR 2021 EXAM
OSWAAL BOOKS®
LEARNING MADE SIMPLE
JEE (MAIN) MOCK TEST 15 SAMPLE QUESTION PAPERS
• PHYSICS • CHEMISTRY • MATHEMATICS

HIGHLY RECOMMENDED
FOR 2022 EXAM
OSWAAL BOOKS®
LEARNING MADE SIMPLE
JEE (MAIN) SOLVED PAPERS
CHAPTERWISE & TOPICWISE
2019 - 2021
MATHEMATICS

- Latest Solved Papers (all 4 Shifts) of 2021 Exam
- Concept based Revision Notes
- Valuable Exam insights, Tips, Tricks & Shortcuts
- Two Sample Practice Papers